On the Computational Complexity of Ordered Subgraph Recognition

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ABSTRACT

Let $(G, <)$ be a finite graph G with a linearly ordered vertex set V. We consider the decision problem $(G, <)$ORD to have as an instance an (unordered) graph Γ and as a question whether there exists a linear order $<$ on $V(\Gamma)$ and an order preserving graph isomorphism of $(G, <)$ onto an induced subgraph of Γ. Several familiar classes of graph are characterized as the yes-instances of $(G, <)$ORD for appropriate choices of $(G, <)$. Here the complexity of $(G, <)$ORD is investigated. We conjecture that for any 2-connected graph G, $G \not\cong K_k$, $(G, <)$ORD is NP-complete. This is verified for almost all 2-connected graphs. Several related problems are formulated and discussed. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

In this paper we shall consider the computational complexity of the following graph recognition problem. Given an ordered graph $(G, <)$ where $G = (V, E)$ is a graph with no loops or multiple edges, and $<$ is a linear ordering of V, we pose this decision problem.

$$(G, <)\textbf{Ord}$$

Instance. A graph $\Gamma = (V, E)$.

Question. Does there exist an ordering $<$ of $V(\Gamma)$ such that

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$$(G,<)\not\leq(\Gamma,<)?$$

By $(G,<)\leq(\Gamma,<)$ we mean that there is an injection $\phi: V(G)\rightarrow V(\Gamma)$ such that $v_i<v_j$ implies $\phi(v_i)<\phi(v_j)$ and ϕ is a graph isomorphism of G onto the induced subgraph of Γ on $\phi(V(G))$. In the future, we often abuse notation by using $<$ for $\prec$ and just say that $(G,<)$ is an induced ordered subgraph of $(\Gamma,<)$.

Several graph characterization problems that at first may not seem to be related to ordered subgraphs may be formulated as $(G,<)$ORD for the appropriate choice of $(G,<)$. Two well-known examples of this are characterizations of comparability graphs and chordal graphs.

A graph $\Gamma=(V,E)$ is said to be the *comparability graph* of the partially ordered set $P=(V,\leq)$, denoted $\Gamma=\mathrm{Comp}(P)$, if for all $a,b\in V$, $\{a,b\}\in E$ if and only if $a<b$ or $b<a$ in P. These graphs have been well studied and many results relating to them appear in [14]. Ordered graphs provide a well-known characterization.

Theorem 1.1. *A graph Γ is a comparability graph if and only if Γ is a satisfying instance of $(G,<)$ORD with $V(G,<)=(1,2,3)$ and $E(G,<)=\{\{1,2\},\{2,3\}\}$.*

Since the problem of determining whether a graph is a comparability graph has been shown to be solvable in polynomial time, this is an example of a $(G,<)$ORD problem which is in P.

Similarly, a graph Γ is said to be a *chordal graph* if every cycle of length 4 or more has a chord. These graphs are also sometimes called triangulated, rigid circuit, monotone transitive, and perfect elimination graphs, and have an extensive literature [14]. The reason for the term perfect elimination graph leads us to a $(G,<)$ORD characterization of chordal graphs as follows. An ordering $(v_1,v_2,\ldots,v_k)$ of the vertices of a graph is said to be a *perfect elimination scheme* if, for every i, $1\leq i\leq k$, $N(v_i)\cap\{v_{i+1},\ldots,v_k\}$ induces a complete subgraph, where $N(v_i)$ is the neighborhood of v_i. Dirac [8] showed that a graph Γ is a chordal graph if and only if it has a perfect elimination scheme. A moment's thought will show that this is equivalent to the following.

Theorem 1.2. *A graph Γ is a chordal graph if and only if Γ satisfies $(G,<)$ORD, where $V(G,<)=(1,2,3)$ and $E(G,<)=\{\{1,2\},\{1,3\}\}$.*

Again the decision problem of chordal graph recognition has been shown to be solvable in polynomial time, so this is another $(G,<)$ORD problem that is in P.

A simple generalization of a greedy algorithm for recognizing chordal graphs given by [11] leads to the following theorem.

Theorem 1.3. *If $(G,<)$ is the ordered graph with vertex set $V(G,<)=(1,2,\ldots,k)$ for any $k\geq 3$ and edge set $E(G,<)=\{\{1,2\},\{1,3\},\ldots,\{1,k\}\}$, then $(G,<)$ORD is in P.*

Another example of $(G,<)$ORD that is in P is given by the complete ordered subgraph $(K_k,<)$, where $V(K_k,<)=(1,2,\ldots,k)$ and $E(K_k,<)$ consists of all of the $\binom{k}{2}$ possible edges. $(K_k,<)$ORD is solvable in polynomial time as it is equivalent to seeing if a graph has a complete subgraph of order k which is easily seen to be solvable in polynomial time as long as k is in our problem definition, not our input.

More examples of ordered graphs for which $(G,<)$ORD is solvable in polynomial time may be obtained from the following observations. The problem $(G,<)$ORD is solvable in polynomial time if and only if either of the problems $(G^c,<)$ORD or $(G,>)$ORD is solvable in polynomial time, where by G^c we mean the complement of G and by $>$ we mean the reverse order of $<$. The first of these observations is true because Γ satisfies $(G,<)$ORD if and only if Γ^c satisfies $(G^c,<)$ORD, and finding the complement of a graph Γ can be done in polynomial time. The second is true because a graph Γ satisfies $(G,<)$ORD if and only if Γ satisfies $(G,>)$ORD. Hence, whatever we deduce about the computational complexity of $(G,<)$ORD is also valid for $(G^c,<)$ORD and $(G,>)$ORD.

We will use somewhat nonstandard notation to represent our different mappings. Given $<$ on $V=(1,2,\ldots,k)$, we represent the mapping that sends v_1 to 1, v_2 to 2, and so on, by $(v_1,v_2,\ldots,v_k)$.

In this paper, we will show that for some large classes of ordered graphs the problem $(G,<)$ORD is NP-complete. First we look at the class of ordered graphs that are 2-connected and have automorphism group consisting only of the two mappings,

$$(1,2,\ldots,k) \qquad \text{and} \qquad (k,k-1,\ldots,1)\,,$$

where $(1,2,\ldots,k)$ is the ordered set of vertices of $(G,<)$.

Theorem 1.4. *If $(G,<)$ is a 2-connected ordered graph with the 2-element automorphism group consisting of the identity automorphism and the mapping*

$$(k,k-1,\ldots,1)\,,$$

then $(G,<)$ORD is NP-complete.

This result is proven by a polynomial reduction from the known NP-complete problem of $(k-1)$-colorability, specifically when we restrict our instances to graphs Γ, where $\Delta(\Gamma)\le d=d(k-1)$, an absolute constant depending only on k. An interesting corollary of this result is the NP-completeness of $(G,<)$ORD for a k-element path with the "natural" order and $k\ge 5$. Notice that since $k=3$ yields the comparability graph recognition problem, an easy one, only the naturally ordered 4-element path remains. In fact, this problem can be shown also to be NP-complete as it contains as a subproblem the known NP-complete problem of deciding if a graph of girth at least 5 is 3-colorable.

Our final result applies to less restricted automorphism groups than the first one and settles the complexity question of $(G,<)$ORD for a much larger set of ordered graphs.

Theorem 1.5. *Let $(G,<)$ be a 2-connected ordered graph, where*

$$V(G,<)=(1,2,\ldots,k)\,,$$

whose automorphism group contains neither of the mappings

$$(2, 3, \ldots, k, 1) \quad \text{or} \quad (k, k-1, \ldots, 1).$$

Then $(G, <)$ORD is NP-complete.

Our proof of this result is much more involved than the first and causes us to define the rather artificial decision problem d-MIDDLE whose NP-completeness must be established before that of $(G, <)$ORD.

It is easy to show that almost all graphs are 2-connected, and it follows from a result of Erdős and Renyi [9] that almost all graphs have the trivial automorphism group. Hence, in a probabilistic sense, almost all 2-connected graphs satisfy the hypotheses of Theorem 1.5. This combined with the earlier mentioned results and our intuition have led us to make the following conjecture on the computational complexity of $(G, <)$ORD.

Conjecture 1.6. *For any ordered graph $(G, <)$, such that $(G, <)$ is neither a complete nor an empty graph, $(G, <)$ORD is NP-complete if either G or its complement is 2-connected.*

While the full conjecture remains unsettled, by the observation above, we have made significant progress. Notice that because of the result on 4-vertex paths, the conjecture cannot be extended to a characterization. This leaves the question open as to whether there is some easy characterization of the $(G, <)$'s for which $(G, <)$ORD is NP-complete.

2. NOTATION AND TERMINOLOGY

We take $X = \{x_1, x_2, \ldots, x_m\}$ to be an unordered m-element set. We define $|X| = m$. By $[n]$ we will denote the set of integers $\{1, 2, \ldots, n\}$. If we wish to give a specific order to a set, we will denote it by $X = (x_1, x_2, \ldots, x_m)$ where here the order is $x_1 < x_2 < \cdots < x_m$. By $\mathcal{P}(X)$ we denote the set of all subsets of X, and by $[X]^t$ the set of all t-element subsets of X. Given a linear order $<$ on a set X and subsets Y and Z of X, we write $Y < Z$ if for every $y \in Y$ and every $z \in Z$, $y < z$. We will also say that $Y <^\delta Z$ if for all subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ such that $|Y'| \geq \delta|Y|$, $|Z'| \geq \delta|Z|$, we have $Z' \not< Y'$. So, $Y \not<^\delta Z$ means that there exist subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ such that $|Y'| \geq \delta|Y|$, $|Z'| \geq \delta|Z|$, and $Z' < Y'$.

By a *hypergraph* $\mathcal{H} = (V, E)$ we mean a set V of *vertices* and a set $E \subseteq \mathcal{P}(V)$ of *edges*. If for every $e \in E$, $|e| = k$ then we say that $\mathcal{H}$ is a *k-uniform hypergraph*. A 2-uniform hypergraph will be called a *graph*. An *ordered graph* $(G, <)$ is a graph G along with a linear order $<$ on $V(G)$. For any $x \in V$ we will define $\deg(x) = |\{e \in E : x \in e\}|$. Further, let $\Delta(\mathcal{H}) = \max_{x \in V}(\deg(x))$.

At some points in this paper we wish to define an orientation on the edges of our hypergraphs. By an *oriented* hypergraph $\mathcal{H} = (V, E)$ we mean a hypergraph where each edge is an ordered set $(v_1, v_2, \ldots, v_k)$. Given a linear order $<$ on V, we say that $e = (v_1, v_2, \ldots, v_k)$ is an *increasing edge* with respect to $<$ if

$$v_1 < v_2 < \cdots < v_k$$

and define it to be a *decreasing edge* with respect to $<$ in the analogous way. We further say that e is *monotone* with respect to $<$ if it is either increasing or decreasing with respect to $<$.

Given a hypergraph $\mathscr{H} = (V, E)$, we denote by a *path* in $\mathscr{H}$ a sequence $P = (v_1, e_1, v_2, e_2, \ldots, e_l, v_{l+1})$ where $v_1, v_2, \ldots, v_{l+1} \in V$, $v_i, v_{i+1} \in e_i \in E$, $1 \leq i \leq l$, and all edges and vertices are distinct except possibly v_1 and v_{l+1}. The vertex v_1 is called the *initial vertex* of P and v_{l+1} is called the *terminal vertex* of P. We say that $l = l(P)$ is the *length* of P. If we have $v_1 = v_{l+1}$, then we say P is a *cycle* and usually denote it by the letter C. If $\mathscr{C}$ is the set of all cycles of a hypergraph $\mathscr{H} = (V, E)$, then we define the *girth* of $\mathscr{H}$, girth($\mathscr{H}$), to be $\min_{C \in \mathscr{C}}\{\text{length}(C)\}$. For a hypergraph $\mathscr{H}$ and any two vertices x and y of $\mathscr{H}$, we define the *distance* from x to y, $\text{dist}_{\mathscr{H}}(x, y)$, to be the length of the shortest path in $\mathscr{H}$ with initial vertex x and terminal vertex y. If no such path exists we say that $\text{dist}_{\mathscr{H}}(x, y) = \infty$.

Given a metric $\rho: V \times V \to \mathbb{Z} \cup \{\infty\}$ on the vertex set of a hypergraph $\mathscr{H} = (V, E)$ we define a *ρ-cycle* to be a sequence $C = \{P_1, P_2, \ldots, P_m\}$ of paths in $\mathscr{H}$ satisfying the following conditions:

(i) If $i \neq j$, then P_i and P_j are disjoint except possibly at the endpoints mentioned below.

(ii) If $v_{i,t}$ is the terminal vertex of P_i and $v_{i+1,1}$ is the initial vertex of P_{i+1}, then $\rho(v_{i,t}, v_{i+1,1}) < \infty$.

(iii) If $v_{m,t}$ is the terminal vertex of P_m and $v_{1,1}$ is the initial vertex of P_1, then $\rho(v_{m,t}, v_{1,1}) < \infty$.

We define the

$$\text{length}(C) = \sum_{i=1}^{t} l(P_i) + \sum_{i=1}^{t-1} \rho(v_{i,t}, v_{i+1,1}) + \rho(v_{m,t}, v_{1,1}) .$$

Throughout the text, we will be referring to the *automorphism group* of an ordered graph. We use the term automorphism in the graph theoretic, rather than order theoretic manner. That is, for an ordered graph $(G, <) = (V, E)$, a mapping $f: V \to V$ belongs to the automorphism group $\text{Aut}(G, <)$ if and only if it is a bijection where

$$\{f(x), f(y)\} \in E \text{ if and only if } \{x, y\} \in E .$$

As indicated earlier, we use somewhat nonstandard notation to represent our different mappings. Given $<$ on $V = (1, 2, \ldots, k)$, we represent the mapping that sends v_1 to 1, v_2 to 2, and so on, by $(v_1, v_2, \ldots, v_k)$. All of the other graph and hypergraph terminology that we will be using is quite standard. Please see [2] for any questions on these.

The proofs in the next section of the paper all depend upon probabilistic arguments. In these arguments we are assuming that we have a probability space with a binomial distribution $\text{Bi}(n, p)$. This distribution may be thought of as a random subset, Y, generated from an n-element set X, where every $x \in X$ is in Y with probability p and all selections are independent of one another. For a more complete discussion of these probabilistic arguments, see [1].

In this paper we will be discussing many different decision problems. The main one, $(G, <)$ORD has been introduced already. The others of interest to us now follow.

Graph 3-colorability (3-COL)

Instance. A graph $\Gamma = (V, E)$.

Question. Does there exist a good 3-coloring of the vertices of Γ?

By a good 3-coloring we mean a function $f: V(\Gamma) \to [3]$ such the for all $\{x, y\} \in E(\Gamma)$, $f(x) \neq f(y)$. This problem can be generalized to the following.

Graph k-Colorability (k-COL)

Instance. A graph $\Gamma = (V, E)$.

Question. Does there exist good k-coloring of the vertices of Γ?

This problem has been shown to be NP-complete for any fixed $k \geq 3$, and for $k = 3$ even if we restrict it to graphs of maximum degree 4 [13], and will be the basis for the proofs of all of our further NP-completeness results. The next decision problem is used as an intermediate step in the proof of Theorem 1.5.

d-MIDDLE

Instance. A structure $\mathcal{D} = (S, E_{MIM}, E_{MOM})$ where S is a finite set, $E_{MIM}, E_{MOM} \subseteq [S]^3$, and $\forall s \in S$, $|\{T \mid T \in E_{MIM} \cup E_{MOM}, s \in T\}| \leq d$.

Question. Does there exist an ordering of S such that

$$\forall (a, b, c) \in E_{MIM}, a < b < c \text{ or } c < b < a,$$

$$\forall (a, b, c) \in E_{MOM}, \neg(a < b < c \text{ or } c < b < a)?$$

With these definitions in mind we are ready to begin with the proofs of some preliminary lemmas.

3. THE GIRTH MACHINE

In order to obtain instances of our ordered graph problem, where we can control the copies of G appearing, we shall first construct an appropriate oriented hypergraph with large girth from one, without necessarily large girth but bounded degree, and then insert a copy of $(G, <)$ into each hypergraph edge so that the ordering of the vertices corresponds to the orientation of the edge. Because we shall assume that G is 2-connected and know that the girth of the hypergraph is larger than $k = |V(G)|$, the only copies of G created will be those confined to single hypergraph edges.

We shall construct our hypergraph piece by piece (Lemma 3.3), showing the existence of each piece by probabilistic methods (Lemma 3.2). As the problem size considered in Lemma 3.2 depends only on fixed constants, the "construction" of an appropriate piece can be done for fixed constants in constant time. Thus we may piece together the entire hypergraph in time linear with respect to the number of edges of our original hypergraph.

Throughout this section we will make use of the following lemma, used to bound the tail of the binomial distribution, which is proven in [17].

Lemma 3.1. *For all* $0<p<1$ *and* $0<\epsilon<1$,

$$\Pr(|\mathrm{Bi}(n, p) - np| \geq \epsilon np) \leq 2\exp(-\eta^2(1-\eta)np_{\min}/2)$$

where $\eta = \min(\epsilon, 2/3)$ *and* $p_{\min} = \min(p, 1-p)$.

Using this, we can prove our first lemma which states the existence of the pieces with the properties we will need for our later constructions. It is a generalization of a lemma proven in [15].

Lemma 3.2. *Given integers* $n \geq 1$, $l \geq 2$, $j_1, \ldots, j_n \geq 1$, *and reals* c_1, $\delta > 0$, *let* $X_1, X_2, \ldots, X_n$ *be pairwise disjoint sets of size* N *and* $\epsilon = 1/l^3$. *Let* $V = \bigcup_{i=1}^n X_i$ *and*

$$\rho : V \times V \to \mathbb{Z} \cup \{\infty\}$$

be a metric satisfying, for every $x \in V$,

$$|\{y \in V : \rho(x, y) = i\}| \leq (c_1 N^\epsilon)^i . \tag{1}$$

Then if $N \geq N_0(c_1, l, n, j_1, \ldots, j_n, \delta)$, *there exists an* m*-uniform hypergraph* $\mathcal{H} = (V, E)$, *where* $m = \sum_{i=1}^n j_i$, *which enjoys the following properties.*

1. *For all edges* $e \in E$, $|X_i \cap e| = j_i$, *for all* $i = 1, 2, \ldots, n$.
2. $\Delta(\mathcal{H}) < 1.5N^\epsilon$.
3. *For every* $X_i' \subseteq X_i$ *such that* $|X_i'| \geq \delta|X_i|$, *for* $i = 1, 2, \ldots, n$,
$$\left|\left[\bigcup_{i=1}^n X_i'\right]^m \cap E\right| \geq c_2 N^{1+\epsilon}$$
for $c_2 = c_2(\delta, l, j_1, \ldots, j_n)$.
4. $\mathcal{H}$ *contains no* ρ*-cycle of length* l *or less.*

Proof. Let $\mathcal{H}(N, p)$ be the random hypergraph on the vertex set $V = \bigcup_{i=1}^n X_i$, $|X_i| = N$, where for every $e \subseteq V$ such that $|e \cap X_i| = j_i$, $i = 1, 2, \ldots, n$, e is an edge of $\mathcal{H}(N, p)$ with probability

$$p = N^{1-m+\epsilon} .$$

We will show below that, with probability greater than 0, $\mathcal{H}(N, p)$ has the following properties provided that N is large enough:

(a) $\Delta(\mathcal{H}(N, p)) \leq 1.5n^\epsilon$.
(b) If $X_i' \subseteq X_i$, such that $|X_i'| \geq \delta|X_i|$ for $i = 1, 2, \ldots, n$, then

$$\left|\left[\bigcup_{i=1}^{n} X_i'\right]^m \cap E'\right| \geq c_2' N^{1+\epsilon},$$

where $c_2' = c_2'(\delta, j_1, \ldots, j_n)$ and $E' = E(\mathscr{H}(N, p))$.

(c) $\mathscr{H}(N, p)$ has fewer than $2N$ ρ-cycles of length at most l, for any metric ρ for which (1) holds.

If we can show this, then there exists a hypergraph $\mathscr{H}'$ in our probability space $\mathscr{H}(N, p)$ with these properties. If we take N large enough that $2N < \frac{c_2'}{2} N^{1+\epsilon}$, let $c_2 = \frac{c_2'}{2}$ and delete an edge from each ρ-cycle of length l or less in $\mathscr{H}'$, we will form a hypergraph $\mathscr{H}$ with the desired properties.

Given any vertex $x \in X_i$,

$$\mathbf{E}(\deg(x)) = \prod_{k \neq i} \binom{N}{j_k} \binom{N-1}{j_i - 1} p \leq N^{\epsilon}$$

and this random variable has binomial distribution. So, by Lemma 3.1, we get, for any $x \in V$,

$$\Pr(\deg(x) \geq 1.5N^{\epsilon}) \leq 2\exp(-cN^{\epsilon}).$$

Hence, we obtain

$$\Pr(\exists x \in V(\mathscr{H}(N, p)) : \deg(x) \geq 1.5N^{\epsilon}) \leq 2nN\exp(-cN^{\epsilon}) \leq 1/4 \tag{2}$$

for N large enough. So we have condition (a) with probability at least 3/4.

Now, for any collection of sets, $X_i' \subseteq X_i$, $|X_i'| \geq \delta |X_i|$, $i = 1, 2, \ldots, n$,

$$\begin{aligned}
\mathbf{E}\left(\left|\left[\bigcup_{i=1}^{n} X_i'\right]^m \cap E'\right|\right) &\geq \prod_{i=1}^{n} \binom{\delta N}{j_i} p \\
&= \frac{\delta^m N^m}{\Pi_{i=1}^n j_i!}(1 + o(1))p \\
&= \frac{\delta^m N^{1+\epsilon}}{\Pi_{i=1}^n j_i!}(1 + o(1)) \\
&\geq \frac{\delta^m N^{1+\epsilon}}{2\Pi_{i=1}^n j_i!} \\
&= cN^{1+\epsilon}
\end{aligned}$$

for N large enough. Since this distribution is also binomial, Lemma 3.1 gives

$$\Pr\left(\left|\left[\bigcup_{i=1}^{n} X_i'\right]^m \cap E'\right| \leq \frac{c}{2} N^{1+\epsilon}\right) \leq 2\exp(-c'N^{1+\epsilon}).$$

Hence,

$$\begin{aligned}
&\Pr\left(\exists X_i' \subseteq X_i, |X_i'| \geq \delta |X_i|, 1 \leq i \leq n: \left|\left[\bigcup_{i=1}^{n} X_i'\right]^m \cap E\right| \leq \frac{c}{2} N^{1+\epsilon}\right) \\
&\leq \binom{N}{\delta N}^n 2\exp(-c'N^{1+\epsilon}) \\
&\leq \left(\frac{e}{\delta}\right)^{n\delta N} 2\exp(-c'N^{1+\epsilon}) \\
&< 1/4 \qquad (3)
\end{aligned}$$

for N large enough. So if we let $c_2' = c/2$, we get condition (b) of our list with probability greater than 3/4.

Notice that since ρ satisfies the condition that, for any $x \in V$,

$$|\{y \in V: \rho(x, y) = i\}| \leq (c_1 N^{\epsilon})^i ,$$

the number of vertices $y \in V$ such that $\rho(x, y) < l$ is at most

$$\sum_{i=0}^{l-1} c_1^i N^{\epsilon i} < N^{1.5/l^2}$$

for N large enough.

This means that for any path $P = \{v_1, e_1, v_2, e_2, \ldots, e_m, v_{m+1}\}$, there are less than $N^{1.5/l^2}$ vertices that could be the initial vertex of the next path in a ρ-cycle containing P of length at most l. So if $\mathbf{X}_{t,s}(\mathcal{H})$ is a random variable counting the number of ρ-cycles in $\mathcal{H}$ of length at most l with t paths containing a total of s edges in these paths, there are at most $(nN(N^{1.5/l^2}))^t$ ways of selecting the terminal and initial vertices of each pair of paths, at most $(nN)^{s-t}$ ways of picking the other intersecting vertices of the edges of each path and at most $(nN)^{(m-2)s}$ ways of picking the other vertices to go on each edge of each path. Hence,

$$\begin{aligned}
\mathbf{E}(\mathbf{X}_{t,s}(\mathcal{H}(N, p))) &\leq (nN(N^{1.5/l^2}))^t (nN)^{s-t} (nN)^{(m-2)s} p^s \\
&= n^{s+t(m-2)} N^{\frac{1.5t}{l^2} + \frac{s}{l^3}} \\
&= c_{s,t} N^{\frac{1.5t}{l^2} + \frac{s}{l^3}} .
\end{aligned}$$

Therefore, if we let $\mathbf{X}_l(\mathcal{H})$ be a random variable counting the number of ρ-cycles in $\mathcal{H}$ of length at most ℓ,

$$\begin{aligned}
\mathbf{E}(\mathbf{X}_l(\mathcal{H}(N, p))) &\leq \sum_{s=2}^{l} \sum_{t=1}^{s} \mathbf{E}(\mathbf{X}_{t,s}(\mathcal{H}(N, p))) \\
&\leq \sum_{s=2}^{l} \sum_{t=1}^{s} cN^{\frac{1.5t}{l^2} + \frac{s}{l^3}} < N
\end{aligned}$$

for N large enough.

So by Markov's Inequality we get that

$$\begin{aligned}
\Pr(\mathbf{X}_l(\mathcal{H}(N, p)) > 2n) &\leq \mathbf{E}(\mathbf{X}_l(\mathcal{H}(N, p)))/2N \\
&< N/2N = \tfrac{1}{2} .
\end{aligned}$$

So the probability that any one of our three conditions listed above does not hold is less than $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$. Hence the probability that all three of them hold is positive and, as noted at the beginning of the proof, we can obtain the desired hypergraph. □

The next lemma gives the main step in our construction of oriented hypergraphs with large girth.

Lemma 3.3 (The Girth Machine). *Let d, l, and k be positive integers at least 2, let $\delta>0$, and let $\mathcal{H}=(V,E)$ be a k-uniform oriented hypergraph with $\Delta(\mathcal{H})\le d$. Then there exists $N=N(d,k,l,\delta)$ and a construction, polynomial in the size of $\mathcal{H}$, of a k-uniform oriented hypergraph $\mathcal{H}'=(V\times X,E')$, where $|X|=N$, and satisfying*:

1. *For all $e'\in E'$, $e'=(v_1',v_2',\ldots,v_k')$, where $v_i'\in X_{v_i}=\{v_i\}\times X$, and $(v_1,v_2,\ldots,v_k)\in E$;*
2. *$\Delta(\mathcal{H}')\le d'N^{\epsilon}$, where $d'=d'(d,k,l,\delta)$ and $\epsilon=\frac{1}{l^3}$;*
3. *For $(v_1,v_2,\ldots,v_k)\in E$, $X'_{v_i}\subseteq X_{v_i}$ with $|X'_{v_i}|\ge\delta|X_{v_i}|$, for all $i\in[n]$,*
$$(X'_{v_1}\times X'_{v_2}\times\cdots\times X'_{v_k})\cap E'\ne\emptyset\,;$$
4. *Girth$(\mathcal{H}')>l$.*

Proof. Let N be an integer large enough that Lemma 3.2 holds with $n=k$, $j_i=j_2=\cdots=j_n=1$, $l=l$, $\delta=\delta$, and $c_1=1.5d(k-1)$. Let $(e_1,\ldots,e_p)$ be a listing of $E(\mathcal{H})$ and let X be a set of N distinct vertices. We inductively construct a sequence of k-uniform oriented hypergraphs $\mathcal{H}_0,\ldots,\mathcal{H}_p$ on $V\times X$ and set

$$\rho_i(x,y)=\mathrm{dist}_{\mathcal{H}_i}(x,y)\,.$$

Initially set $E(\mathcal{H}_0)=\emptyset$ and, so, $\rho_0(x,y)=\infty$ for all $x\ne y$ in $V\times X$.

For $i=1,2,\ldots,p$ we do the following procedure.

> If $e_i=(v_1,v_2,\ldots,v_k)$, then with the parameters of Lemma 3.2 specified above, $\rho=\rho_{i-1}$, and $X_i=X_{v_i}$, $i=1,2,\ldots,k$, let $\mathcal{H}^i$ be the hypergraph guaranteed by Lemma 3.2 and $\mathcal{H}_i=\mathcal{H}_{i-1}\cup\mathcal{H}^i$. Let $\rho_i=\mathrm{dist}_{\mathcal{H}_i}$.

Execution of each step of this construction depends on the verification of the following claim.

Claim 1. *For every i, every $z\in V\times X$ and every $r\le l$,*

$$|\{y\in V\times X:\rho_i(z,y)=r\}|\le(c_1N^{\epsilon})^r\,.$$

Recall that $c_1=1.5d(k-1)$.

Proof (of Claim 1). What we will actually show is that for all i,

$$\Delta(\mathcal{H}_i)\le 1.5dN^{\epsilon}\,.$$

This will imply our claim since every edge of $\mathcal{H}_i$ has cardinality k, which would imply that any vertex z would have at most $1.5d(k-1)N^{\epsilon}$ vertices of distance 1 from it.

Fix i and consider $\mathscr{H}_i$. Let $z=(v,x)\in V\times X$. Let

$$E_v=\{e_q\in E(\mathscr{H}): v\in e_q, q\le i\}\ .$$

Obviously $|E_v|\le d$. For each q such that $e_q\in E_v$, z is a vertex of $\mathscr{H}^q$. Further, these are the only hypergraphs in the collection whose union makes up $\mathscr{H}_i$, which have z as a vertex. Since $\Delta(\mathscr{H}^q)\le 1.5N^{\epsilon}$ for all q, $\deg_{\mathscr{H}_i}(z)\le d\cdot 1.5N^{\epsilon}$.

□ (Claim 1)

Therefore, we can construct $\mathscr{H}_p$. Notice that any cycle in $\mathscr{H}_i$ is a ρ-cycle at the stage where the last edge is added to it. Therefore, the length of any cycle in any of our $\mathscr{H}_i$'s is greater than l. So girth $(\mathscr{H}_i)>l$ for all i. We also get property (2) of the statement of our lemma for $\mathscr{H}_p$ if we let $d'=1.5d$, as well as property (1). Hence $\mathscr{H}_p$ is the hypergraph required. As to the time taken to construct our $\mathscr{H}^l$, since N was determined by d, δ, and k, not by $\mathscr{H}$, finding the hypergraph to define on each vertex set $e_i\times X$ had nothing to do with $\mathscr{H}$, so requires constant time. So, the time taken to construct $\mathscr{H}'$ is linear in terms of the size of $\mathscr{H}$, completing the proof of the lemma. □

4. THE PROOF OF THEOREM 1.4

The proof of Theorem 1.4 depends upon a polynomial reduction from the problem of $(k-1)$-COL where $k=|V(G)|$. This problem was shown to be NP-complete for any $(k-1)\ge 3$ by Garey, Johnson, and Stockmeyer in [13]. In the same paper they showed that it remained NP-complete for $(k-1)=3$ even if we restrict our instances to those graphs Γ with $\Delta(\Gamma)\le 4$. Here is a simple fact which we use to extend this result.

Lemma 4.1. *If k-COL is NP-complete when restricted to instances* Γ *with* $\Delta(\Gamma)\le d$, *then* $(k+1)$-*COL is NP-complete when restricted to instances* Γ *with* $\Delta(\Gamma)\le 2d+1$.

Proof. Given a graph $\Gamma=(V,E)$ with $\Delta(\Gamma)\le d$, we wish to give a polynomial transformation to a graph $\Gamma'=(V',E')$ with $\Delta(\Gamma')\le 2d+1$ such that Γ is k-colorable if and only if Γ' is $(k+1)$-colorable. The construction of Γ' is as follows:

$$\text{let } V'=\{x_1,x_2: x\in V\}\ ;$$

$$\text{let } E'=\{\{x_1,y_1\},\{x_1,y_2\},\{x_2,y_1\}:\{x,y\}\in E\}\cup\{\{x_1,x_2\}:x\in V\}\ .$$

It is not hard to show that this construction has the properties stated above and we will leave it to the reader to verify. □

This then means that for any $k\ge 3$, there exists a constant $d=d(k)$ such that the problem of k-colorability remains NP-complete when we restrict ourselves to graphs Γ with $\Delta(\Gamma)\le d$.

We will use the following description of chromatic number in our arguments. By an ordered n-path in an ordered graph $(G, <)$, we mean a sequence of distinct vertices $\{v_1, \ldots, v_{n+1}\}$, where $v_1 < v_2 < \cdots < v_{n+1}$ and each of the pairs $\{v_i, v_{i+1}\}$, $i \in [n]$, is an edge of G.

Lemma 4.2. *A graph $G = (V, E)$ has $\chi(G) \le k$ if and only if there exists an ordering $<$ of $V(G)$ such that $(G, <)$, contains no ordered k-path.* □

Proof of Theorem 1.4. Let $(G, <)$ be a fixed 2-connected ordered graph with $V(G) = (1, 2, \ldots, k)$ and

$$\mathrm{Aut}(G, <) = \{(1, 2, \ldots, k), (k, k-1, \ldots, 1)\}.$$

We describe a polynomial transformation which takes as input an instance $\Gamma = (V, E)$ of $(k-1)$-COL with $\Delta(\Gamma) \le d$ and gives as output an instance Γ' of $(G, <)$ORD such that

$$\Gamma' \text{ is a yes-instance of } (G, <)\text{ORD if and only if } \chi(\Gamma) \le k-1.$$

First, let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be a list of all $(k-1)$-paths of Γ. Since k is a constant, m is polynomial in $|V(\Gamma)|$; in fact, since $\Delta(\Gamma) \le d$, another constant, $m \le d(d-1)^{k-2}|V(\Gamma)|$ is linear in $|V(\Gamma)|$. Thus, there is an algorithm, polynomial-time in $|V(\Gamma)|$, which provides the list $\mathcal{P}$.

Now, let $\mathcal{H}$ be the k-uniform oriented hypergraph with $V(\mathcal{H}) = V(\Gamma)$ and $E(\mathcal{H}) = \mathcal{P}$, where each edge is oriented by its corresponding path orientation. Notice that $\Delta(\mathcal{H}) \le d' = d(d-1)^{k-2}k$, so we can apply Lemma 3.3 with $d = d'$, $\delta = \frac{1}{k^d}$, and $l = k$. This yields a k-uniform oriented hypergraph $\mathcal{H}' = (V \times X, E')$ with these properties.

1. $\Delta(\mathcal{H}') \le d'$ where d' is some large but fixed constant independent of the size of $\mathcal{H}$.
2. If $(v_1, v_2, \ldots, v_k) \in E$, $X'_{v_i} \subseteq X_{v_i}$, and $|X'_{v_i}| \ge \delta |X_{v_i}|$ for all $i \in [k]$, then
$$(X'_{v_1} \times X'_{v_2} \times \cdots \times X'_{v_k}) \cap E' \ne \emptyset.$$
3. $\mathrm{Girth}(\mathcal{H}') > k$.

We now form our graph Γ' by inserting a copy of $(G, <)$ into every edge of $\mathcal{H}'$ so that its vertices coincide with the orientation of the edge.

Claim 2. *Γ' is a yes-instance of $(G, <)$ORD if and only if Γ is a yes-instance of $(k-1)$-COL.*

Proof (of Claim 2). First notice that since G is 2-connected, for all $x, y \in V(G)$ there is a cycle in G which contains both x and y. Since the girth of $\mathcal{H}'$ is greater than $k = |V(G)|$, each subgraph of Γ' that is isomorphic to G is contained within an edge of $\mathcal{H}'$. So any ordering $<_*$ of $V(\Gamma') = V(\mathcal{H}')$ such that $(G, <) \le (\Gamma', <_*)$ induces a monotonically ordered edge of $\mathcal{H}'$ since $(G, <)$ has only the two automorphisms stated above.

Suppose that $\chi(\Gamma) \le k-1$. Then by Lemma 4.2 there is an ordering $<$ of $V(\Gamma)$

such that $(\Gamma, <)$ does not contain an ordered $(k-1)$-path. This implies that $(\mathscr{H}, <)$ does not contain a monotonically ordered edge. Let $<_*$ be an ordering of $V(\Gamma')$ such that for all $a, b \in V(\mathscr{H})$ with $a < b$, we have $X_a <_* X_b$. Then $(\mathscr{H}', <_*)$ will not contain any monotonically ordered edges. Therefore, by the argument given above, $(G, <) \not\leq (\Gamma', <_*)$ and Γ satisfies $(G, <)$ORD.

Now suppose that Γ' satisfies $(G, <)$ORD and let $<_*$ be an ordering of $V(\Gamma')$ such that $(G, <) \not\leq (\Gamma', <_*)$. Then by the argument above $(\mathscr{H}', <_*)$ does not contain a monotonically ordered edge. We will define our ordering $<'$ on $V(\Gamma) = V(\mathscr{H})$ as follows:

1. Let $(e_1, e_2, \ldots, e_m)$ be a listing of $E(\mathscr{H})$.
2. For all $x \in V(\mathscr{H})$ let $X_x^0 = X_x = \{x\} \times X$. When $x = v_i$, we use $X_i = X_{v_i}$.
3. For $i = 1$ to m do the following.

   (a) If $e_i = (v_1, v_2, \ldots, v_k)$, then there is an ordering $(v_{j_1}, v_{j_2}, \ldots, v_{jk})$ of $(v_1, v_2, \ldots, v_k)$ and sets
   $$X'_{j_l} \subseteq X_{j_l}^{i-1}, \qquad 1 \leq l \leq k$$
   such that
   $$X'_{j_1} <_* X'_{j_2} <_* \cdots <_* X'_{j_k}$$
   in our ordering $<_*$ and
   $$|X'_{j_l}| \geq \frac{1}{k} |X_{j_l}^{i-1}|, \qquad 1 \leq l \leq k .$$
   Let $X_{j_l}^i = X'_{j_l}$, $1 \leq l \leq k$.

   (b) For all $x \notin \{v_1, v_2, \ldots, v_k\}$ let $X_x^i = X_x^{i-1}$.

4. There is a partial order induced on the pairs x, y of $V(\mathscr{H}) = V(\Gamma)$ by $X_x^m <_* X_y^m$. Let $<'$ be any linear extension of that partial order.

Indeed let $S \subseteq V(\Gamma')$ be the smallest initial segment of $(V(\Gamma'), <_*)$ which contains $\frac{|X_j^{i-1}|}{k}$ elements of one of sets $X_1^{i-1}, X_2^{i-1}, \ldots, X_k^{i-1}$. Let j_1 be this j and $X'_{j_1} = X_j^{i-1} \cap S$. To obtain j_2, we let $S' \subseteq V(\Gamma') - S$ be the smallest initial segment containing $\frac{|X_j^{i-1}|}{k}$ elements of one of the sets X_j^{i-1}, $j \neq j_1$. Let j_2 be this j and $X'_{j_2} = X_j^{i-1} \cap S'$. Continue in this manner to obtain $j_3, \ldots, j_k$. We claim that $(\mathscr{H}, <')$ does not contain a monotonically ordered edge, implying that $(\Gamma, <')$ does not contain an ordered $(k-1)$-path, thereby showing that $\chi(\Gamma) \leq k-1$. Suppose that $(v_1, v_2, \ldots, v_k)$ is a monotonically ordered edge of $\mathscr{H}$. Without loss of generality, we may assume that
$$v_1 <' v_2 <' \cdots <' v_k .$$
Hence $X_1^m <_* X_2^m <_* \cdots <_* X_k^m$. But since $\Delta(\mathscr{H}) \leq d$ each X_a has been pruned as in 3(a) at most d times so $|X_a^m| \geq \frac{1}{k^d}|X_a| = \delta |X_a|$. So by Lemma 3.3 there exists $e \in E(\mathscr{H}')$ such that $e \subseteq \prod_{l=1}^k X_{v_l}^m$. This implies that e is monotonically ordered by $<_*$ which is a contradiction. Thus $(\mathscr{H}, <')$ does not contain a monotonically ordered edge and $\chi(\Gamma) \leq k-1$. This completes the proof of the claim.

We have a polynomial transformation from an instance Γ of $(k-1)$-COL to an instance Γ' of $(G,<)$ORD such that Γ satisfies $(k-1)$-COL if and only if Γ' satisfies $(G,<)$ORD, thereby establishing the NP-completeness of $(G,<)$ORD for any 2-connected $(G,<)$ with the specified 2-element automorphism group. □

5. THE NP-COMPLETENESS OF *d*-MIDDLE

As we stated earlier our strategy for proving Theorem 1.5 is to take the known NP-complete problem 3-COL $(\Delta(\Gamma) \le 4)$ and give a polynomial transformation that reduces an instance of 3-COL to an instance of d-MIDDLE for any $d \ge 108$, thus establishing the NP-completeness of d-MIDDLE. In the next section we will use this result in the proof of Theorem 1.5.

In our first transformation we again use the characterization of 3-chromatic graphs given in Lemma 4.2. Given an instance $\Gamma = (V, E)$ of 3-COL, let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be the set of all three-paths of Γ where each $P_i = \{q_i, r_i, s_i, t_i\}$, where $\{\{q_i, r_i\}, \{r_i, s_i\}, \{s_i, t_i\}\}$ are the edges of the path.

Define $\mathcal{D}_\Gamma = (S, E_{MIM}, E_{MOM})$ as follows:

$$S = \{V(\Gamma) \cup \{x_{i,j} : i = 1, 2, \ldots, m,\ j = 1, 2, 3\}\},$$

$$E_{MIM} = \{(q_i, x_{i,1}, r_i), (r_i, x_{i,2}, s_i), (s_i, x_{i,3}, t_i) : i = 1, 2, \ldots, m\},$$

$$E_{MOM} = \{(x_{i,1}, x_{i,2}, x_{i,3}) : i = 1, 2, \ldots, m\}.$$

Theorem 5.1. *$\mathcal{D}_\Gamma = (S, E_{MIM}, E_{MOM})$ is an instance of d-MIDDLE $(d \ge 108)$ which is a yes-instance if and only if Γ is 3-colorable.*

Proof. First we will show that $\mathcal{D}_\Gamma$ is in fact an instance of d-MIDDLE. By their definitions $E_{MIM}, E_{MOM} \subseteq S^3$. We need only to show that, for all $s \in S$,

$$|\{T \mid T \in E_{MIM} \cup E_{MOM}, s \in T\}| \le 108. \tag{4}$$

If $s \in S$, then either $s \in V(\Gamma)$ or $s = x_{i,j}$ for some i and j. If $s = x_{i,j}$, then s occurs in exactly 2 triples, one in E_{MIM} and one in E_{MOM} so (4) is obviously satisfied. If $s \in V(\Gamma)$, then $s \not\in T$ for any $T \in E_{MOM}$ and s is in one triple of E_{MIM} for every 3-path for which it is an end vertex and two triples of E_{MIM} for every 3-path for which it is an interior vertex. Since $\Delta(\Gamma) \le 4$, any vertex of Γ can be the end vertex of at most $4 \cdot 3 \cdot 3 = 36$ 3-paths and an interior vertex of at most $4 \cdot 3 \cdot 3 = 36$ 3-paths. Hence, if $s \in V(\Gamma)$, then s occurs in at most 108 triples of $\mathcal{D}_\Gamma$. Condition (4) is satisfied so $\mathcal{D}_\Gamma$ is an instance of d-MIDDLE for any $d \ge 108$.

Now assume that $<$ is an ordering of S that satisfies d-MIDDLE and consider $<_* = <|_{V(\Gamma)}$. We claim that this is an ordering of $V(\Gamma)$ which does not contain an ordered 3-path which by Theorem 4.2 implies that Γ has chromatic number at most 3. Suppose not, then $(\Gamma, <)$ contains an ordered 3-path. Then, for some $P_i = \{q_i, r_i, s_i, t_i\}$, we have that $q_i < r_i < s_i < t_i$ or $t_i < s_i < r_i < q_i$. Without loss of generality, we may assume the former. But since $<$ is a satisfying order for d-MIDDLE this implies that

$$q_i < x_{i,1} < r_i < x_{i,2} < s_i < x_{i,3} < t_i.$$

This is a contradiction to satisfaction of d-MIDDLE because this means that

$$x_{i,1} < x_{i,2} < x_{i,3} .$$

Hence, if $\mathscr{D}_\Gamma$ satisfies d-MIDDLE then Γ is 3-colorable.

Conversely, if Γ is 3-colorable then, by Lemma 4.2, we know that there is an ordering $<$ of $V(\Gamma)$ such that $(\Gamma, <)$ does not contain an ordered 3-path. We will now define a series of extensions of $<$ which will eventually lead to a partial order $\leq_*$ on the set S. We will then show that there is a linear extension $<_*$ of $\leq_*$ which satisfies d-MIDDLE.

To define $\leq_*$ take each $\{a, b\} \in E(\Gamma)$ such that $a > b$. Then for every P_i such that $\{a, b\} \in \{q_i, r_i\}$, $\{r_i, s_i\}$, $\{s_i, t_i\}\}$ define $a \geq_* x_{i,j} \geq_* b$ for $j = 1, 2$ or 3 depending on the position of $\{a, b\}$ in P_i. If we do this for every edge in $E(\Gamma)$, then the transitive closure of $\leq_*$ is obviously an extension of $<$ to a partial order on S. We now need to show that this partial order has a satisfying linear extension.

For each $P_i \in \mathscr{P}$ either $q_i < r_i$, and $q_i < r_i < s_i < t_i$ does not hold, or $q_i > r_i$ and $q_i > r_i > s_i > t_i$ does not hold. In either case there are 11 possible linear orders on q_i, r_i, s_i, and t_i and it is easy to check that for each of them $\leq_* |_{P_i}$ can be extended to a linear order $<_i$ such that neither $x_{i,1} <_i x_{i,2} <_i x_{i,3}$ nor $x_{i,3} <_i x_{i,2} <_i x_{i,3}$ hold. Let $<_* = \bigcup_{i=1}^{k} <_i$. Since $<_* |_{V(G)} = <$, it is easy to see that the transitive closure of $<_*$ is a desired satisfying order on S.

We have shown that Γ satisfies 3-COL if and only if $\mathscr{D}_\Gamma$ satisfies d-MIDDLE thereby establishing the NP-completeness of d-MIDDLE for $d \geq 108$. □

6. BLOCK DECOMPOSITIONS

Given a graph, G, we can define a binary relation $\sim$ on $V(G)$ in the following manner. For all $x, y \in V(G)$,

$$x \sim y \Leftrightarrow N(x) - y = N(y) - x ,$$

where $N(x) = \{y \in V(G) \colon \{x, y\} \in E(G)\}$. With this definition in hand, we can easily prove the following two propositions.

Proposition 6.1. *The binary relation* $\sim$ *is an equivalence relation on* $V(G)$.

Proof. It is obvious by the definition of $\sim$ that it is both symmetric and reflexive so let us verify that it is transitive.

Suppose $x \sim y$ and $y \sim z$. We show that $N(x) - z = N(z) - x$. So suppose that $t \in N(x) - z$. First we will assume that $t \neq y$, in this case we know that $t \in N(x) - y = N(y) - x$, but since $t \neq x$, z, we know that $t \in N(y) - z = N(z) - y$. Hence $t \in N(z) - x$. Now suppose that $t = y$. Then $y \in N(x)$, which implies that $x \in N(y)$. But, since $N(y) - z = N(z) - y$, we get that $x \in N(z)$ so $z \in N(x)$. But, since $N(x) - y = N(y) - x$ meaning $z \in N(y)$, we get that $y \in N(z) - x$. Therefore, $x \sim z$. □

Proposition 6.2. *The subgraph induced by the vertices of any equivalence class* C *of* $\sim$ *is either complete or empty.*

Proof. If the order of C is either 1 or 2 then the proposition is obvious. So suppose that $|C| \geq 3$. Then, for any $x, y, z \in C$, if $x \in N(y)$ then since $y \sim z$ which implies $N(y) - z = N(z) - y$, and $x \in N(z)$. If x is adjacent to any vertex in C then it must be adjacent to all of them and the proposition holds. □

Now, if we consider the ordered graph $(G, <)$, we can define the *block decomposition* $(B_1, B_2, \ldots, B_t)$ of $(G, <)$ to be the decomposition of $V(G)$ into maximum consecutive sets of vertices of the same equivalence class. Formally, if $V(G, <) = (v_1, v_2, \ldots, v_k)$, then

$$B_1 = (v_1, v_2, \ldots, v_{i_1}) \Leftrightarrow v_1 \sim v_j\,, \qquad \text{for all } j,\ 1 \leq j \leq i_1\,,$$

and

$$v_1 \not\sim v_{i_1+1}\,.$$

Similarly,

$$B_j = (v_{i_{j-1}+1}, \ldots, v_{i_j}) \Leftrightarrow v_{i_{j-1}+1} \sim v_k\,, \qquad \text{for all } k, i_{j-1} + 1 \leq k \leq i_j\,,$$

and

$$v_{i_{j-1}+1} \not\sim v_{i_j+1}\,.$$

Notice that if $x \in B_i$, $y \in B_j$, and x is adjacent to y, then every vertex of B_i is adjacent to every vertex of B_j. This allows definition of the block graph $(G_{\sim}, <)$ of $(G, <)$ to be the ordered graph with vertex set $(B_1, \ldots, B_t)$, where B_i is adjacent to B_j if and only if the vertices of B_i are adjacent to the vertices of B_j in $(G, <)$.

7. ANOTHER PROBABILISTIC RESULT—THE XY LEMMA

Here we employ probabilistic methods to obtain the final tool needed for our proof of Theorem 1.5. The basic step in that proof is a transformation of an instance $\mathscr{D}$ of d-MIDDLE to an instance Γ of $(G, <)$ORD in a manner similar to the one used in verifying Theorem 1.4. First, we form a sequence of hypergraphs and then insert copies of $(G, <)$ into the edges of the last hypergraph, obtaining the graph Γ. Here the new part of each successive hypergraph in the sequence corresponds to a triple in $E_{MIM} \cup E_{MOM}$, the edge set of $\mathscr{D}$. Actually the ordering of the first three blocks of the block decomposition of $(G, <)$ in each edge of the final hypergraph will be determined by the ordering of the corresponding triple.

Since the graph $(G, <)$ may have more than three blocks, we need a way of ensuring that the rest of the copy of $(G, <)$ in each edge is properly ordered. This is where the XY Lemma (Lemma 7.1) comes into play. It enables us to force, for any ordering $<$ of Γ not containing $(G, <)$, that two subsets of vertices X and Y satisfy $X <^{\delta} Y$.

Some special terminology is required for this section. Let V be a set, let $V = X \cup Y$ be a partition, and let $<$ be a fixed order on V with $Y < X$. Given a second order $<$ on V, we say that a k-set $e = \{y, x_1, x_2, \ldots, x_{k-1}\}$ with $y \in Y$, $x_i \in X$, and order

$$y < x_1 < x_2 < \cdots < x_{k-1}$$

is *almost* $<$*-monotone* if either

$$y < x_1 < \cdots < x_{k-1} \qquad \text{or} \qquad y < x_{k-1} < \cdots < x_1 .$$

In the former case, say that e is $<$*-increasing.*

Lemma 7.1 (The XY-Lemma). *Let $\delta > 0$ and let c_1, $k \geq 2$ be integers. Then there exists an integer $N = N(k, \delta, c_1)$ satisfying the following. Given a set $V = X \cup Y$, $X \cap Y = \emptyset$, $|X| = |Y| = N$, an order $<$ on V with $Y < X$, and a metric*

$$\rho : V^2 \to \mathbb{Z} \cup \{\infty\}$$

such that for all $v \in V$, $\epsilon = \frac{1}{k^3}$, and $i \leq k$,

$$|\{u \in V : \rho(u, v) = i\}| \leq (c_1 N^{\epsilon})^i ,$$

there exists a k-uniform hypergraph $\mathcal{H} = (V, E)$ satisfying these conditions.

(1) For every $e \in E(\mathcal{H})$, $|e \cap X| = k - 1$, $|e \cap Y| = 1$.
(2) $\Delta(\mathcal{H}) \leq 1.5N^{\epsilon}$.
(3) $\mathcal{H}$ contains no ρ-cycle of length at most k.
(4) For all orders $<$ on V with $X \not<^{\delta} Y$, E contains at least $c_2 N^{1+\epsilon}$ almost $<$-monotone edges for some $c_2 = c_2(\delta, k)$.
(5) There is an orientation $\bar{E} = \{\bar{e} \mid e \in E\}$ of E such that:

(i) for all $e = \{y, x_1, x_2, \ldots, x_{k-1}\} \in E$, with $y < x_1 < \cdots < x_{k-1}$, either

$$\bar{e} = (y, x_1, x_2 \cdots x_{k-1}) \qquad \text{or} \qquad \bar{e} = (y, x_{k-1}, \ldots, x_1) ,$$

(ii) for all orders $<$ on V with $X \not<^{\delta} Y$, $\bar{E}$ contains an $<$-increasing oriented edge.

Proof. First we will obtain a hypergraph satisfying (1)–(4).

Consider the random hypergraph $\mathcal{H}(N, p)$ defined as in the proof of Lemma 3.2, now with $n = 2$, $l = k$, $j_1 = k - 1$, and $j_2 = 1$. Recall that $p = N^{1-k+\epsilon}$. Let $X = X_1$ and $Y = X_2$. We claim that

(a′) $\Delta(\mathcal{H}(N, p)) \leq 1.5N^{\epsilon}$, with probability greater than $\frac{3}{4}$.
(b′) $\mathcal{H}(N, p)$ contains at most $2N$ ρ-cycles of length at most k, with probability greater than $\frac{1}{2}$.

Statement (a′) is verified just as (a) in the proof of Lemma 3.2, with argument culminating in Eq. (2). Statement (b′) follows in the same manner as (c) in the proof of Lemma 3.2, which ends with Eq. (3).

We shall now prove:

(c′) For all orders $<$ of V such that $X \not<^{\delta} Y$, $E(\mathcal{H}(N, p))$ contains at least $c_2' N^{1+\epsilon}$ almost $<$-monotone edges, where $c_2' = c_2'(\delta, k)$, with probability exceeding $\frac{3}{4}$.

Once this is done, we know that with positive probability, $\mathcal{H}(N, p)$ satisfies (a'), (b'), and (c'). Take N large enough that $2N < \frac{c_2'}{2} N^{1+\epsilon}$. Select $\mathcal{H}' \in \mathcal{H}(N, p)$ satisfying (a'), (b'), and (c') and delete an edge from each ρ-cycle of length less than k in $\mathcal{H}'$. The resulting hypergraph $\mathcal{H}$ satisfies (1)–(4).

Let $<$ be an order of V. Then for all $X' \subseteq X$, with $|X'| \geq \delta |X|$, let $\mathcal{F}_{X'}$ be the family of all $k-1$ element subsets of S of X' such that $(S, <)$ is the same as or the dual to $(S, <)$. It is easy to show that

$$|\mathcal{F}_{X'}| \geq \frac{\binom{\delta N}{k-1}}{\binom{(k-2)^2+1}{k-1}}.$$

Indeed by the well-known theorem of Erdős and Szekeres [10], any subset $Y \subseteq X'$, $|Y| = (k-2)^2 + 1 = l$ contains a member of $\mathcal{F}_{X'}$. Thus

$$|\mathcal{F}_{X'}| \binom{\delta N - k + 1}{l - k + 1} \geq \binom{\delta N}{l}$$

holds. Simple computation then will obtain the desired result. Now assume that $X \not<^{\delta} Y$. This means that the $<$-first δN elements of Y precede the $<$-last δN elements of X. So if $\mathbf{X}_<$ is a random variable that counts the number of almost $<$-monotone edges of $\mathcal{H}$ with respect to $<$,

$$\begin{aligned}
\mathbf{E}(\mathbf{X}_<(\mathcal{H}(N, p))) &\geq \delta N \frac{\binom{\delta N}{k-1}}{\binom{(k-2)^2+1}{k-1}} p \\
&= \frac{\delta N p}{\binom{(k-2)^2+1}{k-1}} \frac{\delta^{k-1} N^{k-1}}{(k-1)!} (1 + o(1)) \\
&= \frac{\delta^k}{(k-1)! \binom{(k-2)^2+1}{k-1}} N^{1+\epsilon} (1 + o(1)) \\
&\geq \frac{\delta^k}{2(k-1)! \binom{(k-2)^2+1}{k-1}} N^{1+\epsilon} \\
&= c_2' N^{1+\epsilon}
\end{aligned}$$

for N large enough, $c_2' = c_2'(\delta, k)$. So once again by Lemma 3.1 we get

$$\Pr\left(\mathbf{X}_<(\mathcal{H}(N, p)) \leq \frac{c_2'}{2} N^{1+\epsilon}\right) \leq 2 \exp\left(\frac{-c_2'}{16} N^{1+\epsilon}\right).$$

Thus,

$$\begin{aligned}
&\Pr\left(\exists <: X \not<^{\delta} Y \quad \text{and} \quad \mathbf{X}_<(\mathcal{H}(N, p)) < \frac{c_2'}{2} N^{1+\epsilon}\right) \\
&\leq N! 2 \exp\left(\frac{-c_2'}{16} N^{1+\epsilon}\right) \\
&< 1/4
\end{aligned}$$

for N large enough.

We now show that, for N large enough, but again depending only on k, δ and c_1, a choice of $\mathcal{H} \in \mathcal{H}(N, p)$ satisfying (1)–(4) also satisfies (5). So, take $\mathcal{H} \in \mathcal{H}(N, p)$, $\mathcal{H} = (V, E)$, and define the random oriented hypergraph $\vec{H} = (V,$

E) to be the probability space over the orientations of the edges of $\mathscr{H}$ where for every edge $e = \{y, x_1, \ldots, x_{k-1}\}$ with

$$x_1 < x_2 < \cdots < x_{k-1} ,$$

$$\Pr(\vec{e} = (y, x_1, \ldots, x_{k-1})) = 1/2 ,$$

and

$$\Pr(\vec{e} = (y, x_{k-1}, \ldots, x_1)) = 1/2 .$$

Given an order $<$ on V with $X \not<^{\delta} Y$ define $A_<$ to be the event that $(\vec{H}, <)$ does not contain an $<$-increasing edge. Then

$$\Pr(A_<) \leq 2^{-c_2 N^{1+\epsilon}}$$

by (4). Let B be the event that there exists an order $<$ on V with $X \not<^{\delta} Y$ and $A_<$ holds. Then

$$\Pr(B) \leq (2N)! 2^{-c_2 N^{1+\epsilon}} < 1 ,$$

for N large enough. So there is an orientation of E satisfying (5)(i) and (5)(ii). □

8. THE PROOF OF THEOREM 1.5

Throughout this section we will assume that $(G, <)$ is a 2-connected ordered graph such that

$$(v_2, v_3, \ldots, v_k, v_1), (v_k, v_{k-1}, \ldots, v_1) \not\in \mathrm{Aut}(G, <) ,$$

where $(v_1, v_2, \ldots, v_k)$ is an ordered listing of $V(G)$. Let $(B_1, B_2, \ldots, B_t)$ be the block decomposition of $(G, <)$.

We say that a block B_i is *distinguishable* from another block B_j if at least one of the following conditions holds.

1. $|B_i| \neq |B_j|$.
2. B_i and B_j are neither both complete nor both empty as induced subgraphs.
3. $N_{G_{\sim}}(B_i) - B_j \neq N_{G_{\sim}}(B_j) - B_i$.

In other words, two blocks are distinguishable in $G, <)$ if switching them in the order of $(G, <)$ gives a nonisomorphic ordered graph.

Our strategy for the rest of this section will be to first prove the theorem where $(G, <)$ is an ordered graph with three or more blocks in which the second block is distinguishable from the first and third, then to outline the proof in the case where $(G, <)$ has three or more blocks and the second is indistinguishable from either the first or the third, and finally outline the proof when $(G, <)$ has only two blocks. The second two cases are very similar to the first so we only outline the differences without going into great detail. The case where the ordered graph has just one block does not apply as these are either complete or empty graphs.

8.1. Case 1: B_2 Is Distinguishable

We give a polynomial transformation of instances of d-MIDDLE to instances of $(G, <)$ORD, for a fixed integer $d \geq 108$ and a fixed ordered graph $(G, <)$, with $|V(G)| = k$, block decomposition $(G, <) = (B_1, B_2, \ldots, B_t)$, $|B_i| = b_i$, $t \geq 3$, and B_2 distinguishable from B_1 and B_3. Yes-instances will correspond under the transformation.

Let N be an integer large enough that each of these hold.

(i) Lemma 7.1 with $\delta = \frac{1}{3^{d_t}}$, $k = |V(G)|$, $c_1 = 3d(4k-1)$.
(ii) Lemma 3.2 with $\delta = \frac{1}{3^{d_t}}$, $l = k$, $n = t$, $c_1 = 3d(4k-1)$, $j_1 = b_1 + 2b_2 + b_3$, $j_2 = 2b_1 + 2b_3$, $j_3 = j_1$ and $j_i = 4b_i$, $4 \leq i \leq t$.
(iii) Lemma 3.2 with $\delta = \frac{1}{3^{d_t}}$, $l = k$, $n = t$, $c_1 = 3d(4k-1)$, $j_1 = b_1 + b_3$, $j_2 = 2b_2$, $j_3 = j_1$ and $j_i = 2b_i$, $4 \leq i \leq t$.

Let $\mathscr{D} = (S, E_{MIM}, E_{MOM})$ be an instance of d-MIDDLE, with $(e_1, \ldots, e_p)$ a listing of $E_{MIM} \cup E_{MOM}$. Set

$$V = \bigcup_{a \in S} X_a \cup \bigcup_{i=1}^{p} \bigcup_{j=4}^{t} X_{i,j},$$

where $|X_a| = |X_{i,j}| = N$ for all a, i, j and all X's are disjoint, and fix some ordering $<$ on each X_a and $X_{i,j}$ for use as part of the hypothesis of Lemma 7.1. We inductively construct a sequence $\mathscr{H}_0, \mathscr{H}_1, \ldots, \mathscr{H}_p$ of hypergraphs on V, and set

$$\rho_i(x, y) = \operatorname{dist}_{\mathscr{H}_i}(x, y).$$

Initially, set $E(\mathscr{H}_0) = \emptyset$ and, so, $\rho_0(x, y) = \infty$ for all $x \neq y$ in V.

Assume that $\mathscr{H}_{i-1}$ and, thus, ρ_{i-1} have been obtained. The method of construction of $\mathscr{H}_i$ will depend on whether $e_i \in E_{MIM}$ or $e_i \in E_{MOM}$.

MIM. Let $e_i = (a, b, c) \in E_{MIM}$ (see Fig. 1).

Step (0) With the parameters of Lemma 3.2 specified by (ii) and $\rho = \rho_{i-1}$, $X_1 = X_a$, $X_2 = X_b$, $X_3 = X_c$ and $X_j = X_{i,j}$ $(j = 4, \ldots, t)$, let $\mathscr{H}_{i,0}$ be the hypergraph guaranteed by Lemma 3.2 and $\mathscr{H}_i^0 = \mathscr{H}_{i-1} \cup \mathscr{H}_{i,0}$. Let $\rho_i^0 = \operatorname{dist}_{\mathscr{H}_i^0}$.
Step (1) With the parameters of Lemma 7.1 specified by (i) and $\rho = \rho_i^0$, $X = X_a$, $Y = X_{i,4}$, let $\mathscr{H}_{i,1}$ be the hypergraph given in Lemma 7.1 and $\mathscr{H}_i^1 = \mathscr{H}_i^0 \cup \mathscr{H}_{i,1}$. Let $\rho_i^1 = \operatorname{dist}_{\mathscr{H}_i^1}$.
Step (2) As (1) but replace $\rho = \rho_i^1$, $X = X_b$, resulting in $\mathscr{H}_{i,2}$ from Lemma 7.1, and $\mathscr{H}_i^2 = \mathscr{H}_i^1 \cup \mathscr{H}_{i,2}$, $\rho_i^2 = \operatorname{dist}_{\mathscr{H}_i^2}$.
Step (3) As (1) but replace $\rho = \rho_i^2$, $X = X_c$, resulting in $\mathscr{H}_{i,3}$ from Lemma 7.1, and $\mathscr{H}_i^3 = \mathscr{H}_i^2 \cup \mathscr{H}_{i,3}$, $\rho_i^3 = \operatorname{dist}_{\mathscr{H}_i^3}$.
Step (j) With the parameters of Lemma 7.1 specified by (i), $\rho = \rho_i^{j-1}$, $X = X_{i,j}$, $Y = X_{i,j+1}$, let $\mathscr{H}_{i,j}$ be the hypergraph given in Lemma 7.1 and $\mathscr{H}_i^j = \mathscr{H}_i^{j-1} \cup \mathscr{H}_{i,j}$, $\rho_i^j = \operatorname{dist}_{\mathscr{H}_i^j}$.

With $j = 4, \ldots, t-1$, we let $\mathscr{H}_i = \mathscr{H}_i^{t-1}$ and $\rho_i = \rho_i^{t-1}$.

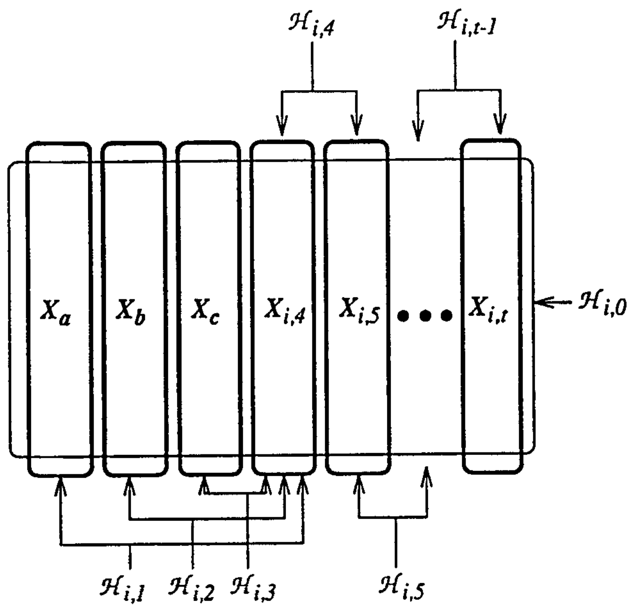

Fig. 1. Hypergraphs added to form $\mathcal{H}_i$.

MOM. Let $e_i = (a, b, c) \in E_{MOM}$.

Steps (0)–($t-1$) as above, *except* in Step (0), the parameters of Lemma 3.2 are given by (iii) .

Execution of the construction specified in **MIM** and **MOM** depends on the verification of the following claim.

Claim 1. *For each $\rho = \rho_i^j$, for each $x \in V$ and $r \leq k$*

$$|\{y \in V : \rho(x, y) = r\}| \leq (c_1 N^\epsilon)^r .$$

Recall that $c_1 = 3d(4k-1)$.

Proof. Our approach here will be to show that, for any i, j,

$$\Delta(\mathcal{H}_i^j) \leq 3dN^\epsilon .$$

This will prove our claim since any edge of any $\mathcal{H}_i^j$ has cardinality at most $4k$, which implies that any vertex $x \in V$ has at most $3d(4k-1)N^\epsilon$ vertices of distance 1 from it, and hence at most $(3d(4k-1)N^\epsilon)^r$ vertices of distance r from it.

Fix i, j and consider $\mathcal{H}_i^j$. Let $x \in V$. Then either $x \in X_a$ for some $a \in S$, or $x \in X_{l,m}$ for some l and m. First, assume that $x \in X_a$ for some $a \in S$. Let

$$E_a = \{e_q \in E_{MIM} \cup E_{MOM} : a \in e_q, q \leq i\} .$$

Obviously $|E_a| \leq d$. For each $e_q \in E_a$, x is a vertex of $H_{q,0}$ as well as one of the hypergraphs $\mathcal{H}_{q,1}$, $\mathcal{H}_{q,2}$, $\mathcal{H}_{q,3}$. Further, these are the only hypergraphs in the collection whose union makes up $\mathscr{H}_i^j$ which have x as a vertex. Since each of these $\mathscr{H}_{q,r}$'s has $\Delta(\mathscr{H}_{q,r}) \leq 1.5N^\epsilon$,

$$\deg_{\mathscr{H}_i^j}(x) \leq d \cdot 2 \cdot 1.5N^\epsilon = 3dN^\epsilon .$$

Now assume that $x \in X_{l,m}$ for some l and m. Then x is a vertex only of the following hypergraphs in the collection whose union is $\mathscr{H}_i^j$:

$\mathscr{H}_{l,0}$;

if $m = 4$, $\mathscr{H}_{l,1}$, $\mathscr{H}_{l,2}$, $\mathscr{H}_{l,3}$ and $\mathscr{H}_{l,4}$;

if $m = t$, $\mathscr{H}_{l,t-1}$;

if $m \neq 4$ and $m \neq t$, $\mathscr{H}_{l,m-1}$ and $\mathscr{H}_{l,m}$.

Since each $\mathscr{H}_{l,m}$ has $\Delta(\mathscr{H}_{l,m}) \leq 1.5N^\epsilon$,

$$\deg_{\mathscr{H}_i^j}(x) \leq 5 \cdot 1.5N^\epsilon = 7.5N^\epsilon .$$

Since $d \geq 108$, $3dN^\epsilon > 7.5N^\epsilon$ so

$$\Delta(\mathscr{H}_{i,j}) \leq 3dN^\epsilon$$

for all i and j. □

Therefore, we are able to construct all of our $\mathscr{H}_i$'s. Notice that, at each step, all we are doing is placing hypergraphs satisfying certain properties onto our vertices, and these properties depend only on d and $(G, <)$, not on the size of our instance $\mathscr{D}$ of d-MIDDLE. Hence, we can think of each step as only taking constant time. Therefore we construct $\mathscr{H}_p$ in time linear with respect to the number of edges of $\mathscr{H}$.

Claim 2. *For every* $i = 1, 2, \ldots, p$, girth$(\mathscr{H}_i) > k = |V(G)|$.

Proof. The proof is by induction on i. For $i = 0$, since $E(\mathscr{H}_0) = \emptyset$ the claim is obvious.

Assume the girth$(\mathscr{H}_{i-1}) > k$ and consider $\mathscr{H}_i$. Recall that

$$\mathscr{H}_i = \mathscr{H}_{i-1} \cup \mathscr{H}_{i,0} \cup \mathscr{H}_{i,1} \cup \cdots \cup \mathscr{H}_{i,t-1}$$

and

$$\mathscr{H}_i^0 = \mathscr{H}_{i-1} \cup \mathscr{H}_{i,0} , \qquad \mathscr{H}_i^j = \mathscr{H}_i^{j-1} \cup \mathscr{H}_{i,j} .$$

We must show that none of the edges added to $\mathscr{H}_{i-1}$ to form $\mathscr{H}_i$ create a cycle of length at most k in $\mathscr{H}_i$. Consider first the edges of $\mathscr{H}_{i,0}$ added to form $\mathscr{H}_i^0$. Since ρ_{i-1}, the metric used in the construction of $\mathscr{H}_{i,0}$, was the distance metric from $\mathscr{H}_{i-1}$, each cycle in $\mathscr{H}_i^0$ which contains edges from $\mathscr{H}_{i,0}$ corresponds to a metric cycle in $\mathscr{H}_{i,0}$. By the induction hypothesis, any cycle in $\mathscr{H}_i^0$ which has length k or less must contain edges from $\mathscr{H}_{i,0}$. But this is not possible since Lemma 3.2 guarantees that $\mathscr{H}_{i,0}$ will contain no metric cycles of length k or less. Hence, girth$(\mathscr{H}_i^0) > k$. A similar argument holds for all other $\mathscr{H}_i^j$'s so girth$(\mathscr{H}_i) > k$. □

When constructing Γ from $\mathcal{H}_p$, we will insert copies of G inside the oriented edges of $\mathcal{H}_p$ in various ways. Sometimes we will want the orientation of the edge to correspond to the ordering of $(G, <)$, but, more often, we will want it to correspond to a slightly different ordering. To aid us in this process, we develop the following terminology.

Recall $(B_1, B_2, \ldots, B_t)$ is the block decomposition of $(G, <)$ and $b_i = |B_i|$. Thinking of the B_i's as labels on their respective vertex sets, for any permutation (p, q, r) of $(1, 2, 3)$, we define the ordered graph $(G, <_{pqr})$ to be the ordered copy of G with block ordering,

$$(B_p, B_q, B_r, B_4, \ldots, B_t).$$

For example, $(G, <_{123}) = (G, <)$.

Construct Γ from $\mathcal{H}_p$ as follows.

1. $V(\Gamma) = V(\mathcal{H}_p)$
2. For each $e_i = (a, b, c) \in E_{MIM}$, define $E(\Gamma)$ so that there are four vertex disjoint copies of G contained in the vertex set of each oriented edge e of the hypergraph $\mathcal{H}_{i,0}$. Define the edge set so that the four copies of G, if ordered to correspond to the orientation of e, are

$$(G, <_{213}), \qquad (G, <_{231}), \qquad (G, <_{132}), \qquad \text{and} \qquad (G, <_{312}).$$

   The vertex set of each $(G, <_{pqr})$ should contain b_p vertices from X_a, b_q vertices from X_b, b_r vertices from X_c, and b_s vertices from $X_{i,s}$, $4 \le s \le t$ (see Fig. 2).
3. For each $e_i = (a, b, c) \in E_{MOM}$, define $E(\Gamma)$ so that there are two vertex disjoint copies of G contained in the vertex set of each oriented edge e of $\mathcal{H}_{i,0}$. Define $E(\Gamma)$ in such a way that if the vertices are ordered to correspond to the orientation of e the copies of G are ordered $(G, <_{123})$

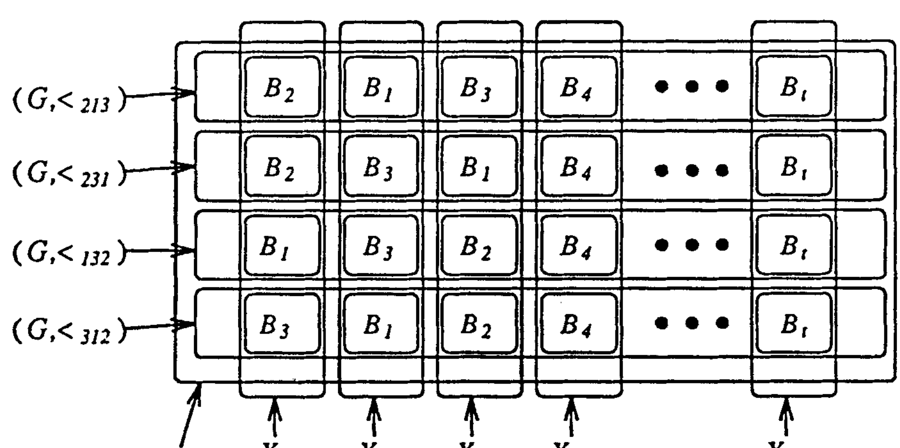

Fig. 2. Edge set of Γ for $e_i \in E_{MIM}$.

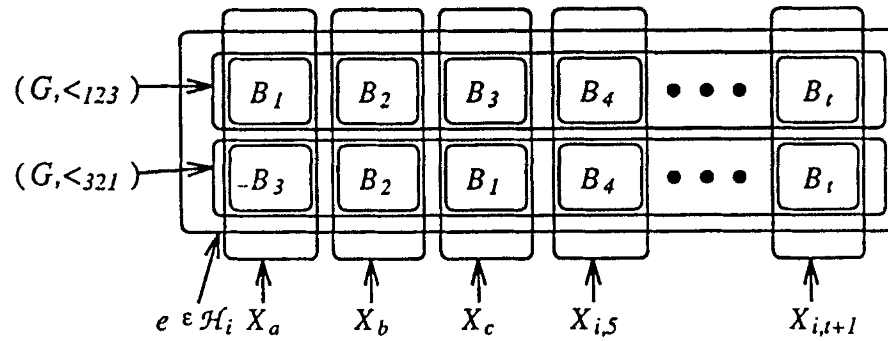

Fig. 3. Edge set of Γ for $e_i \in E_{MOM}$.

and $(G, <_{321})$. The vertex set of each $(G, <_{pqr})$ will be selected just as in step 1 (see Fig. 3).

4. For every oriented edge $e = (v_1, v_2, \ldots, v_k)$ of each of the hypergraphs satisfying Lemma 7.1 used in constructing $\mathscr{H}_p$, we put in $E(\Gamma)$ the proper edges so that $(v_1, \ldots, v_k)$ is a copy of $(G, <)$.
5. The only edges in $E(\Gamma)$ are those specified in 2–4.

Claim 3. *$\Gamma = (V, E)$ is a yes-instance of $(G, <)ORD$ if and only if $\mathscr{D}$ is a yes-instance of d-MIDDLE.*

Proof. By Claim 2, the girth of $\mathscr{H}_p$ is greater than $k = |V(G)|$. Hence any cycle of length less than or equal to k in Γ must be entirely contained in a single hyperedge of $\mathscr{H}_p$. Since we assume that $(G, <)$ is 2-connected, which means that any two vertices of $(G, <)$ are contained on a cycle, each subgraph of Γ which is isomorphic to G must be contained within a single hyperedge of $\mathscr{H}_p$.

Suppose that $<'$ is a good ordering of S for d-MIDDLE. Then define $<_*$ on $V(\Gamma)$ in the following manner.

1. If $x \in X_a$, $y \in X_b$ then $x <_* y \Leftrightarrow a <' b$.
2. If $x \in X_a$, $y \in X_{i,j}$ then $x <_* y$.
3. If $x \in X_{i,j}$, $y \in X_{k,l}$ then $x <_* y \Leftrightarrow i < k$ or $i = k$ and $j < l$.
4. Order each X_a and each $X_{i,j}$ by ordering $<$ from (5) of Lemma 7.1.

Subclaim 3.1. *$<_*$ is a good order of $V(\Gamma)$ for $(G, <)ORD$, [i.e., $(G, <) \not\leq (\Gamma, <_*)$].*

Proof (of Subclaim 3.1). Notice that each of the $\mathscr{H}_{i,j}$'s, $j \neq 0$, is ordered so that each hypergraph is oriented $(v_2, v_3, \ldots, v_k, v_1)$ or $(v_k, v_{k-1}, \ldots, v_1)$, neither of which will contain a subgraph of $(\Gamma, <_*)$ isomorphic to $(G, <)$ by our assumptions on $\mathrm{Aut}(G, <)$.

Consider the hyperedges of each $\mathscr{H}_{i,0}$.

If $e_i = (a, b, c) \in E_{MIM}$, then since $<$ is a satisfying order for d-MIDDLE, we know that $a < b < c$ or $c < b < a$. Hence each copy of $(G, <)$ contained in each hyperedge of $\mathscr{H}_{i,0}$ is ordered by blocks in one of the following ways:

$$B_1 <_* B_3 <_* B_2 <_* B_4 <_* \cdots <_* B_t ,$$

$$B_3 <_* B_1 <_* B_2 <_* B_4 <_* \cdots <_* B_t ,$$

$$B_2 <_* B_1 <_* B_3 <_* B_4 <_* \cdots <_* B_t ,$$

or

$$B_2 <_* B_3 <_* B_1 <_* B_4 <_* \cdots <_* B_t .$$

Each of these orderings, by our assumption that B_2 is distinguishable, is not isomorphic to $(G, <)$.

Similarly, if $e_i = (a, b, c) \in E_{MOM}$, then $b < a < c$, $b < c < a$, $a < c < b$, or $c < a < b$, and we get the same possible block orderings of our copies of $(G, <)$ in each hyperedge; hence, none of them is isomorphic to $(G, <)$. Therefore $(G, <) \not\leq (\Gamma, <_*)$ so $<_*$ is a good order of Γ for $(G, <)$ORD.

□ (Subclaim 3.1)

Suppose that $<'$ is an ordering of $V(\Gamma)$ such that

$$(G, <) \not\leq (\Gamma, <') .$$

Now we define an ordering, $<_*$, on S.

1. Let $(e_1, e_2, \ldots, e_p)$ be a list of $E(\mathscr{D})$.
2. For all $x \in S$ let $X_x^0 = X_x$.
3. For $i = 1$ to p do the following:

   (a) If $e_i = (a, b, c)$, then there is an ordering (j, k, l) of (a, b, c) and sets

   $$X_j' \subseteq X_j^{i-1}, X_k' \subseteq X_k^{i-1}, X_l' \subseteq X_l^{i-1}$$

   such that

   $$X_j' <' X_k' <' X_l'$$

   and

   $$|X_j'] \geq \tfrac{1}{3}|X_j^{i-1}| , \qquad |X_k'| \geq \tfrac{1}{3}|X_k^{i-1}| , \qquad |X_l'| \geq \tfrac{1}{3}|X_l^{i-1}| .$$

   Let $X_j^i = X_j'$, $X_k^i = X_k'$, $X_l^i = X_l'$.

   (b) For all $x \not\in \{a, b, c\}$ let $X_x^i = X_x^{i-1}$.

4. There is a partial order induced on the pairs x, y of S by $X_x^p <' X_y^p$. Let $<_*$ be any linear extension of that partial order.

The following subclaim will complete the proof of Claim 3.

Proof (of Subclaim 3.2). Suppose not and let

$$e_i = (a, b, c) \in E_{MIM} \cup E_{MOM}$$

be an edge which is not correctly ordered.

If $e_i \in E_{MIM}$, then this implies that

$$a <_* c <_* b , \qquad c <_* a <_* b , \qquad b <_* a <_* c ,$$

or $b <_* c <_* a$. Assume that $a <_* c <_* b$ (the rest of the cases follow a similar argument). This implies that $X_a^p <' X_c^p <' X_b^p$.

Let

$$Z_1 = X_a^p, \qquad Z_2 = X_c^p, \qquad Z_3 = X_b^p, \qquad Z_j = X_{i,j}.$$

It is easily seen that for each j there exists $Z_j' \subseteq Z_j$ such that $|Z_j'| \geq \frac{1}{t}|Z_j|$ and

$$Z'_{\pi(1)} <' Z'_{\pi(2)} <' \cdots <' Z'_{\pi(t)}$$

for some permutation π of $[t]$. We claim that this permutation must be the identity, for otherwise

$$Z_j' <' Z_{j-1}'$$

for some $j \geq 4$. Recall that Lemma 7.1, the XY-Lemma, was applied with $X \supseteq Z_{j-1}$ and $Y \supseteq Z_j$ to form $\mathscr{H}_{i,j}$. Since $|Z_j'| \geq \delta N$ and $|Z_{j-1}'| \geq \delta N$, $\delta = \frac{1}{3^{d_t}}$, there exists an ordered hyperedge $(v_1, \ldots, v_k)$ with $v_1 <' \cdots <' v_k$, which would correspond to a copy of $(G, <)$ in Γ.

Since $|Z_j'| \geq \delta N$ for all j, Lemma 3.2 guarantees the existence of a hyperedge from $\mathscr{H}_{i,0}$ in $\Pi_{j=1}^{t} Z_j'$. Since π is the identity, though, this edge is ordered so that it will contain a copy of $(G, <)$ (see Fig. 2). Hence e_i must not be ordered $a <_* c <_* b$.

This same argument will hold for each of the other bad orderings of e_i as well as for any bad ordering of an edge in E_{MOM}. Thus we deduce that $<_*$ must be a good order for $\mathscr{D}$. □

Since we proved the NP-completeness of d-MIDDLE earlier, we have now established Theorem 1.5 in the case where $(G, <)$ is 2-connected and has B_2 distinguishable from B_1 and B_3.

8.2. Case 2: B_2 Is Indistinguishable

Here we will outline the proof of Theorem 1.5 for ordered graphs $(G, <)$ where B_2 is not distinguishable from at least one of B_1 and B_3. As we noted earlier, this proof is very similar to the last one, so we will just emphasize the differences here.

The construction of Section 8.1 depended upon the fact that interchanging B_1 and B_2 or B_2 and B_3 resulted in a different graph. We will modify the construction of Section 8.1 and describe how this works when B_1 and B_2 are indistinguishable.

Let $(G, <)$ be an ordered graph where B_1 and B_2 are indistinguishable and B_2 and B_3 are distinguishable. This means that the following conditions hold:

(1) $B_1 \cong B_2$;
(2) $N_{G_-}(B_1) - B_2 = N_{G_-}(B_2) - B_1$.

Notice that this implies that $|B_1| = |B_2| \neq 1$: were $\{x\} = B_1$, $\{y\} = B_2$, then by condition (2), $x \sim y$, implying that they are in the same block. Similar arguments show that if B_1 and B_2 induce complete graphs, the blocks are not adjacent and if they induce empty graphs, they are adjacent. Thus, the subgraph induced by the

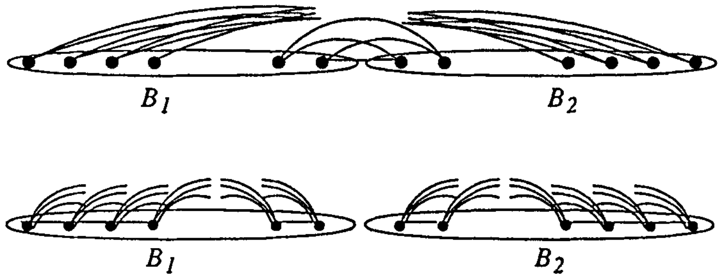

Fig. 4. Possible indistinguishable blocks.

first two blocks is one of the two graphs pictured in Figure 4. Also, every vertex of these two blocks is adjacent to the exact same set of vertices outside of the two blocks.

Let B_1' be the set of the first $\lceil \frac{b_1}{2} \rceil$ vertices in B_1 and $B_1'' = B_1 - B_1'$. Then,

$$(B_1', B_1'', B_2, \ldots, B_t)$$

is isomorphic to $(G, <)$. More importantly, if B_1'' and B_2 are switched, the new ordered graph is not isomorphic to $(G, <)$.

Here is the modification of the transformation from d-MIDDLE of Section 8.1. Let N be an integer large enough that each of these hold:

(i) Lemma 7.1, the XY-Lemma, with $\delta = \frac{1}{3^d(t+1)}$, $k = |V(G)|$, $c_1 = 4.5d(4k - 1)$;

(ii) Lemma 3.2 with $\delta = \frac{1}{3^d(t+1)}$, $l = k$, $n = t + 1$, $c_1 = 4.5d(4k - 1)$, $j_1 = 4\lceil \frac{b_1}{2} \rceil$, $j_2 = \lfloor \frac{b_1}{2} \rfloor + 2b_2 + b_3$, $j_3 = 2\lfloor \frac{b_1}{2} \rfloor + 2b_3$, $j_4 = j_2$, and $j_i = 4b_{i-1}$, $5 \le i \le t + 1$;

(iii) Lemma 3.2 with $\delta = \frac{1}{3^d(t+1)}$, $l = k$, $n = t + 1$, $c_1 = 4.5d(4k - 1)$, $j_1 = 2\lceil \frac{b_1}{2} \rceil$, $j_2 = \lfloor \frac{b_1}{2} \rfloor + b_3$, $j_3 = 2b_2$, $j_4 = j_1$, and $j_i = 2b_{i-1}$, $5 \le i \le t + 1$.

Let $\mathscr{D} = (S, E_{MIM}, E_{MOM})$ be an instance of d-MIDDLE, with $(e_1, \ldots, e_p)$ a listing of $E_{MIM} \cup E_{MOM}$. Set

$$V = \bigcup_{a \in S} X_a \cup \bigcup_{i=1}^{p} X_{i,1} \cup \bigcup_{i=1}^{p} \bigcup_{j=5}^{t+1} X_{i,j} ,$$

where $|X_a| = |X_{i,j}| = N$ for all a, i, j, and all X's disjoint. We inductively construct a sequence of $\mathscr{H}_0, \mathscr{H}_1, \ldots, \mathscr{H}_p$ of hypergraphs on V, and set

$$\rho_i(x, y) = \text{dist}_{\mathscr{H}_i}(x, y) .$$

Initially, set $E(\mathscr{H}_0) = \emptyset$ and, so, $\rho_0(x, y) = \infty$ for all $x \ne y$ in V.

Assume that $\mathscr{H}_{i-1}$ and, thus, ρ_{i-1}, have been obtained. The construction of $\mathscr{H}_i$ will depend on whether $e_i \in E_{MIM}$ or $e_i \in E_{MOM}$.

MIM. Let $e_i = (a, b, c) \in E_{MIM}$ (see Fig. 5).

Step (0) With the parameters of Lemma 3.2 specified by (ii) and $\rho = \rho_{i-1}$,

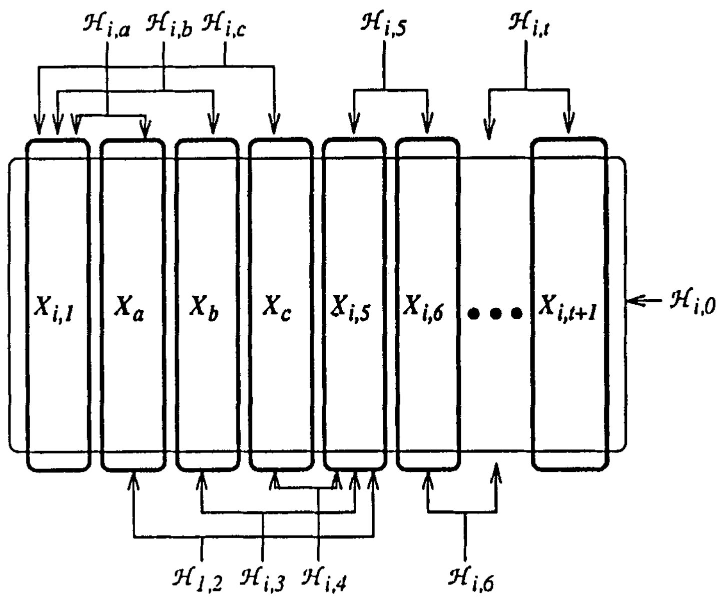

Fig. 5. Hypergraphs added to form $\mathcal{H}_i$.

$X_1 = X_{i,1}$, $X_2 = X_a$, $X_3 = X_b$, $X_4 = X_c$, and $X_j = X_{i,j}$ ($j = 5, \ldots, t+1$), let $\mathcal{H}_{i,0}$ be the hypergraph guaranteed by Lemma 3.2 and $\mathcal{H}_i^0 = \mathcal{H}_{i-1} \cup \mathcal{H}_{i,0}$. Let $\rho_i^0 = \text{dist}_{\mathcal{H}_i^0}$.

Step (1) (a) With the parameters of Lemma 7.1 specified by (i) and $\rho = \rho_i^0$, $X = X_{i,1}$, $Y = X_a$, let $\mathcal{H}_{i,a}$ be the hypergraph given in Lemma 7.1 and $\mathcal{H}_i^a = \mathcal{H}_i^0 \cup \mathcal{H}_{i,a}$. Let $\rho_i^a = \text{dist}_{\mathcal{H}_i^a}$.

(b) With the parameters of Lemma 7.1 specified by (i) and $\rho = \rho_i^a$, $X = X_{i,1}$, $Y = X_b$, let $\mathcal{H}_{i,b}$ be the hypergraph given in Lemma 7.1 and $\mathcal{H}_i^b = \mathcal{H}_i^a \cup \mathcal{H}_{i,b}$. Let $\rho_i^b = \text{dist}_{\mathcal{H}_i^b}$.

(c) With the parameters of Lemma 7.1 specified by (i) and $\rho = \rho_i^b$, $X = X_{i,1}$, $Y = X_c$, let $\mathcal{H}_{i,c}$ be the hypergraph given in Lemma 7.1 and $\mathcal{H}_i^c = \mathcal{H}_i^b \cup H_{i,c}$. Let $\rho_i^c = \text{dist}_{\mathcal{H}_i^c}$.

Step (2) With the parameters of Lemma 7.1 specified by (i) and $\rho = \rho_i^c$, $X = X_a$, $Y = X_{i,5}$, let $\mathcal{H}_{i,2}$ be the hypergraph given in Lemma 7.1 and $\mathcal{H}_i^2 = \mathcal{H}_i^c \cup \mathcal{H}_{i,2}$. Let $\rho_i^2 = \text{dist}_{\mathcal{H}_i^2}$.

Step (3) As (2) but replace $\rho = \rho_i^2$, $X = X_b$, resulting in $\mathcal{H}_{i,3}$ from Lemma 7.1, and $\mathcal{H}_i^3 = \mathcal{H}_i^2 \cup \mathcal{H}_{i,3}$, $\rho_i^3 = \text{dist}_{\mathcal{H}_i^3}$.

Step (4) As (2) but replace $\rho = \rho_i^3$, $X = X_c$, resulting in $\mathcal{H}_{i,4}$ from Lemma 7.1, and $\mathcal{H}_i^4 = \mathcal{H}_i^3 \cup \mathcal{H}_{i,4}$, $\rho_i^4 = \text{dist}_{\mathcal{H}_i^4}$.

Step (j) With the parameters of Lemma 7.1 specified by (i), $\rho = \rho_i^{j-1}$, $X = X_{i,j}$, $Y = X_{i,j+1}$, let $\mathcal{H}_{i,j}$ be the hypergraph given in Lemma 7.1 and $\mathcal{H}_i^j = \mathcal{H}_i^{j-1} \cup \mathcal{H}_{i,j}$, $\rho_i^j = \text{dist}_{\mathcal{H}_i^j}$.

With $j = 5, \ldots, t$, we let $\mathcal{H}_i = \mathcal{H}_i^t$ and $\rho_i = \rho_i^t$.

MOM. Let $e_i = (a, b, c) \in E_{MOM}$.

Steps (0)–(t) as above, *except* in Step (0), the parameters of Lemma 3.2 are given by (iii) .

Just as in Section 8.1, execution of the construction specified in **MIM** and **MOM** depends on the verification of the following claim:

Claim 1. *For each $\rho = \rho_i^j$, for each $x \in V$ and $r \le k$*

$$|\{y \in V : \rho(x, y) = r\}| \le (c_1 N^{\epsilon})^r .$$

Recall that $c_1 = 4.5d(4k - 1)$. □

When constructing Γ from $\mathcal{H}_p$, we will again insert copies of G inside the oriented hyperedges of $\mathcal{H}_p$. We slightly alter the terminology of the last section.

Recall that $(B_1', B_1'', B_2, \ldots, B_t)$ is our modified block decomposition of $(G, <)$. For any permutation (p, q, r) of $(1, 2, 3)$, we define the ordered graph $(G, <_{pqr})$ to be the ordered copy of G with block ordering

$$(B_1', B_p, B_q, B_r, B_4, \ldots, B_t) ,$$

where, for the element 1 of $\{p, q, r\}$, we set (abusing notation slightly) $B_1 = B_1''$.

Construct Γ from $\mathcal{H}_p$ as follows, letting $b_0 = |B_1'|$ and $b_1 = |B_1''|$:

1. $V(\Gamma) = V(\mathcal{H}_p)$.
2. For each $e_i = (a, b, c) \in E_{MIM}$, define $E(\Gamma)$ so that there are four vertex disjoint copies of G contained in the vertex set of each oriented hyperedge e of the hypergraph $\mathcal{H}_{i,0}$. The four copies of G, ordered to correspond to the orientation of e, are $(G, <_{213})$, $(G, <_{231})$, $(G, <_{132})$, and $(G, <_{312})$. The vertex set of each $(G, <_{pqr})$ contains b_0 vertices from $X_{i,1}$, b_p vertices from X_a, b_q vertices from X_b, b_r vertices from X_c, and b_{s-1} vertices from $X_{i,s}$, $5 \le s \le t + 1$ (see Fig. 6).
3. For each $e_i = (a, b, c) \in E_{MOM}$, define $E(\Gamma)$ so that there are two vertex

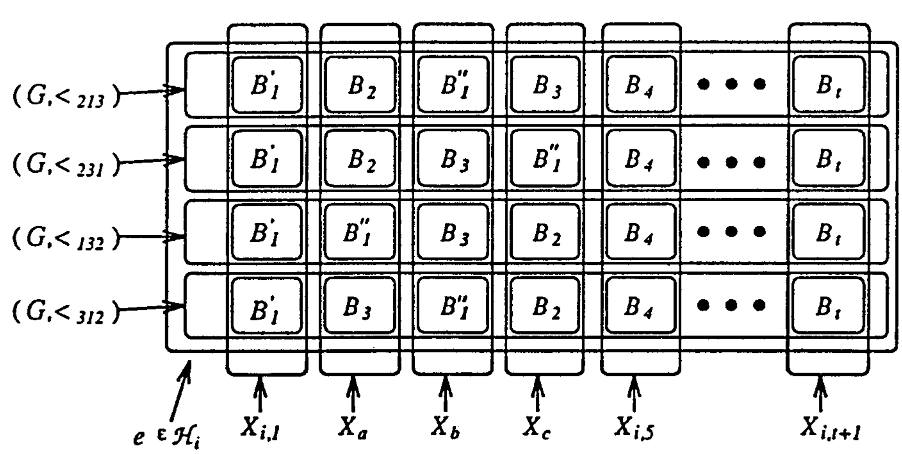

Fig. 6. Edge set of Γ for $e_i \in E_{MIM}$.

disjoint copies of G contained in the vertex set of each oriented hyperedge e of $\mathscr{H}_{i,0}$. The copies of G, ordered to correspond to the orientation of e, are ordered $(G, <_{123})$ and $(G, <_{321})$. The vertex set of each $(G, <_{pqr})$ will be selected just as in step 2 (see Fig. 7).

4. For each hypergraph $\mathscr{H}_{i,j}$ obtained from the XY-Lemma and for each oriented hyperedge $e = (v_1, v_2, \ldots, v_k)$ in $\mathscr{H}_{i,j}$, $E(\Gamma)$ contains all edges necessary for Γ to induce a copy of $(G, <)$ on $(v_1, v_2, \ldots, v_k)$. In other words, $\Gamma|_{(v_1, v_2, \ldots, v_k)}$ is isomorphic as an ordered graph to $(G, <)$.
5. The only edges in $E(\Gamma)$ are those specified in 2–4.

The proof of the following claim follows the same reasoning as the corresponding claim in Section 8.1 and so is left to the reader.

Claim 2. *Γ is a yes-instance of $(G, <)$ORD if and only if $\mathscr{D}$ is a yes-instance of d-MIDDLE.* □

The case where B_2 and B_3 are indistinguishable follows the obvious symmetric steps as when B_1 and B_2 are indistinguishable, simply subdivide B_3 into B_3' and B_3''. Also, the case when both B_1 and B_3 are indistinguishable from B_2 follows from the obvious combination of the first two, with subdivisions of both B_1 and B_3. We have verified the NP-completeness of $(G, <)$ORD for any ordered graph $(G, <)$ with three or more blocks.

8.3. Case 3: 2-Block Graphs

We assume throughout this section that $(G, <)$ is a 2-connected graph with only two blocks in its block decomposition.

The two blocks must be connected to each other since our graph G is 2-connected. The blocks must both induce empty subgraphs or one induce an empty subgraph and one a complete subgraph. Notice that neither block may consist of a single vertex since this would be a cut vertex for the graph.

We will first look at the case where both of the blocks have size two and then generalize to larger blocks.

8.3.1. $|B_1| = |B_2| = 2$. Here our graph is one of the two pictured in Fig. 8.

In either case, we will show the NP-completeness of $(G, <)$ORD by the following polynomial transformation from instances of d-MIDDLE (for some

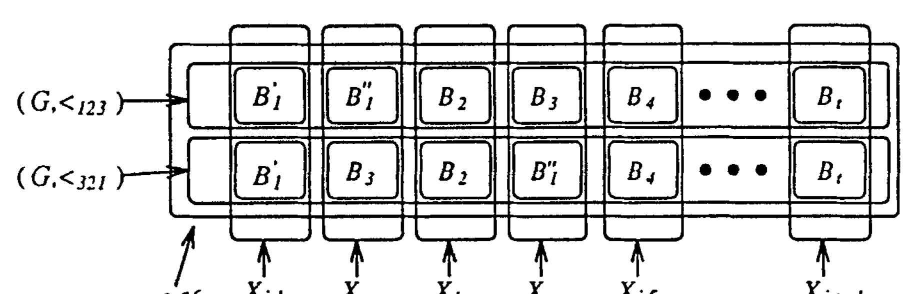

Fig. 7. Edge set of Γ for $e_i \in E_{MOM}$.

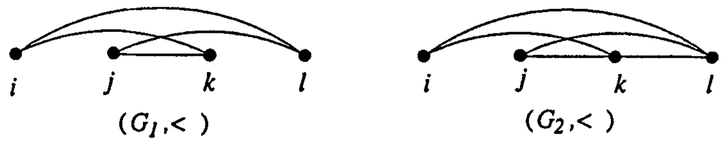

Fig. 8. Possible 4 vertex, 2-block graphs.

fixed $d \geq 108$). Again, yes-instances will correspond. Notice that $(G_1, <)$ of Figure 8 does not satisfy the hypotheses of Theorem 1.5. However, $(G_1, <)$ORD is NP-complete and the transformation that shows this gives the basic idea we will use later in the more general case. [We note that the assumption about the automorphisms of $(G, <)$ in Theorem 1.5 is essential only when we apply the XY-Lemma, a result which is not needed here, so we need not worry that the graphs in Figure 8 fail to satisfy the hypothesis.]

If $(G, <) = (G_1, <)$ then let N be an integer large enough that each of these hold:

(i) Lemma 3.2 with $\delta = \frac{1}{3^d}$, $c_1 = 7.5d$, $l = 4$, $n = 3$, $j_1 = 1$, $j_2 = 2$, $j_3 = 1$.
(ii) Lemma 3.2 with $\delta = \frac{1}{3^d}$, $c_1 = 7.5d$, $l = 4$, $n = 3$, $j_1 = 2$, $j_2 = 2$, $j_3 = 2$.

If $(G, <) = (G_2, <)$, then let N be an integer large enough that each of these hold:

(i′) Lemma 3.2 with $\delta = \frac{1}{3^d}$, $c_1 = 15d$, $l = 4$, $n = 3$, $j_1 = 2$, $j_2 = 4$, $j_3 = 2$.
(ii′) Lemma 3.2 with $\delta = \frac{1}{3^d}$, $c_1 = 15d$, $l = 4$, $n = 3$, $j_1 = 3$, $j_2 = 4$, $j_3 = 3$.

Let $\mathcal{D} = (S, E_{MIM}, E_{MOM})$ be an instance of d-MIDDLE with $(e_1, \ldots, e_p)$ a listing of the edges of $E_{MIM} \cup E_{MOM}$. Set

$$V = \bigcup_{a \in S} X_a,$$

where $|X_a| = N$ and all X_a's are disjoint. We will inductively construct a sequence $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_p$ of hypergraphs on V and set $\rho_i(x, y) = \mathrm{dist}_{\mathcal{H}_i}(x, y)$.

Initially, set

$$E(\mathcal{H}_0) = \emptyset,$$

and, hence, $\rho_0(x, y) = \infty$ for all $x \neq y$.

Assume that $\mathcal{H}_{i-1}$ and ρ_{i-1} have been obtained. The construction of $\mathcal{H}_i$ will depend on whether $e_i \in E_{MIM}$ or $e_i \in E_{MOM}$ as well as if $(G, <) = (G_1, <)$ or $(G, <) = (G_2, <)$.

MIM. Let $e_i = (a, b, c) \in E_{MIM}$.

With the parameters of Lemma 3.2 specified by (i) if $G = G_1$ or (i′) if $G = G_2$, $\rho = \rho_{i-1}$, $X_1 = X_a$, $X_2 = X_b$, and $X_3 = X_c$, let $\mathcal{H}^i$ be the hypergraph guaranteed by Lemma 3.2 and $\mathcal{H}_i = \mathcal{H}_{i-1} \cup \mathcal{H}^i$.

MOM. Let $e_i = (a, b, c) \in E_{MOM}$.

The same condition as in **MIM** except the parameters of Lemma 3.2 are given by (ii) or (ii′).

Again, execution of each step of our construction depends on the following claim.

Claim 1. *For every ρ_i, every $x \in V$, and every $r \leq 4$,*

$$|\{y \in V : \rho_i(x, y) = r\}| \leq (c_1 N^{\epsilon})^r .$$

Proof. As before, it suffices to prove that, for every i,

$$\Delta(\mathcal{H}_i) \leq 1.5dN^{\epsilon} .$$

Here is the reasoning. If $(G, <) = (G_1, <)$, then every edge of each $\mathcal{H}_i$ has cardinality $j_1 + j_2 + j_3 \leq 6$ so every vertex has at most $(6-1) \cdot 1.5dN^{\epsilon} = 7.5dN^{\epsilon}$ neighbors. If $(G, <) = (G_2, <)$, then every edge of each $\mathcal{H}_i$ has cardinality at most 8, giving each vertex at most $(8-1) \cdot 1.5dN^{\epsilon} = 10.5dN^{\epsilon}$ neighbors.

Fix i and consider $\mathcal{H}_i$. Let $x \in V$, then $x \in X_a$ for some $a \in S$. Let

$$E_a = \{e_q \in E_{MIM} \cup E_{MOM} : a \in e_q, q \leq i\} .$$

Obviously $|E_a| \leq d$. For each $e_q \in E_a$, x is a vertex of $\mathcal{H}^q$. Further, these are the only $\mathcal{H}^q$'s, $q \leq i$, of which x is a vertex. Since $\Delta(\mathcal{H}^q) \leq 1.5N^{\epsilon}$ for every q, we get

$$\deg_{\mathcal{H}_i}(x) \leq 1.5dN^{\epsilon} .$$

This establishes the claim. □

Construct Γ from $\mathcal{H}_p$ as follows.

1. Let $V(\Gamma) = V$.
2. For each $e_i = (a, b, c)$, define $E(\Gamma)$ so that every hyperedge of $\mathcal{H}^i$ contains the appropriate graph from Figure 9 if $(G, <) = (G_1, <)$, depending on if

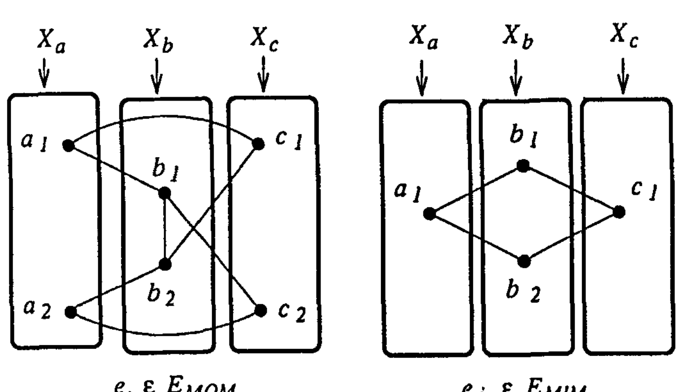

Fig. 9. Insertions if $(G, <) = (G_1, <)$.

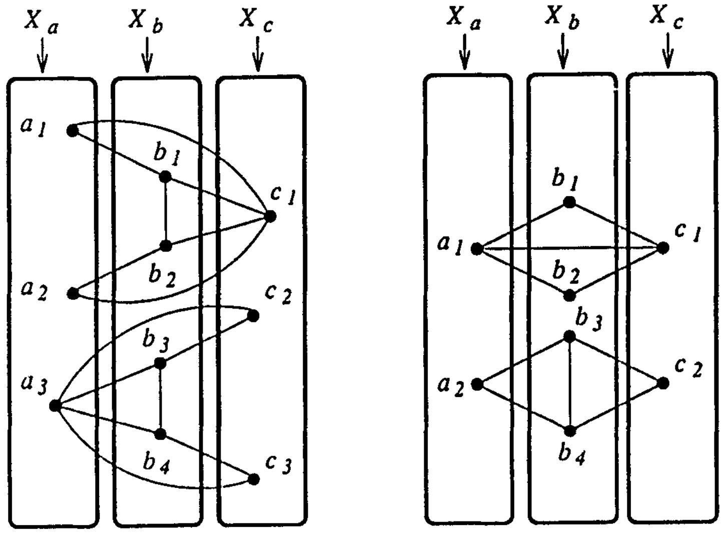

Fig. 10. Insertions if $(G, <) = (G_2, <)$.

$e_i \in E_{MIM}$ or $e_i \in E_{MOM}$, or from Figure 10 if $(G, <) = (G_2, <)$. These are the only edges that are to be in $E(\Gamma)$.

Again, notice that the size of each $\mathcal{H}^i$ depends only upon d and $(G, <)$, not upon the size of our instance $\mathcal{D}$ of d-MIDDLE. Thus, each step in our construction of Γ takes only constant time, and, hence, our construction is polynomial in the size of $\mathcal{D}$. Therefore, the following claim will prove the NP-completeness of $(G, <)$ORD for these two ordered graphs.

Claim 2. *Γ is a yes-instance of $(G, <)$ORD, for $(G, <) = (G_1, <)$ or $(G, <) = (G_2, <)$, if and only if $\mathcal{D}$ is a yes-instance of d-MIDDLE.*

Proof. The girth of $\mathcal{H}_p$ is greater than 4. This is proven in exactly the same manner as Claim 2 of Section 8.1. Hence, any cycle of length 4 or less in Γ must be contained within a single hyperedge on $\mathcal{H}_p$. Since any two vertices of either G_1 or G_2 are on a cycle of length 4, this implies that any copy of G in Γ must be contained within a single edge of $\mathcal{H}_p$.

Suppose $\mathcal{D} = (S, E_{MIM}, E_{MOM})$ is a yes-instance of d-MIDDLE, and let $<$ be a good ordering of S. We define an ordering $<'$ on $V(\Gamma)$ in the following manner:

1. $\forall a, b \in S$, if $a < b$, then define $X_a <' X_b$.
2. Order each X_a arbitrarily.

Subclaim 2.1. *$<'$ is a good order of $V(\Gamma)$ for $(G, <)$ORD* [*i.e.*, $(G, <) \not\leq (\Gamma, <')$].

Proof (of Subclaim 2.1). First assume that $(G, <) = (G_1, <)$. Then $(G, <)$ is a

4-cycle ordered (i, j, k, l) so that the missing edges are $\{i, j\}$ and $\{k, l\}$. The graph inserted into each hyperedge of each $\mathcal{H}^i$, where $e_i \in E_{MOM}$ consists of two 4-cycles which intersect on the edge labeled $\{b_1, b_2\}$ in Figure 9. The first one has vertex set $\{a_1, b_1, b_2, c_1\}$ and is missing edges $\{a_1, b_2\}$ and $\{b_1, c_1\}$ while the second one has vertex set $\{a_2, b_1, b_2, c_2\}$ and is missing edges $\{a_2, b_1\}$ and $\{b_2, c_2\}$. Also, the graph inserted into each hyperedge of each $\mathcal{H}^i$, where $e_i \in E_{MIM}$ is a 4-cycle with missing edges $\{a_1, c_1\}$ and $\{b_1, b_2\}$.

Suppose that $e_i = (a, b, c) \in E_{MOM}$. Since $<$ is a good ordering of S, neither $a < b < c$ nor $c < b < a$ are true. In other words, b is not between a and c when ordered by $<$. By definition of $<'$, b_1 and b_2 are either the first two or the last two vertices of the subgraph induced by a hyperedge of $\mathcal{H}^i$. But this implies that neither of the 4-cycles containing $\{b_1, b_2\}$ are ordered as a copy of $(G, <)$, since b_1 and b_2 are connected in these cycles and neither the first two nor the last two vertices of $(G, <)$ are connected.

If $e_i = (a, b, c) \in E_{MIM}$, then when ordered by $<$ either $a < b < c$ or $c < b < a$. This implies that each of the graphs inserted in the hyperedges of $\mathcal{H}^i$ is ordered by $<'$ so that b_1 and b_2 are between a_1 and c_1. This again implies that none of these graphs are ordered as a $(G, <)$ since $\{b_1, b_2\}$ is not an edge of each of these graphs while the second and third vertices of $(G, <)$ are connected. Hence, if $(G, <) = (G_1, <)$, then $<'$ is a satisfying order for $(G, <)$ORD.

Now assume that $(G, <) = (G_2, <)$. Then $G = K_4 - e$, where the missing edge is between the first two vertices (see Fig. 8). Let $e_i \in E_{MOM}$. The graph defined on the vertex set of each hyperedge of $\mathcal{H}^i$ is described in Figure 10. In the top component of this graph, the copies of $K_4 - e$ are on vertex set $\{a_1, b_1, b_2, c_1\}$ with missing edge $\{a_1, b_2\}$, and on vertex set $\{a_2, b_1, b_2, c_1\}$ with missing edge $\{a_2, b_1\}$. If $e_i \in E_{MIM}$, then the graph defined on each hyperedge of $\mathcal{H}^i$ is obviously two disjoint copies of $K_4 - e$, one with vertex set $\{a_1, b_1, b_2, c_1\}$ and missing edge $\{b_1, b_2\}$, and one with vertex set $\{a_2, b_3, b_4, c_2\}$ and missing edge $\{a_2, c_2\}$.

Now consider $<'$. If $e_i = (a, b, c) \in E_{MOM}$, then it is not ordered $a < b < c$ or $c < b < a$. Thus, the top component of the graph induced by the vertices of each hyperedge of $\mathcal{H}^i$ is ordered with b_1 and b_2 as the first two or the last two vertices. In either case, it is easy to see that neither copy of $K_4 - e$ will contain a copy of $(G, <)$. A similar argument holds for the other component.

If $e_i \in E_{MIM}$, then e_i is ordered either $a < b < c$ or $c < b < a$. In either case the missing edge in each of the copies of $K_4 - e$ will be between the first and last or the second and third vertices when ordered by $<'$. In neither case will this give us a copy of $(G, <)$. Hence if $\mathcal{D}$ is a yes-instance of d-MIDDLE then Γ is a yes-instance of $(G, <)$ORD. □ (Subclaim 2.1).

We now establish the other implication of Claim 2. Suppose that $<$ is an ordering of $V(\Gamma)$ such that

$$(G, <) \not\leq (\Gamma, <) .$$

Now define an ordering $<'$ on $S = V(\mathcal{D})$.

1. Let $(e_1, e_2, \ldots, e_p)$ be a list of $E_{MIM} \cup E_{MOM} = E(\mathcal{D})$.
2. For all $s \in S$ let $X_s^0 = X_s$.

3. For $i = 1, 2, \ldots, p$, we do the following.

   (a) If $e_i = (a, b, c)$, then there is ordering (j, k, l) of (a, b, c) and sets

$$X'_j \subseteq X_j^{i-1}, \qquad X'_k \subseteq X_k^{i-1}, \qquad X'_l \subseteq X_l^{i-1}$$

   such that

$$X'_j < X'_k < X'_l$$

   and

$$|X'_j| \geq \tfrac{1}{3}|X_j^{i-1}|, \qquad |X'_k| \geq \tfrac{1}{3}|X_k^{i-1}|, \qquad |X'_l| \geq \tfrac{1}{3}|X_l^{i-1}|.$$

   Let $X_j^i = X'_j$, $X_k^i = X'_k$, $X_l^i = X'_l$.

   (b) For all $x \notin \{a, b, c\}$ let $X_x^i = X_x^{i-1}$.

4. There is a partial order induced on the pairs x, y of S by $X_x^p < X_y^p$. Let $<'$ be any linear extension of that partial order.

The following subclaim validates Claim 2.

Subclaim 2.2. *$<'$ is a good ordering of S for d-MIDDLE.*

Proof (of Subclaim 2.2). We obtain a contradiction by assuming $e_i = (a, b, c) \in E_{MIM} \cup E_{MOM}$ is a badly ordered edge. If $e_i \in E_{MOM}$, this implies that either $a <' b <' c$ or $c <' b <' a$. Assume that $a <' b <' c$. This means that $X_a^p < X_b^p < X_c^p$. Since $\deg_{E_{MIM} \cup E_{MOM}}(x) \leq d$ for all x in S, $|X_x^p| \geq \frac{1}{3^d}|X_x|$. So by Lemma 3.2 there is a hyperedge of $\mathscr{H}^i$ contained in $X_a^p \cup X_b^p \cup X_c^p$. We claim that there is an ordered copy of $(G, <)$ on vertices within this hyperedge.

Say $(G, <) = (G_1, <)$. Then using the labeling of Figure 9, we know that $\{a_1, a_2\} <' \{b_1, b_2\} <' \{c_1, c_2\}$. We have two cases to consider. If $b_1 <' b_2$, then $\{a_2, b_1, b_2, c_2\}$ is ordered as $(G, <)$. If $b_2 <' b_1$, then $\{a_1, b_2, b_1, c_1\}$ is a properly ordered $(G, <)$. A similar argument holds for $e_i \in E_{MIM}$ as well as for both cases when $(G, <) = (G_2, <)$. This contradiction shows that every edge of $E_{MIM} \cup E_{MOM}$ must be properly ordered. Hence we get the NP-completeness of $(G, <)$ORD for these two ordered graphs, $(G_1, <)$ and $(G_2, <)$. □

8.3.2. Other 2-Block Graphs. Now we shall consider the other 2-block, 2-connected ordered graphs. These ordered graphs are generalizations of the two discussed in the last subsection and are pictured in Figure 11. The methods that we use to show the NP-completeness of $(G, <)$ORD for these graphs combine method of 8.3.1 with the XY-Lemma. We will give an outline of the steps, followed by the details of the construction. The details of the proofs, which are omitted, follow the same reasoning as earlier proofs.

If $(G, <) = (G_1, <)$, then, in order for the hypotheses of Theorem 1.5 to be satisfied, we must have $|B_1| \neq |B_2|$. This is not necessary for $(G, <) = (G_2, <)$. Notice that no block of these graphs may have only one vertex. Thus the subgraph induced by the last two vertices of B_1 and the first two vertices of B_2 corresponds to one of the ordered graphs of 8.3.1.

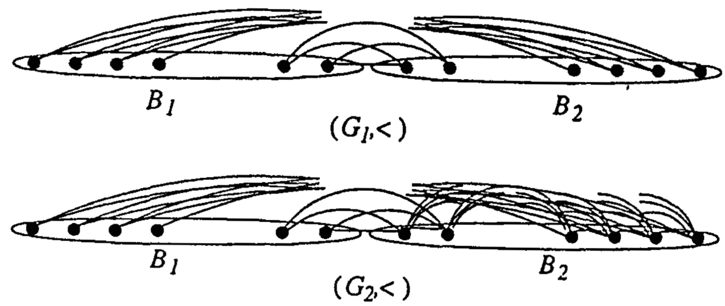

Fig. 11. Possible 2-block graphs.

We give a polynomial transformation of an instance $\mathscr{D} = (S, E_{MIM}, E_{MOM})$ of d-MIDDLE to an instance Γ of $(G,<)$ORD. Here is an outline of the construction of Γ. Let $e_1, e_2, \ldots, e_p$ be a listing of $E_{MIM} \cup E_{MOM}$. We obtain a sequence $\mathscr{H}_0, \mathscr{H}_1, \ldots, \mathscr{H}_p$ of hypergraphs, where each $\mathscr{H}_i$ is formed from $\mathscr{H}_{i-1}$ by adding a hypergraph $\mathscr{H}_i^0$ satisfying Lemma 3.2 and hypergraphs $\mathscr{H}_i^j$, $j = 1, 2, \ldots, 6$, satisfying Lemma 7.1, the XY-Lemma. If $e_i = (a, b, c)$, $\mathscr{H}_i^0$ has vertex set

$$X_{i,1} \cup X_a \cup X_b \cup X_c \cup X_{i,5}$$

(see Fig. 12). The graphs inserted into each hyperedge of $\mathscr{H}_i^0$, when restricted to $X_a \cup X_b \cup X_c$, are identical to the ones placed in the corresponding hyperedges in 8.3.1 (see Figs. 13, 14, 9, and 10). Thus, if e_i is ordered incorrectly for d-MIDDLE and X_a, X_b, and X_c are ordered likewise, then these sets will always contain an ordered copy of the last two vertices of B_1 and the first two vertices of B_2. The remaining vertices of B_1 are in $X_{i,1}$ and of B_2 are in $X_{i,5}$. These sets are forced to be ordered before and after the X_x's if we are to avoid $(G,<)$'s by the XY-Lemma (see Fig. 12). Thus any time that X_a, X_b, and X_c are ordered

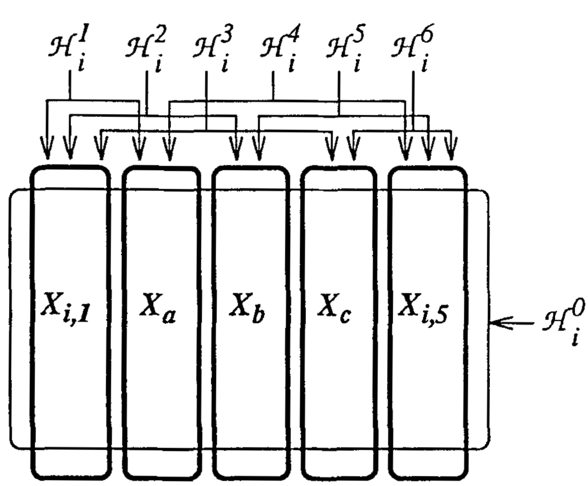

Fig. 12. Hypergraphs added to form $\mathscr{H}_i$.

incorrectly for d-MIDDLE, there must somewhere in Γ be a copy of $(G,<)$. It also may be shown that if X_a, X_b, and X_c may be correctly ordered for d-MIDDLE for all e_i's, then there is an ordering of $V(\Gamma)$ which does not contain a $(G,<)$.

We now will give the details of the construction. If $(G,<)=(G_1,<)$, then let N be an integer large enough that each of the following holds:

(i) Lemma 7.1 holds with $\delta=\frac{1}{3^d 5}$, $c_1=4.5\cdot 4(b_1+b_2)N^\epsilon$, and $k=|V(G)|$.
(ii) Lemma 3.2 holds with $\delta=\frac{1}{3^d 5}$, $c_1=4.5\cdot 4(b_1+b_2)N^\epsilon$, $l=|V(G)|$, $n=5$, $j_1=2(b_1-2)$, $j_2=2$, $j_3=4$, $j_4=2$, and $j_5=2(b_2-2)$.
(iii) Lemma 3.2 holds with $\delta=\frac{1}{3^d 5}$, $c_1=4.5\cdot 4(b_1+b_2)N^\epsilon$, $l=|V(G)|$, $n=5$, $j_1=4(b_1-2)$, $j_2=4$, $j_3=4$, $j_4=4$, and $j_5=4(b_2-2)$.

Similarly, if $(G,<)=(G_2,<)$, then let N be an integer large enough that each of the following hold:

(i′) Lemma 7.1 holds with $\delta=\frac{1}{3^d 5}$, $c_1=4.5\cdot 4(b_1+b_2)N^\epsilon$, and $k=|V(G)|$.
(ii′) Lemma 3.2 holds with $\delta=\frac{1}{3^d 5}$, $c_1=4.5\cdot 4(b_1+b_2)N^\epsilon$, $l=|V(G)|$, $n=5$, $j_1=2(b_1-2)$, $j_2=2$, $j_3=4$, $j_4=2$, and $j_5=2(b_2-2)$.
(iii′) Lemma 3.2 holds with $\delta=\frac{1}{3^d 5}$, $c_1=4.5\cdot 4(b_1+b_2)N^\epsilon$, $l=|V(G)|$, $n=5$, $j_1=4(b_1-2)$, $j_2=3$, $j_3=4$, $j_4=3$, and $j_5=4(b_2-2)$.

Let $\mathcal{D}=(S, E_{MIM}, E_{MOM})$ be an instance of d-MIDDLE with $e_1, e_2, \ldots, e_p$ a list of $E_{MIM}\cup E_{MOM}$. We will form a sequence of hypergraphs $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_p$ of hypergraphs on the vertex set

$$V=\bigcup_{a\in S} X_a \cup \bigcup_{i=1}^{p} [X_{i,1}\cup X_{i,5}],$$

where each X_a and each $X_{i,j}$ are sets of N vertices. Again we define $\rho_i(x,y)=\mathrm{dist}_{\mathcal{H}_i}(x,y)$.

Initially, set

$$E(\mathcal{H}_0)=\emptyset$$

and, hence $\rho_0(x,y)=\infty$ for all $x\neq y$.

Assume $\mathcal{H}_{i-1}$ and ρ_{i-1} have been obtained. The construction of $\mathcal{H}_i$ depends on whether $e_i\in E_{MIM}$ or $e_i\in E_{MOM}$.

MIM. Let $e_i=(a,b,c)\in E_{MIM}$:

(0) With the parameters of Lemma 3.2 specified by (ii) if $(G,<)=(G_1,<)$ or, (ii′) if $(G,<)=(G_2,<)$, $\rho=\rho_{i-1}$, $X_1=X_a$, $X_2=X_b$, and $X_3=X_c$, let $\mathcal{H}_i^0$ be the hypergraph guaranteed by Lemma 3.2. Let $\mathcal{H}_{i,0}=\mathcal{H}_{i-1}\cup\mathcal{H}_i^0$ and $\rho_{i,0}=\mathrm{dist}_{\mathcal{H}_{i,0}}$.
(1) With the parameters of Lemma 7.1 given by (i) or (i′), $\rho=\rho_{i,0}$, $X=X_{i,1}$, and $Y=X_a$, let $\mathcal{H}_i^1$ be the hypergraph guaranteed by Lemma 7.1. Let $\mathcal{H}_{i,1}=\mathcal{H}_{i,0}\cup\mathcal{H}_i^1$ and $\rho_{i,1}=\mathrm{dist}_{\mathcal{H}_{i,1}}$.

(2) As (1) but replace $\rho = \rho_{i,1}$, $Y = X_b$, resulting in $\mathcal{H}_i^2$ from Lemma 7.1 and $\mathcal{H}_{i,2} = \mathcal{H}_{i,1} \cup \mathcal{H}_i^2$, $\rho_{i,2} = \text{dist}_{\mathcal{H}_{i,2}}$.
(3) As (1) but replace $\rho = \rho_{i,2}$, $Y = X_c$, resulting in $\mathcal{H}_i^3$ from Lemma 7.1 and $\mathcal{H}_{i,3} = \mathcal{H}_{i,2} \cup \mathcal{H}_i^3$, $\rho_{i,3} = \text{dist}_{\mathcal{H}_{i,3}}$.
(4) As (1) but replace $\rho = \rho_{i,3}$, $X = X_a$, $Y = X_{i,5}$, resulting in $\mathcal{H}_i^4$ from Lemma 7.1 and $\mathcal{H}_{i,4} = \mathcal{H}_{i,3} \cup \mathcal{H}_i^4$, $\rho_{i,4} = \text{dist}_{\mathcal{H}_{i,4}}$.
(5) As (1) but replace $\rho = \rho_{i,4}$, $X = X_b$, $Y = X_{i,5}$, resulting in $\mathcal{H}_i^5$ from Lemma 7.1 and $\mathcal{H}_{i,5} = \mathcal{H}_{i,4} \cup \mathcal{H}_i^5$, $\rho_{i,5} = \text{dist}_{\mathcal{H}_{i,5}}$.
(6) As (1) but replace $\rho = \rho_{i,5}$, $X = X_c$, $Y = X_{i,5}$, resulting in $\mathcal{H}_i^6$ from Lemma 7.1 and $\mathcal{H}_i = \mathcal{H}_{i,5} \cup \mathcal{H}_i^6$, $\rho_i = \text{dist}_{\mathcal{H}_{i,6}}$.

MOM. Let $e_i = (a, b, c) \in E_{MOM}$.

Steps (0)–(6) from **MIM** *except* the parameters of Lemma 3.2 are given by (iii) or (iii′).

Again, verification of the following claim is all that is needed to ensure that each step may in fact be executed. The proof follows the same reasoning as Claim 2 in Section 8.2 and so is omitted.

Claim 1. *For each $\rho = \rho_i^j$, for each $x \in V$ and $r \leq k$*

$$|\{y \in V : \rho(x, y) = r\}| \leq (c_1 N^\epsilon)^r .$$

Recall that $c_1 = 4.5 \cdot 4(b_1 + b_2)N^\epsilon$. □

This means we can form our hypergraph $\mathcal{H}_p$. Construct Γ from $\mathcal{H}_p$ as follows:

1. Let $V(\Gamma) = V$.
2. Define $E(\Gamma)$ according to the following.

   (a) For every $\mathcal{H}_i^0$ each hyperedge induces the appropriate graph from Figure 13 if $(G, <) = (G_1, <)$, depending on whether $e_i \in E_{MOM}$ or $e_i \in E_{MIM}$, or from Figure 14 if $(G, <) = (G_2, <)$, again depending on whether $e_i \in E_{MOM}$ or $e_i \in E_{MIM}$.
   (b) For every $\mathcal{H}_i^j$, $1 \leq j \leq 6$, each hyperedge induces a copy of $(G, <)$ if the vertices on the hyperedge are ordered to correspond to the orientation of the hyperedge.

The procedure takes polynomial time because each step of the construction depends only upon d and $(G, <)$, not upon the size of $\mathcal{D}$. Therefore, the following claim establishes the NP-completeness of $(G, <)$ORD for 2-block graphs.

Claim 2. *Γ is a yes-instance of $(G, <)$ORD if and only if $\mathcal{D}$ is a yes-instance of d-MIDDLE.*

Proof. The details of the proof of this claim are similar to those of earlier claims and so are suppressed. We will only outline the proof here.

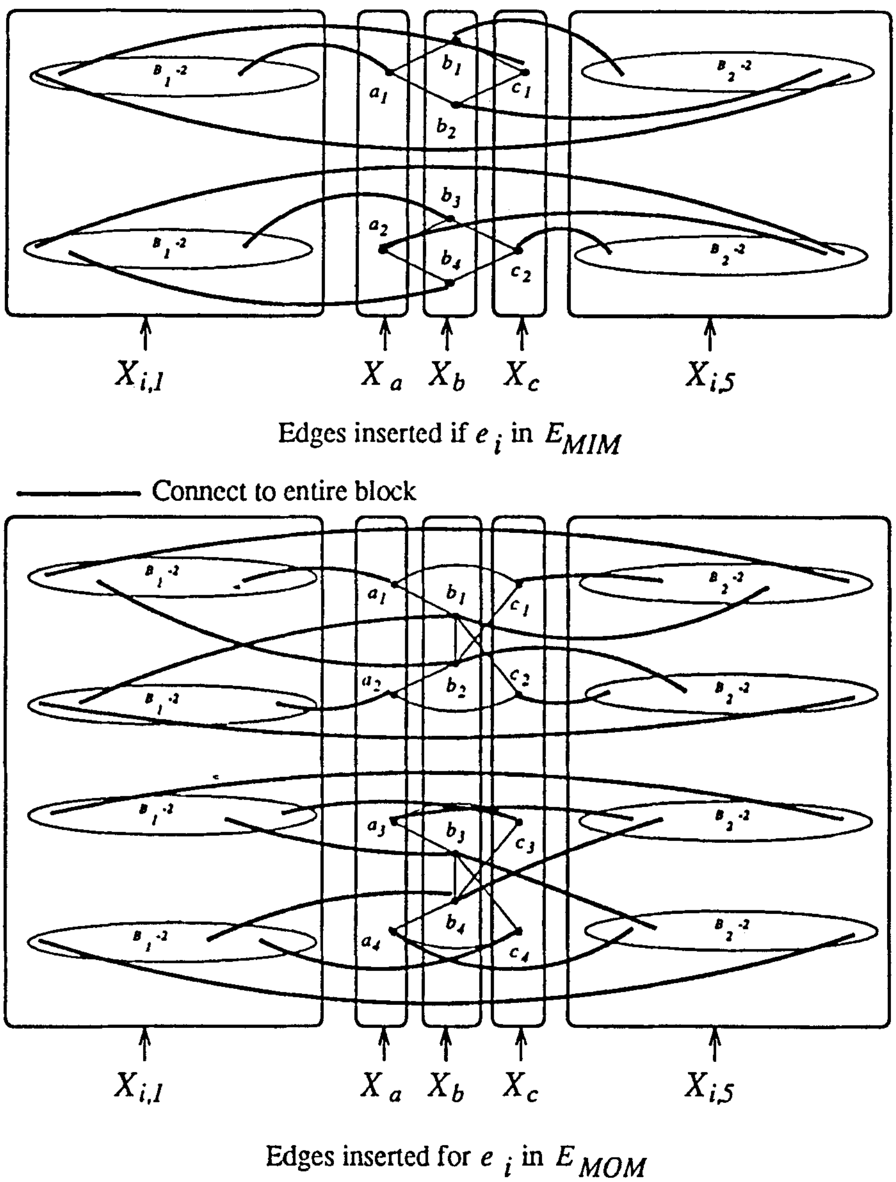

Fig. 13. Insertions for $(G_1, <)$.

Let $\mathcal{D}$ be a yes-instance of d-MIDDLE. We order Γ according to the following rules. All of the $X_{i,1}$'s are ordered before all of the X_a's, which are ordered the same way as a good ordering of the a's in $\mathcal{D}$ and before all of the $X_{i,5}$'s. At the same time, we order each $X = X_a$, $X_{i,j}$ in the ordering provided by Lemma 7.1. One can check that $(\Gamma, <)$ will not contain a $(G, <)$.

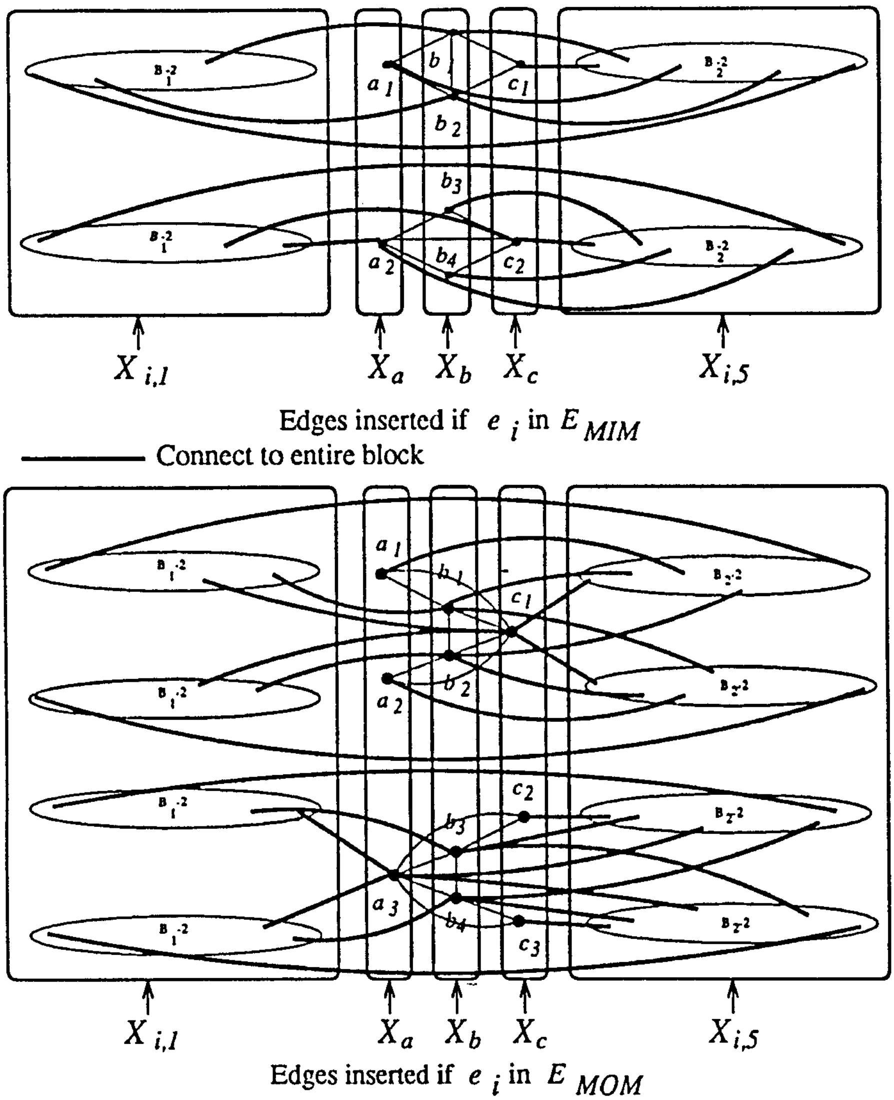

Fig. 14. Insertions for $(G_2, <)$.

Conversely, if Γ is a yes-instance of $(G, <)$ORD, then, defining an ordering on $\mathscr{D}$ from a good ordering of Γ in the same way as in Section 8.1, a badly ordered edge of $\mathscr{D}$ would correspond to a copy of $(G, <)$ in $(\Gamma, <)$ and hence does not exist. This verifies the claim. □

With the NP-completeness of $(G, <)$ORD for all 2-block, 2-connected ordered graphs now established, we have completed the proof of Theorem 1.5.

9. FURTHER DIRECTIONS

9.1. Some Open Problems

There are many open problems left in this area.

The first, and most obvious, of these is to settle the conjecture made in the introduction on the sufficiency of 2-connectedness for the NP-completeness of $(G, <)$ORD.

Conjecture 1.6. *For any ordered graph $(G, <)$, such that $(G, <)$ is neither a complete nor an empty graph, $(G, <)$ORD is NP-complete if either G or its complement is 2-connected.*

One could also ask the following question.

Problem 9.1. *What are some other large classes of ordered graphs $(G, <)$ for which $(G, <)ORD \in P$?*

With the exception of a few small graphs G, $(G, <)$ORD is known to be in P only for G complete or $G(, <)$ a star with the cut vertex first or last in the order.

At the same time, if we assume that Conjecture 1.6 is true, the graphs remaining to be considered for NP-completeness are those for which neither G nor its complement are 2-connected. These graphs are characterized by Rödl and Sauer in [24]. We give their description here. Denote by $\mathcal{M}$ the class of graphs G with the property that there is a cut vertex $u \in V(G)$ which is adjacent to every other vertex of G. Denote by $\mathcal{K}$ the class of graphs G with $u \in V(G)$ which is adjacent to every other vertex of G except one. Finally, for any class $\mathcal{G}$ of graphs, we denote by $\mathcal{G}^c$ the class of complements of $\mathcal{G}$.

Theorem 9.2. *If neither G nor G^c is 2-connected, then,*

$$G \in \mathcal{M} \cup \mathcal{M}^c \cup \mathcal{K} \cup \mathcal{K}^c .$$

This characterization leads us to our next problem, whose solution would shed considerable light on the computational complexity of $(G, <)$ORD for ordered graphs which have the property that neither they nor their complements are 2-connected. Given an ordered graph $(G, <) = (V, E, <)$, we will denote by $(\hat{G}, <)$ the ordered graph obtained from $(G, <)$ by adding a vertex x, joining it to every vertex of G, and placing it last in the ordering.

Problem 9.3. *For an ordered graph $(G, <)$ decide the following. Does the NP-completeness of $(G, <)ORD$ imply the NP-completeness of $(\hat{G}, <)ORD$?*

A positive answer to this question would give a large class of ordered graphs which are not 2-connected but for which $(G, <)$ORD is still NP-complete.

More generally one may pose the following.

Problem 9.4. *If $(G_1, <)$ORD is NP-complete and $(G_2, <) \supseteq (G_1, <)$, does this imply that $(G_2, <)$ORD is NP-complete?*

If the answer to this question is "yes," then indeed for nearly all ordered graphs $(G, <)$, $(G, <)$ORD would be NP-complete.

It would also be interesting to know if changing the ordering of $V(G)$ could change the complexity. Thus we ask the following question.

Problem 9.5. *Does there exist a graph G and two orderings of $V(G)$, $<_1$ and $<_2$, such that $(G, <_1)$ORD is NP-complete and $(G, <_2)$ORD is in P?*

9.2. The Problem $(\mathcal{G}, <)$ORD

A natural generalization of $(G, <)$ORD is the problem $(\mathcal{G}, <)$ORD where one asks for an ordering which avoids a whole class of graphs instead of just a single graph. There are well studied classes of graphs which may be characterized by $(\mathcal{G}, <)$ORD (cf. [7]). Here we will list a few of them.

Threshold Graph. A graph $G = (V, E)$ such that there is a weight assignment $w : V \to \mathbb{Z}^+$ and an integer t such that for all sets $\{x, y\} \in [V]^2$, $\{x, y\} \in E$ if and only if $w(x) + w(y) > t$.

Split Graph. A graph $G = (V, E)$ whose vertex set V may be partitioned into two sets S and K in such a way that S is an independent set and K induces a complete graph.

Interval Graph. A graph whose vertex set may be represented as intervals on the real line in such a way that two vertices are adjacent if and only if their intervals intersect.

Permutation Graph. A graph $G = (V, E)$ with $|V| = n$ such that there is a permutation $\pi \in S_n$ and a labeling $(1, 2, \ldots, n)$ of V such that

$$\{i, j\} \in E \Leftrightarrow (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0 .$$

The following theorem may be inferred from [7].

Theorem 9.6. *If $V(G_i, <) = (1, 2, 3)$ for $i = 1, 2, 3$ and*

$$E(G_1, <) = \{\{1,2\}, \{1,3\}\} ,$$

$$E(G_2, <) = \{\{1,2\}, \{2,3\}\} ,$$

and

$$E(G_3, <) = \{\{1,2\}, \{1,3\}, \{2,3\}\} ,$$

then Γ is a yes-instance of $(\mathcal{G}, <)$ORD if and only if:

(*i*) *Γ is a threshold graph, for $(\mathcal{G}, <) = \{(G_1, <), (G_1^c, <)\}$,*
(*ii*) *Γ is a split graph, for $(\mathcal{G}, <) = \{(G_1, <), (G_1^c, >)\}$,*
(*iii*) *Γ is an interval graph, for $(\mathcal{G}, <) = \{(G_1, <), (G_2^c, <)$,*

(*iv*) Γ *is a permutation graph, for* $(\mathcal{G}, <) = \{(G_2, <), (G_2^c, <)\}$,
(*v*) Γ *is a forest, for* $(\mathcal{G}, <) = \{(G_1, <), (G_3, <)\}$,
(*vi*) Γ *is a bipartite graph, for* $(\mathcal{G}, <) = \{(G_2, <), (G_3, <)\}$,
(*vii*) Γ *is a threshold graph, for* $(\mathcal{G}, <) = \{(G_1, <), (G_2, <), (G_1^c, >)\}$.

All of the classes of graphs characterized in Theorem 9.6 are recognizable in polynomial time [14, 19]. Hence for all of the classes $(\mathcal{G}, <)$ mentioned in the theorem, $(\mathcal{G}, <)$ORD is in *P*.

In [7], the author also lists $(\mathcal{G}, <)$ORD characterizations for many other classes, including circular arc graphs and outerplanar graphs.

Another interesting class of graphs that can be characterized by $(\mathcal{G}, <)$ORD are perfectly orderable graphs, first defined by Berge and Chvátal in [3]. The problem of recognizing if a graph is perfectly orderable was shown to be NP-complete by Middendorf and Pfeiffer [20]. The $(\mathcal{G}, <)$ORD characterization of perfectly orderable graphs allows us to state their result as follows.

Theorem 9.7. *If* $(\mathcal{G}, <) = \{(G_1, <), (G_2, <), (G_3, <\})$, *where*

$$V(G_i, <) = (1, 2, 3, 4)$$

for all i, and

$$E(G_1, <) = \{\{1, 2\}, \{2, 4\}, \{3, 4\}\},$$

$$E(G_2, <) = \{\{1, 3\}, \{2, 4\}, \{3, 4\}\},$$

and

$$E(G_3, <) = \{\{1, 4\}, \{2, 3\}, \{3, 4\}\},$$

then $(\mathcal{G}, <)$*ORD is NP-complete.*

Graphs of bounded arboricity constitute a class of graphs which can be "approximated" by $(\mathcal{G}, <)$ORD classes. The *arboricity* of a graph G, $\Upsilon(G)$ is defined to be the minimum numbers of edge-disjoint spanning forests into which G can be decomposed [16]. The following theorem by Nash-Williams [21] allows us to give a $(\mathcal{G}, <)$ORD characterization for approximating the arboricity of a graph.

Theorem 9.8. *Let G be a graph, then*

$$\Upsilon(G) = \max_{H \subseteq G} \left\{ \frac{|E(H)|}{|V(H)| - 1} \right\}.$$

Using this, we can obtain the following $(\mathcal{G}, <)$ORD characterization to approximate the arboricity of a graph within a factor of 2.

Theorem 9.9. *Let* $(\mathcal{D}, <)$ *be the class of all ordered graphs on d vertices and* $(\mathcal{G}, <) = \{(\hat{G}, <) : G \in \mathcal{D}\}$. *[Refer to* 9.1 *for the definition of* $(\hat{G}, <)$.*] If a graph* Γ *is a yes-instance of* $(\mathcal{G}, <)$*ORD, then it has*

$$\Upsilon(\Gamma) < d,$$

and if Γ *is a no-instance of* $(\mathcal{G}, <)ORD$, *then*

$$\Upsilon(\Gamma) > d/2 .$$

A similar result may be obtained for p-arrangeable graphs which are considered with a problem in Ramsey theory [4]. To define a p-arrangeable graph, we first need the following terminology. Given a graph $G = (V, E)$ and $A, B \subseteq V$, we define $N_B(A) = (\cup_{a \in A} N(a)) \cap B$. For an ordered graph $(G, <)$ with $V = (v_1, v_2, \ldots, v_k)$ and any $i \in [k]$, we define $L_i = (v_1, \ldots, v_i)$ and $R_i = (v_{i+1}, \ldots, v_k)$. A graph G of order n is said to be p-arrangeable if there exists an ordering $(v_1, \ldots, v_n)$ of the vertices of G such that for each $1 \leq i \leq n-1$

$$|N_{L_i}(N_{R_i}(v_i))| \leq p .$$

It is not hard to see that there is a natural $(\mathcal{G}, <)$ORD characterization of p-arrangeable graphs. We omit, however, the somewhat tedious description of this characterization. To further investigate the general $(\mathcal{G}, <)$ORD problem, the first, most general question to ask is the following.

Problem 9.10. *For what classes of ordered graphs* $(\mathcal{G}, <)$ *is* $(\mathcal{G}, <)ORD$ *NP-complete and for what classes is it in P?*

A starting point for this type of question could be the following.

Problem 9.11. *If* $(G_1, <_1)ORD$ *and* $(G_2, <_2)ORD$ *are both NP-complete and*

$$(\mathcal{G}, <) = \{(G_1, <_1), (G_2, <_2)\} ,$$

then is $(\mathcal{G}, <)ORD$ *NP-complete?*

Perhaps this question has a negative answer, but we can get a partial positive answer to this problem by adapting some of the proofs that we have used for earlier results.

Theorem 9.12. *Let ordered graphs*

$$(G_1, <), (G_2, <), \ldots, (G_k, <) ,$$

satisfy each of the following:

1. *Each* G_i *is 2-connected.*
2. $G_i \not\subseteq G_j$ *for* $i \neq j$.
3. *For some* i, $(G_i, <)ORD$ *is one of the problems that we have shown is NP-complete.*

Then with

$$(\mathcal{G}, <) = \{(G_1, <), (G_2, <), \ldots, (G_k, <)\} ,$$

$(\mathcal{G}, <)ORD$ *is NP-complete.*

This theorem does not apply if we take different orderings of the same graph to define our class. This is the type of class used to characterize perfectly orderable graphs in Theorem 9.7 and may give other useful characterizations. Hence we pose the following final problem.

Problem 9.13. *If* $(\mathcal{G}, <)$ *is a class of ordered graphs obtained by taking different orderings of a simple graph* G*, what is the computational complexity of* $(\mathcal{G}, <)ORD$*? In particular, how is this related to the computational complexity of* $(G, <)ORD$ *for the various orderings* $<$ *of* $V(G)$*?*

Adapting the technique we used in proving Theorem 1.5, we can again give a partial answer. For a given ordered graph $(G, <)$, where

$$V(G, <) = (v_1, v_2, \dots, v_k)\,,$$

we denote by $(G, <^f)$ the ordered copy of G, where

$$V(G, <^f) = (v_k, v_{k-1}, \dots, v_1)\,,$$

and by $(G, <^r)$ the ordered copy of G where,

$$V(G, <^r) = (v_2, v_3, \dots, v_k, v_1)\,.$$

Our result may now be stated as follows.

Theorem 9.14. *Let* G *be a 2-connected graph and let*

$$(\mathcal{G}, <) = \{(G, <_1), (G, <_2), \dots, (G, <_m)\}\,.$$

If for some $i \in [m]$

$$(G, <_i^f) \not\cong (G, <_j)$$

and

$$(G, <_i^r) \not\cong (G, <_j)$$

for all $j \in [m]$*, then* $(\mathcal{G}, <)ORD$ *is NP-complete.*

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