JOURNAL OF COMBINATORIAL THEORY, Series A 65, 334–342 (1994)

Note

An Explicit 1-Factorization in the Middle of the Boolean Lattice

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Communicated by the Managing Editors

Recieved June 19, 1991

An explicit definition of a 1-factorization of $\mathbf{B}_k$ (the bipartite graph defined by the k- and $(k+1)$-element subsets of $[2k+1]$), whose constituent matchings are defined using addition modulo $k+1$, is introduced. We show that the matchings are invariant under rotation (mapping under $\sigma = (1, 2, 3, ..., 2k+1)$), describe the effect of reflection (mapping under $\rho = (1, 2k+1)(2, 2k) \cdots (k, k+2)$), determine that there are no other symmetries which map these matchings among themselves, and prove that they are distinct from the lexical matchings in $\mathbf{B}_k$. © 1994 Academic Press, Inc.

1. Introduction

For a fixed k, denote the collection of j-element subsets of $[2k+1] = \{1, 2, ..., 2k+1\}$ by $\mathbf{R}_j$ and let $\mathbf{B}_k$ be the bipartite graph defined on the vertex set $\mathbf{R}_k \cup \mathbf{R}_{k+1}$ by letting A be adjacent to B iff $A \subset B$ or vice versa. In [KT] the second author and Trotter introduced an explicit 1-factorization $\{\boldsymbol{l}_0, ..., \boldsymbol{l}_k\}$ of $\mathbf{B}_k$, called the *lexical* factorization, and determined its behavior under the automorphisms of $\mathbf{B}_k$. In this article we report on

* Supported by Office of Naval Research Grant N0004-85-K-0769.

† Supported by Office of Naval Research Grant N00014-90-J-1206.

0097-3165/94 $6.00

another explicit 1-factorization $\{\mathbf{m}_1, ..., \mathbf{m}_{k+1}\}$ of $\mathbf{B}_k$, which we call the *modular* factorization, and whose component matchings we call *modular* matchings. The origins of the modular factorization are quite murky; it has apparently been rediscovered several times. We first learned of it in 1986, defined in terms of lattice paths, from Robinson [R], who asked if it was the same as the lexical factorization. Here we show that this is not the case: for $k \geqslant 4$, no modular matching $\mathbf{B}_k$ is a lexical matching. However, they behave remarkably similarly under the automorphisms of $\mathbf{B}_k$. We show that the stabilizer of each modular matching $\mathbf{m}_i$ is the same as the stabilizer of each lexical matching $\mathbf{l}_{i-1}$.

The odd graph $\mathbf{O}_k$ is defined on the vertices of $\mathbf{R}_k$ by letting A be adjacent to B iff $A \cap B = \emptyset$. In the case where k is even, Kierstead and Trotter used the $k/2$-lexical matching $\mathbf{l}_{k/2}$ to obtain an explicit perfect matching in $\mathbf{O}_k$. Here we show that the same construction, with $\mathbf{l}_{k/2}$ replaced by $\mathbf{m}_{k/2+1}$, gives a new explicit perfect matching of $\mathbf{O}_k$. For more results on this subject, the reader is referred to the third author's dissertation [S].

Given a j-subset A of $[2k+1]$, we take $A = \langle a_1, a_2, ..., a_j \rangle$ to mean $a_1 < a_2 < ...$, and $A^C = [2k+1] \backslash A = \langle \bar{a}_1, \bar{a}_2, ..., \bar{a}_{(2k+1)-j} \rangle$ to mean $\bar{a}_1 > \bar{a}_2 > ...$. When there is no chance for confusion, we simply write a_i for the ith smallest element of A and $\bar{a}_i$ for the ith largest element of A^C. Also, $\sum A$ denotes the sum of the elements in A. We denote the symmetric group on $[2k+1]$ by S_{2k+1}. A perfect matching in $\mathbf{B}_k$ is a collection $\mathbf{M}$ of edges such that each vertex of $\mathbf{B}_k$ is incident to exactly one edge of $\mathbf{M}$. A 1-factorization of $\mathbf{B}_k$ is a collection of $k+1$ disjoint perfect matchings of $\mathbf{B}_k$. However, it will be convenient for us to consider a perfect matching to be an injection $\mathbf{m}: \mathbf{R}_k \to \mathbf{R}_{k+1}$ such that A is adjacent to $\mathbf{m}(A)$, for all $A \in \mathbf{R}_k$.

For the sake of completeness we include the definition of the lexical factorization given in [KT]. Let $[y, x)$ denote the set $\{y, y+1, ..., x-2, x-1\}$, where addition is modulo $2k+1$. For S, $R \subset [2k+1]$, let $R/S = |R \cap S| - |R \cap S^C|$ and $d_S(x) = |\{y \in S^C - \{x\}: [y, x)/S < 0\}|$. It can be shown that for each set $S \in \mathbf{R}_k$ and each $i = 0, ..., k$, there exists a unique $x \in S^C$ such that $d_S(x) = i$. The i-lexical matching $\mathbf{l}_i: \mathbf{R}_k \to \mathbf{R}_{k+1}$ is defined by $\mathbf{l}_i(S) = S \cup \{x\}$, where $d_S(x) = i$.

2. Modular Matchings

We are now ready to define the modular factorization. The ith perfect matching $\mathbf{m}_i$, $i = 1, ..., k+1$, in the modular factorization is defined by $\mathbf{m}_i: \mathbf{R}_k \to \mathbf{R}_{k+1}$, with

$$\mathbf{m}_i(A) = A \cup \{\bar{a}_y\}, \qquad \text{where} \quad y \equiv \left(i + \sum A\right) (\bmod k+1). \tag{1}$$

Our first task is to verify that this is indeed a factorization.

THEOREM 1. *For* $i=1, ..., k+1$, $\mathbf{m}_i$ *is a matching in* $\mathbf{B}_k$ *and* $\{\mathbf{m}_1, ..., \mathbf{m}_{k+1}\}$ *is a* 1-*factorization of* $\mathbf{B}_k$.

Proof. The second assertion is an immediate consequence of the first and the definition of $\mathbf{m}_i$ via (1). To see that $\mathbf{m}_i$ is a matching, we find a rule $\mathbf{b}_i: \mathbf{R}_{k+1} \to \mathbf{R}_k$ such that $\mathbf{b}_i \circ \mathbf{m}_i = \text{id}$. Define $\mathbf{b}_i$ by $\mathbf{b}_i(B) = B - \{b_x\}$, where

$$x \equiv \left(i + \sum B\right) \qquad (\text{mod } k+1). \tag{2}$$

Suppose that for some $S \in \mathbf{R}_k$, $\mathbf{m}_i(S) = S \cup \{\bar{s}_y\}$. Then there are $(2k+1) - \bar{s}_y$ elements of $[2k+1]$ larger than $\bar{s}_y$,

$y-1$ are in $\bar{S}$, so

$(2k+1) - \bar{s}_y - (y-1)$ are in S, so

$k((2k+2) - (\bar{s}_y + y))$ elements of S are less than $\bar{s}_y$.

Computing modulo $k+1$, we get

$$\begin{aligned} k - ((2k+2) - (\bar{s}_y + y)) &\equiv k + \bar{s}_y + y \\ &\equiv k + \bar{s}_y + \left(i + \sum S\right) \\ &\equiv -1 + i + \sum (S \cup \{\bar{s}_y\}), \end{aligned}$$

and thus $\bar{s}_y$ is the $i + \sum (S \cup \{\bar{s}_y\})$ smallest element of $(S \cup \{\bar{s}_y\})$. Hence, $\mathbf{b}_i(\mathbf{m}_i(S)) = \mathbf{b}_i(S \cup \{\bar{s}_y\}) = S$, by (2). ∎

We say that a function $\mathbf{f}$ on $\mathbf{O}_k$ is a perfect matching if, for all $S \in \mathbf{R}_k$, both (1) S is adjacent to $\mathbf{f}(S)$ and (2) $\mathbf{f}^2(S) = S$. The second condition is needed since $\mathbf{O}_k$ is not bipartite.

THEOREM 2. *Suppose that k is even. Let* $\mathbf{f}$ *be the function defined on the vertices, the sets in* $\mathbf{R}_k$, *of the odd graph* $\mathbf{O}_k$ *by* $\mathbf{f}(S) = (\mathbf{m}_i(S))^{\mathrm{C}}$, *where* $i = k/2 + 1$. *Then* $\mathbf{f}$ *is a perfect matching of* $\mathbf{O}_k$.

Proof. Clearly $\mathbf{f}$ satisfies (1) of the definition above. We check (2). Fix $S \in \mathbf{R}_k$ and let $S^{\mathrm{C}} = \langle s_1^{\mathrm{c}}, ..., s_{k+1}^{\mathrm{c}} \rangle$. First note that $\mathbf{f}^{-1}(S) = \mathbf{b}_i(S^{\mathrm{C}})$ and $\sum S^{\mathrm{C}} = -\sum S (\text{mod } k+1)$. It suffices to check that $\mathbf{f}(S) = \mathbf{f}^{-1}(S)$:

$$\mathbf{f}(S)=(\mathbf{m}_i(S))^{\mathrm{C}}=S^{\mathrm{C}}-\{\bar{s}_x\},$$
$$\text{where} \quad x\equiv i+\sum S \,(\mathrm{mod}\, k+1);$$
$$\mathbf{f}^{-1}(S)=\mathbf{b}_i(S^{\mathrm{C}})=S^{\mathrm{C}}-\{s_y^{\mathrm{c}}\},$$
$$\text{where} \quad y\equiv i+\sum S^{\mathrm{C}}\equiv i-\sum S \,(\mathrm{mod}\, k+1).$$

Since $x+y\equiv 1 \pmod{k+1}$, it follows that $\bar{s}_x=s_y^{\mathrm{c}}$, and we are done. ∎

3. Orbits of Modular Matchings

It is well known (cf. [DHR]) that the automorphisms of $\mathbf{B}_k$ which preserve levels are just those induced by the action of S_{2k+1} on $[2k+1]$. We denote, therefore, automorphisms of $\mathbf{B}_k$ by permutations on $[2k+1]$. Given a permutation α, that a subgraph H of $\mathbf{B}_k$ is α-invariant means that AB is an edge in H if and only if $A^\alpha B^\alpha$ is an edge in H. In other words, α is a member of the stabilizer of H. A matching $\mathbf{a}$ of $\mathbf{B}_k$ is α-invariant means $\mathbf{a}(S^\alpha)=(\mathbf{a}(S))^\alpha$ for all $S\in\mathbf{R}_k$. In general, let $\mathbf{a}^\alpha$ be the matching given by the rule $\mathbf{a}^\alpha(S^\alpha)=(\mathbf{a}(S))^\alpha$.

Let $\sigma=(1, 2, 3, ..., 2k+1)\in S_{2k+1}$. We call a permutation of the form σ^i a *rotation*.

THEOREM 3. *For* $i=1, ..., k+1$, $\mathbf{m}_i$ *is a σ-invariant matching, i.e.*, $\mathbf{m}_i^\sigma=\mathbf{m}_i$.

Proof. Let $S\in\mathbf{R}_k$, $S=\langle s_1, s_2, ..., s_k\rangle$ and take $S^\sigma=\langle s_1', s_2', ..., s_k'\rangle$. Let

$\mathbf{m}_i(S)=S\cup\{\bar{s}_y\}$, i.e., $y\equiv i+\sum S \pmod{k+1}$, and
$\mathbf{m}_i(S^\sigma)=S^\sigma\cup\{\bar{s}_z'\}$, i.e., $z\equiv i+\sum S^\sigma \pmod{k+1}$.

If $2k+1\notin S$ then $\sum S^\sigma\equiv\sum S-1 \pmod{k+1}$, so $z\equiv y-1 \pmod{k+1}$. Because $\bar{s}_1=2k+1$ and $\bar{s}_1^\sigma=1$, $\bar{s}_y^\sigma$ is the $y-1$st largest element of $(S^{\mathrm{C}})^\sigma$. Thus, $\bar{s}_y^\sigma=\bar{s}_z'$.

If $2k+1\in S$ then $\sum S^\sigma\equiv\sum S \pmod{k+1}$, so $z\equiv y \pmod{k+1}$. Therefore, $\bar{s}_y^\sigma=\bar{s}_y+1$ is the yth largest element of $(S^{\mathrm{C}})^\sigma$ and $\bar{s}_y^\sigma=\bar{s}_y+1=\bar{s}_z'$. ∎

Next we consider the permutation $\rho=(1, 2k+1)(2, 2k)\cdots(k, k+2)$, which we call a *reflection*. The action of ρ on the $\mathbf{m}_i$'s is also easily described.

THEOREM 4. *For* $i, j=1, ..., k+1$, $\mathbf{m}_i^\rho=\mathbf{m}_j$, *where* $i+j\equiv 1 \pmod{k+1}$.

Proof. First observe that for all $S \in \mathbf{R}_k$, $\sum S^\rho \equiv -(\sum S) \pmod{k+1}$. It is enough to prove that for any $S \in \mathbf{R}_k$, $\mathbf{m}_i(S)^\rho = \mathbf{m}_j(S^\rho)$. Taking $S^\rho = \{s'_1, ..., s'_k\}$, let

$\mathbf{m}_i(S) = S \cup \{\bar{s}_y\}$, where $y \equiv (i + \sum S) \pmod{k+1}$, and
$\mathbf{m}_j(S^\rho) = S^\rho \cup \{\bar{s}'_z\}$, where $z \equiv (j + \sum S^\rho) \pmod{k+1}$.

We must show that $\bar{s}^\rho_y = \bar{s}'_z$. Computing modulo $k+1$, we get

$$\begin{aligned} z \equiv j + \sum S^\rho &\equiv j - \sum S \\ &\equiv (1-i) - \sum S \\ &\equiv 1 - (i + \sum S) \\ &\equiv 1 - y. \end{aligned}$$

Since $\bar{s}_y$ is the yth largest element of S^{C}, $\bar{s}^\rho_y$ is the yth smallest element of $(S^{\mathrm{C}})^\rho = (S^{\mathrm{C}})^\rho$. But then, as $z + y \equiv 1 \pmod{k+1}$, $\bar{s}^\rho_y$ is the zth largest element of $\overline{S^\rho}$. Hence $\bar{s}^\rho_y = \bar{s}'_z$. ∎

Now we show that relations among the matchings $\mathbf{m}_1, ..., \mathbf{m}_{k+1}$ induced by S_{2k+1} are confined to those described in Theorems 2 and 3, except for the special case $k = 2$, where $\mathbf{m}_2$ is also τ-invariant, for $\tau = (1, 3, 2, 5)$. Suppose that $\mathbf{m} = \mathbf{m}^\alpha_i$, for some $\alpha \in S_{2k+1}$ and some $i = 1, ..., k+1$. For $x, y \in [2k+1]$ write

$$x \sim_\alpha y \qquad \text{if} \quad |\alpha^{-1}(x) - \alpha^{-1}(y)| \equiv 1 \pmod{2k+1},$$

meaning that $\alpha^{-1}(x)$ and $\alpha^{-1}(y)$ are consecutive in the cyclic permutation $1, 2, ..., 2k+1$. We show that the relation $\sim_\alpha$ on $[2k+1]$ is determined by $\mathbf{m}$ and so α is determined up to rotation and reflection. Since each $\mathbf{m}_i$ is σ-invariant, $\mathbf{m}$ determines $\mathbf{m}_i$ up to the action of ρ. In particular, we have

THEOREM 5. *For all i, j, $\mathbf{m}^\alpha_i = \mathbf{m}_j$ implies that either $\alpha = \sigma^p$ and $i = j$, or $\alpha = \sigma^p \rho$ and $i + j \equiv 1 \pmod{k+1}$, or $k = 2$, $i = j = 2$, and $\alpha = \tau$.*

Proof. For $k = 1$ or 2, the result follows by inspection. Now assume that $k \geq 3$. Given a matching $\mathbf{m}: \mathbf{R}_k \to \mathbf{R}_{k+1}$ and $z \in [2k+1]$, say that $u \in [2k+1]$ is *special for* the pair $(z, \mathbf{m})$ if, for all $S \in \mathbf{R}_{k-1}$ and all $u, v \notin S$, $\mathbf{m}(S \cup \{u\}) = S \cup \{u, z\}$ and $\mathbf{m}(S \cup \{v\}) = S \cup \{v, z\}$ imply $v = u$.

CLAIM A. *Let $k \geq 3$. For all $i = 1, ..., k+1$ and $u \in [k-1]$, u is not special for $(2k+1, \mathbf{m}_i)$.*

Proof. It suffices to construct $S \in \mathbf{R}_{k-1}$ such that $u, u+k+1, 2k+1 \notin S$ and

$$\sum S \equiv 1-(i+u) \qquad (\text{mod } k+1),$$

since then both $\mathbf{m}_i(S \cup \{u\}) = S \cup \{u, 2k+1\}$ and $\mathbf{m}_i(S \cup \{u+k+1\}) = S \cup \{u+k+1, 2k+1\}$.

If $u > 1$ we set

$$U_r = \{u+1, ..., u+k\} \setminus \{u+r\}, \text{ for } r = 1, ..., k \text{ and}$$
$$U_{k+1} = \{u-1\} \cup \{u+2, ..., u+k-2\} \cup \{u+k\}.$$

Then $u, u+k+1 \notin U_r$, for all r. Also $\sum U_{r+1} \equiv \sum U_r - 1$ (mod $k+1$), for $r = 1, ..., k$. This gives $k+1$ distinct residues modulo $k+1$, so $S = U_r$ works, for some $r = 1, ..., k+1$.

If $u = 1$ we set

$$U_r = \{u+1, ..., u+k\} \setminus \{u+r\}, \text{ for } r = 1, ..., k \text{ and}$$
$$U_{k+1} = \{u+1\} \cup \{u+3, ..., u+k-1\} \cup u+k+2\}$$

Again, u, $u+k+1 \notin U_r$, for all r, and $\sum U_{r+1} \equiv \sum U_r - 1$ (mod $k+1$), for $r = 1, ..., k$. Thus $S = U_r$ works, for some $r = 1, ..., k+1$. Note that we need $k+3 \leqslant 2k$, so $3 \leqslant k$, to define U_{k+1}. ∎

Conversely, we have

CLAIM B. *For all* $i = 1, ..., k+1$ *and* $u \in \{k, k+1\}$, u *is special for* $(2k+1, \mathbf{m}_i)$.

Proof. Suppose $u = k$. Then $\mathbf{m}_i(T \cup \{k\}) = T \cup \{k, 2k+1\}$ means that

$$\sum (T \cup \{k\}) + i \equiv 1 \text{ (mod } k+1).$$

If $\mathbf{m}_i(T \cup \{v\}) = T \cup \{v, 2k+1\}$, then $k \equiv v$ (mod $k+1$). Since $v \neq 2k+1$, $v = k$.

With $u = k+1$, we also obtain $v = k+1$, since no other element of $[2k+1]$ is congruent to $k+1$ modulo $k+1$. ∎

CLAIM C. *Let* $\mathbf{m} = \mathbf{m}_i^\alpha$ *for some* i *and* α. *For all* $x, y \in [2k+1]$, $x \sim_\alpha y$ *if and only if there exists* $z \in [2k+1]$ *such that both* x *and* y *are special for* $(z, \mathbf{m})$.

Proof. Assume $x \sim_\alpha y$. Then without loss of generality, there exists $u \in [2k+1]$ such that $u^\alpha = x$ and $(u+1)^\alpha = y$. Let $t = u+k+1$ (mod $2k+1$). We first show that both u and $u+1$ are special for $(t, \mathbf{m}_i)$. Suppose that

$$\mathbf{m}_i(S \cup \{u\}) = S \cup \{u, t\} \qquad \text{and} \qquad \mathbf{m}_i(S \cup \{v\}) = S \cup \{v, t\}. \tag{3}$$

By the σ-invariance of $\mathbf{m}_i$, with $\sigma^j(u)=k$ and $T=S^{\sigma^j}$,

$$\mathbf{m}_i(T \cup \{k\}) = T \cup \{k, 2k+1\}$$

and (4)

$$\mathbf{m}_i(T \cup \{\sigma^j(v)\}) = T \cup \{\sigma^j(v), 2k+1\}.$$

Applying Claim B, we see that $\sigma^j(v)=k$, and so $v=u$. Thus, u is special for $(t, \mathbf{m}_i)$.

Now suppose that (3) holds with u replaced by $u+1$. Let $\sigma^j(u+1) = k+1$ and $T=S^{\sigma^j}$. Then (4) holds with k replaced by $k+1$. Again, Claim B yields $\sigma^j(v)=k+1$ and $v=u+1$. Thus, $u+1$ is special for $(t, \mathbf{m}_i)$.

By applying α, we have that $x=u^\alpha$ and $y=(u+1)^\alpha$ are special for $(t^\alpha, \mathbf{m}_i^\alpha)$.

Conversely, assume that there exists z such that x and y are special for $(z, \mathbf{m})$. Let $x=u^\alpha$, $y=v^\alpha$, $z=w^\alpha$. Then both u and v are special for $(w, \mathbf{m}_i)$. Choosing l such that $w^{\sigma^l}=2k+1$, and using σ-invariance, both u^{σ^l} and v^{σ^l} are special for $(2k+1, \mathbf{m}_i)$. By Claim A, u^{σ^l}, $v^{\sigma^l} \in \{k, k+1\}$, so $|u-v| \equiv 1 \pmod{2k+1}$. This means that $x \sim_\alpha y$. ∎

Claim D. *If* $\sim_\alpha = \sim_{\text{id}}$ *then* $\alpha = \sigma^p \rho^q$.

Now suppose $\mathbf{m}_i^\alpha = \mathbf{m}_j$. By Claim C, $x \sim_\alpha y$ iff there exists z such that both x and y are special for $(z, \mathbf{m}_i^\alpha) = (z, \mathbf{m}_j)$ *iff* $x \sim_{\text{id}} y$. Thus by Claim D, $\alpha = \sigma^p \rho^q$. If q is even, then $\alpha = \sigma^p$, so $i=j$, by Theorems 1 and 3. If q is odd then $\alpha = \sigma^p \rho$ and $i+j \equiv 1 \pmod{k+1}$ by Theorems 1, 3, and 4. ∎

Consider the orbit $\mathbf{M}_i = \{\mathbf{a}_i^\alpha : \alpha \in S_{2k+1}\}$ of $\mathbf{m}_i$. By Theorem 5, if $2i \neq k+2$, then the stabilizer of $\mathbf{m}_i$ is the cyclic group Z_{2k+1} and thus $|\mathbf{M}_i| = (2k)!$; otherwise the stabilizer of $\mathbf{m}_i$ is the dihedral group D_{2k+1} and thus $|\mathbf{M}_i| = (2k)!/2$. Also for $1 \leqslant i < j \leqslant \lfloor k/2 \rfloor + 1$, $\mathbf{M}_i \cap \mathbf{M}_j = \varnothing$. Thus it makes sense to call any matching in $\mathbf{M}_i$ an i-modular matching.

We show that for $k \geqslant 3$ the modular matchings $\mathbf{m}_i$, $i=1, ..., k+1$, are different from the lexical matchings $\mathbf{l}_0, \mathbf{l}_1, ..., \mathbf{l}_k$ obtained in [KT]. Let $\mathbf{L}_i = \{\mathbf{l}_i^\alpha : \alpha \in S_{2k+1}\}$ be the orbit of $\mathbf{l}_i$. We shall need some terminology from [KT]. For a matching $\mathbf{m}$ of $\mathbf{B}_k$ and $x \in [2k+1]$, we call $S \in \mathbf{L}_k$ an *x-vertex of* $\mathbf{m}$ if $\mathbf{m}(S) = S \cup \{x\}$. Call $F \subseteq [2k+1]$ an *x-filter of* $\mathbf{m}$ if $F \cap S \neq \varnothing$ for every x-vertex S of $\mathbf{m}$. In [KT] it is shown that $\mathbf{L}_i = \mathbf{L}_j$ iff $i+j=k$. By Theorem 5, we need only prove that $\mathbf{m}_i \notin \mathbf{L}_j$ for $i=1, ..., \lfloor k/2 \rfloor + 1$ and $j=0, ..., \lfloor k/2 \rfloor$.

Lemma 6. *For* $k>4$ *and* $j=0, 1, ..., \lfloor k/2 \rfloor$ *there are three distinct* $(2k+1)$*-vertices* S_1, S_2, S_3 *of* $\mathbf{l}_j$ *such that* $|S_1 \cap S_2 \cap S_3| = k-1$. *Moreover,* $S_p \neq S_q^{\sigma^l}$, *when* $p \neq q$. *The same holds for* $k=4$ *and* $j=0, 1$.

Proof. Let

$$S_1 = \{k-j+1, ..., 2k-j\},$$
$$S_2 = (S_1 - \{k-j+1\}) \cup \{k-j\},$$
$$S_3 = (S_2 - \{k-j\}) \cup \{k-j-1\}.$$

Then the fact that $\mathbf{l}_i(S_p) = S_p \cup \{2k+1\}$ for $p = 1, 2, 3$, follows directly from the definition of $\mathbf{l}_i$ in [KT], except in the case where $k=4$ and $j=2$. Also $S_1 \cap S_2 \cap S_3 = \{k-j+2, k-j+3, ..., 2k-j\}$ and $S_p \neq S_q^{\sigma^l}$. ▮

THEOREM 7. *For $k \geqslant 4$ and all i and j, $\mathbf{m}_i \notin \mathbf{L}_j$.*

Proof. First assume that $k > 4$ or $k = 4$ and $j \neq 2$. Since S_0, S_1, S_2, as given in Lemma 6, belong to distinct σ-classes in $\mathbf{R}_k$, for $\mathbf{m}_i$ to belong to $\mathbf{L}_j$ would require that there exist three $(2k+1)$-vertices T_1, T_2, T_3 in distinct σ-classes such that $|T_1 \cap T_2 \cap T_3| = k-1$. By definition of $\mathbf{m}_i$ this would mean that for $p = 1, 2, 3$,

$$\sum T_p + i \equiv 1 \qquad (\text{mod } k+1).$$

As $|T_1 \cap T_2 \cap T_3| = k-1$ and $|T_p| = k$, this requires $x_1, x_2, x_3 \notin [2k]$ distinct yet pairwise congruent modulo $k+1$, which is nonsense.

Now suppose $k=4$. We must show that $\mathbf{m}_i \notin \mathbf{L}_2$ for $i = 1, 2, 3$. Here it is convenient to define for each σ-invariant matching $\mathbf{m}: \mathbf{R}_k \to \mathbf{R}_{k+1}$ the distribution vector $\bar{d}(\mathbf{m})$ by

$$\bar{d}(\mathbf{m})_i = |\{S \in \mathbf{R}_k : i \in S \text{ and } \mathbf{m}(S) = S \cup \{2k+1\}\}|.$$

Direct computation shows that

$$\bar{d}(\mathbf{m}_1) = (5, 7, 7, 7, 7, 7, 7, 9),$$
$$\bar{d}(\mathbf{m}_2) = (6, 7, 7, 7, 7, 7, 7, 8),$$
$$\bar{d}(\mathbf{m}_3) = (7, 7, 7, 7, 7, 7, 7, 7),$$

while

$$\bar{d}(\mathbf{l}_2) = (7, 7, 7, 7, 7, 7, 7, 7).$$

So the only matching $\mathbf{m}_i$ whose distribution vector is (a permutation of) $\mathbf{l}_2$'s is $\mathbf{m}_3$. Examination of the $\binom{8}{3} = 56$ 3-subsets of [8] shows that the 9-vertices of $\mathbf{l}_2$ contain 10 9-filters of size 3, while the 9-vertices of $\mathbf{m}_3$ contain 11 9-filters of size 3. Hence, $\mathbf{m}_3 \notin \mathbf{L}_2$. ▮

For $k \leqslant 3$ we have the following special cases.

Proposition 8. *If* $k=1$, *then* $\mathbf{m}_1=\mathbf{l}_0$ *and* $\mathbf{m}_2=\mathbf{l}_1$; *if* $k=2$, *then* $\mathbf{L}_2=\mathbf{L}_0=\mathbf{M}_3=\mathbf{M}_1\neq\mathbf{M}_2=\mathbf{L}_1$; *and if* $k=3$, *then* $\mathbf{M}_1=\mathbf{M}_4=\mathbf{L}_1=\mathbf{L}_2$ *and* $\mathbf{M}_2=\mathbf{M}_3\neq\mathbf{L}_0=\mathbf{L}_3$.

Proof. The case $k=1$ is clear by inspection. Suppose $k=2$. Using Theorem 4 and its analog in [KT] for lexical matchings, we have $\mathbf{M}_1=\mathbf{M}_3$ and $\mathbf{L}_0=\mathbf{L}_2$, while direct calculation shows $\mathbf{l}_0^{\tau}=\mathbf{m}_3$, with $\tau=(1, 3, 2, 5)$, as before. Moreover $\mathbf{l}_1=\mathbf{m}_2$, and $\mathbf{L}_0\neq\mathbf{L}_1$ by [KT], finishing the case. Now suppose $k=3$. By inspection we have $\mathbf{m}_4=\mathbf{l}_1^{(1,4,2)(3,5,6)}$. To see that $\mathbf{M}_3\neq L_0$, we compute the sizes of the smallest x-filters in $\mathbf{m}_3$ and $\mathbf{l}_0$, which are 2 and 1. As in the case above, Theorem 4 and the corresponding result in [KT] establish the rest. ∎

4. Questions

Given the unexpected identical behavior of the lexical and modular factorizations under the automorphisms of $\mathbf{B}_k$, one is led naturally to look for an explanation. Could it be that every factorization of $\mathbf{B}_k$ into σ-invariant matchings behaves the same way? In particular, if $\mathbf{m}$ is a matching in such a factorization, is $\mathbf{m}^{\rho}$ also in the factorization? If $\mathbf{m}$ is a matching of $\mathbf{B}_k$, which is both σ-invariant and ρ-invariant, is $f(S)=(\mathbf{m}(S))^C$ a matching in $\mathbf{O}_k$?

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