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COLORING ORDERED SETS TO AVOID MONOCHROMATIC MAXIMAL CHAINS

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ABSTRACT. This paper is devoted to settling the following problem on (infinite, partially) ordered sets: *Is there always a partition (2-coloring) of an ordered set X so that all nontrivial maximal chains of X meet both classes (receive both colors)?* We show this is true for all countable ordered sets and provide counterexamples of cardinality $\aleph_3$. Variants of the problem are also considered and open problems specified.

0. **Introduction.** It is obvious that if a finite ordered set contains no one-element maximal chains (isolated points) then the set can be 2-colored so that every maximal chain receives both colors—let the maximal elements be blue and the rest, red. More generally, the *top half* of a finite ordered set can be made blue, and the bottom, red: every maximal chain makes an appearance in each half. But in the infinite case, can we halve in a similar manner and guarantee that every maximal chain intersects both halves?

QUESTION 1. Given an ordered set X without isolated points, is there a 2-coloring of the elements of X by blue and red so that each maximal chain receives both colors and so that the blue set is a final segment and the red, an initial segment of X?

In Section 1 we show that the answer is yes for countable orders but that in general, there fails to be such a partition. We can ask for somewhat less.

QUESTION 2. Given X, without isolated points, is there a 2-coloring so that each maximal chain receives both colors?

In Section 2 we consider examples of partial orders for which the answer to Question 2 is positive, including the counterexamples to the first question. For instance a finite product of scattered chains admits a good 2-coloring. It is not the case that all scattered orders have good 2-colorings (Example 2 of Section 4 settles this). However, we do not know whether all finite products of arbitrary chains admit such colorings.

Section 3 is comprised of some small examples (of size the continuum) showing that, for them, it is at least consistent that good 2-colorings always exist.

In Section 4, we settle Question 2 in the negative by providing examples of orders for which all 2-colorings have monochromatic maximal chains. Indeed we show that for

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each cardinal μ there is an order P for which each μ-coloring leaves a monochromatic maximal chain. These examples are, in fact, products of two orders which admit good colorings. The counterexamples in the section are uncomfortably large because they depend on applications of the partition calculus. These results, and the theorem in the countable case, leave as the most interesting open question whether there are counterexamples of cardinality the continuum.

To avoid constant reference to isolated points, we shall assume that all ordered sets under discussion have none. As used above, the term *good coloring* refers to a coloring of the elements of an ordered set that leaves no maximal chain monochromatic.

1. **The countable case.** Here is the main result of the section.

THEOREM 1. *If X is a countable ordered set then X has a good 2-coloring with one color class an initial segment and the other, a final segment.*

PROOF. Let $X = \{x_n \mid n < \omega\}$. We construct two increasing sequences of subsets of X, B_n and R_n so that for each n

B_n is a final segment,
R_n is an initial segment,
$B_n \cap R_n = \emptyset$, and
$x_n \in B_n \cup R_n$.

To initiate the sequences set $B_0 = \max(X)$ and $R_0 = \min(X)$ and remember that no isolated points appear in X. Let $[x, \longrightarrow)$ be the closed final segment $\{y \in X \mid x \leq y\}$ and let $(\longleftarrow, x) = \{y \in X \mid y < x\}$, the open initial segment generated by x.

With B_n and R_n defined, for $n \geq 0$ set

$$B_{n+1} = B_n \cup ([x_n, \longrightarrow) - R_n), \text{ and}$$
$$R_{n+1} = R_n \cup ((\longleftarrow, x_n) - B_n).$$

Notice that B_{n+1} can replace B_n in the definition of R_{n+1} without change, so $B_{n+1} \cap R_{n+1} = \emptyset$.

It is evident that the conditions above are satisfied. With $B = \bigcup B_n$ and $R = \bigcup R_n$, B is an initial segment of X, R is a final segment and $X = B \cup R$. We now argue that there are no maximal chains of X contained in B (blue chains) and leave out the (essentially) dual argument for red chains.

Let C be a maximal chain of X. If C has a minimum element then $C \cap R \neq \emptyset$, so we assume it lacks a minimum. Let n be least such that $x_n \in C$. For $m < n$, x_m is incomparable with some element of C. Therefore the final segment C_m of those elements of C which dominate x_m is not all of C. As there are only finitely many such segments C_m, their union is not all of C. The initial segment of C not in their union is contained in R_{n+1}. ■

We understand that Igor Kriz has found this result independently. In Section 3, this argument will be generalized to cover some uncountable orders satisfying further conditions.

This countable result cannot be extended to provide a positive answer to Question 1 in general. Here is a nice example Bill Sands obtained from two familiar chains.

Let ω^d denote the dual of ω, let ω_1 be the first uncountable ordinal and let $P = \omega^d \times \omega_1$, where this is the usual coordinate-wise product of relations. Suppose that P has been partitioned into an initial segment R and a final segment B. For each $n \in \omega^d$ let α_n be the least member of ω_1 such that $(n, \alpha_n) \in B$. For each n this α_n exists—just consider the maximal chain $\{(m, 0) \mid m < n \text{ in } \omega^d\} \cup \{(n, \alpha) \mid \alpha < \omega_1\}$. Let α be the supremum of $\{\alpha_n\}$ in ω_1. It is now easy to see that B contains a maximal chain:

$$\{(m, \alpha) \mid m \in \omega^d\} \cup \{(0, \beta) \mid \alpha \leq \beta\}.$$

Of course, a good 2-coloring of this example is readily constructed using parity. We exploit this for a variety of orders in the next section.

2. **Positive results based on alternating colorings.** Recall that an ordered set X is *scattered* if the rational chain cannot be embedded in X.

The main result of the section is

THEOREM 2. *A finite product of scattered chains admits a good 2-coloring.*

The proof of Theorem 2 follows easily from these observations.

PROPOSITION 1. *A scattered chain admits a 2-coloring so that consecutive elements receive different colors.*

PROOF. Let C be a scattered chain. Call elements x and y equivalent if there are only finitely many elements of C between them. (This is the usual finite condensation). Pick an element in each equivalence class; color each element of the class according to the parity of its distance from the chosen point. ■

A 2-coloring of an ordered set X is called *alternating* if whenever x is covered by y in X, x and y receive different colors.

PROPOSITION 2. *If P and Q are ordered sets which each have alternating 2-colorings then $P \times Q$ has an alternating 2-coloring.*

PROOF. Let $c: P \longrightarrow 2$ and $d: Q \longrightarrow 2$ be alternating 2-colorings. Define $f: P \times Q \longrightarrow 2$ by

$$f((p, q)) = \begin{cases} 0 & \text{if } c(p) = d(q) \\ 1 & \text{otherwise} \end{cases}$$

The coloring f is alternating because a covering pair in a product occurs exactly when there are equal entries in one coordinate and covering in the other. ■

PROPOSITION 3. *A finite product of scattered orders is scattered.*

This is well known.

PROOF OF THEOREM 2. Let P be a finite product of scattered chains. Propositions 1 and 2 yield an alternating 2-coloring c of P. We claim that c is a good coloring. To see this let C be a maximal chain of P and let $x < y$ in C. Now C is scattered so there must be a $x \leq x' < y' \leq y$ such that y' covers x' in P. ■

The method of proof of Theorem 2 breaks down for an arbitrary scattered ordered set X as the graph defined on the elements of X with edges the covering pairs in X fails to be bipartite. (Again, Example 2 below provides a scattered order with no good 2-coloring). This result does suggest another problem, at this point not fully resolved.

PROBLEM 1. Does every finite product of chains admit a good 2-coloring?

Problem 1 may well prove difficult: the examples of Section 4 are of the form $A \times B$ where both A and B have good 2-colorings by final and initial segments.

When looking for natural examples of orders not easily 2-colored and, hence, without maximal or minimal elements, one is led to consider this lattice. Let X be an infinite set and let $\mathbf{X}$ denote the set of subsets of X which are both infinite and coinfinite, with the containment order. Define an equivalence relation Δ on $\mathbf{X}$ by $A \Delta B$ if the symmetric difference of A and B is finite. Now 2-color $\mathbf{X}$ by fixing an element from each Δ-class and letting sets whose symmetric difference with the fixed set is odd be colored 1 and the others, 0. We need only see that every maximal chain contains two sets differing by exactly one element. Let $\mathbf{C}$ be a maximal chain in $\mathbf{X}$ and let

$$A = \bigcup \mathbf{C} \text{ and } B = \bigcap \mathbf{C}.$$

Since $\mathbf{C}$ is infinite, so is $A - B$. Let $x \in A - B$. Let

$$Y = \bigcup \{S \in \mathbf{C} \mid x \notin S\},$$
$$Z = \bigcap \{S \in \mathbf{C} \mid x \in S\}.$$

Since $\mathbf{C}$ is a maximal chain, both Y and Z are in $\mathbf{C}$. Moreover, Y and Z must differ in precisely the element x.

Parity can be used to obtain good 2-colorings in some dense ordered sets. We illustrate this with an infinite family of examples based on the η_α-chains.

Hausdorff (cf. [Ro]) defined these chains: for an ordinal α, a chain A is an η_α-chain if for any two subsets X and Y of A of cardinality less than $\aleph_\alpha$ and with each element of X less than each of Y, there exists $a \in A$ such that $x < a < y$ for all x in X and y in Y. To construct examples, let $\aleph_\alpha$ be a regular cardinal and let Q_α denote the set of all ω_α -sequences $\mathbf{a}$ of 0 and 1 for which there exists δ such that $a_\delta = 1$ and $a_\beta = 0$ for all $\beta > \delta$. With the lexicographic order Q_α is an η_α-chain (cf. [Ro]). Here is the property of Q_α which is important to us: for all $\mathbf{a}$ in Q_α, the cofinality (coinitiality) of $(\longleftarrow, \mathbf{a})$ (respectively, $(\mathbf{a}, \longrightarrow)$) is ω_α.

With $\aleph_\alpha$ and $\aleph_\beta$ distinct regular cardinals, let $P_{\alpha\beta} = Q_\alpha \times Q_\beta$ have the coordinatewise order.

To obtain a good 2-coloring f of $P_{\alpha\beta}$, let $(\mathbf{a}, \mathbf{b}) \in P_{\alpha\beta}$, suppose that δ is maximum in α with $a_\delta = 1$ and ε is maximum in β with $b_\varepsilon = 1$; define

$$f((\mathbf{a}, \mathbf{b})) = \begin{cases} 0 & \text{if } \delta \text{ and } \varepsilon \text{ have the same parity} \\ 1 & \text{otherwise.} \end{cases}$$

(Every ordinal is the sum of a limit and an integer; the ordinal's parity is that of the integer). Let C be a maximal chain in $P_{\alpha\beta}$ and let $(\mathbf{a},\mathbf{b})$ be in C. With $C_i = \pi_i(C)$, for $i = 1, 2$, it is easy to see that the maximality of C forces

$$\inf\big(C_1 \cap (\mathbf{a}, \rightarrow)\big) = \mathbf{a} = \sup\big(C_1 \cap (\leftarrow, \mathbf{a})\big), \text{ and}$$
$$\inf\big(C_2 \cap (\mathbf{b}, \rightarrow)\big) = \mathbf{b} = \sup\big(C_2 \cap (\leftarrow, \mathbf{b})\big).$$

We claim that there is $\mathbf{a}' > \mathbf{a}$ such that $(\mathbf{a}',\mathbf{b})$ is in C or that there is $\mathbf{b}' > \mathbf{b}$ such that $(\mathbf{a},\mathbf{b}')$ is in C (and dually). Otherwise, the projections π_1 and π_2 are both $1-1$ on an initial segment of $C \cap \big((\mathbf{a},\mathbf{b}), \rightarrow\big)$. This would cause the final segments $(\mathbf{a}, \rightarrow)$ and $(\mathbf{b}, \rightarrow)$ to have the same coinitiality.

To finish, it suffices to consider the case that there is some $\mathbf{a}' > \mathbf{a}$ such that $(\mathbf{a}',\mathbf{b})$ is in C. Notice that C contains every element of the form $(\mathbf{a}'',\mathbf{b})$ where $\mathbf{a} < \mathbf{a}'' < \mathbf{a}'$. This easily gives two elements of C with distinct colors. Indeed, this coloring is such that every proper interval of every maximal chain of $P_{\alpha\beta}$ receives both colors. This observation raised the question of whether Question 2 could be settled affirmatively with "dense" colorings. The answer is yes for arbitrary chains but fails in $P = P_{00}$, the product of two rational chains.

PROPOSITION 4. *Every chain C admits a 2-coloring such that each nontrivial interval receives both colors.*

PROOF. Assume first that C is a dense chain. Let $\{X_\alpha \mid \alpha < \kappa\}$ be an enumeration of the nonempty open intervals of C. We define a coloring of C with red and blue as follows. At step α:

- if I_α has no element of color red (blue) choose a point x_α (respectively, y_α) in I_α and color it red (respectively, blue);
- if I_α has no colored points choose distinct elements x_α and y_α of the interval and color them red and blue, respectively;
- if I_α has points of both colors, do nothing.

Color all uncolored points arbitrarily. We claim that each interval receives both colors. If not, there is a least α so that I_α is monochromatic, say all its points are red. By definition of the coloring, I_α has an enumeration $\{x_{\beta_\gamma} \mid \beta_\gamma < \alpha, \gamma < |I_\alpha|\}$. There is some γ such that x_{β_γ} is between x_{β_0} and x_{β_1}. As x_{β_γ} was colored red and $\beta_0, \beta_1 < \beta_\gamma$, the interval I_{β_γ} is properly between x_{β_0} and x_{β_1} and, so, is properly contained in I_α. But then the guaranteed blue point of I_{β_γ} is a point of I_α.

Now, let C be any chain. Let $\sim$ denote the finite condensation relation on C: $x \sim y$ if the interval with endpoints x and y is finite. By the argument above the dense chain $C/\!\sim$ has a coloring so that all nontrivial intervals are 2-colored. To endow C with such a coloring, choose an arbitrary element from each $\sim$-class, give it the color of the class in $C/\!\sim$ and color alternate elements of the class with alternate colors, starting from the representative. ■

The argument above was suggested by Eric Milner and is more direct than the original.

PROPOSITION 5. *Let c be a 2-coloring of P, the product of two rational chains, such that each proper subinterval of P receives both colors. Then P has monochromatic maximal chains in each color.*

PROOF. Call the colors red and blue. We shall obtain a maximal chain C consisting of red elements. Enumerate the blue elements $b_n (n < \omega)$ of P. Construct finite chains C_n with these properties:

$$\text{if } (x, y) < (u, v) \text{ in } C_n \text{ then } x < u \text{ and } y < v,$$
$$C_n \subseteq C_{n+1} \text{ for all } n, \text{ and}$$
$$b_n \text{ is incomparable to some element of } C_n.$$

Once we manage this, any chain C which is maximal and contains $\bigcup C_n$ will have all elements red.

This observation is required for the construction. Given an interval $I = [(x, y), (x_1, y_1)]$ where $x < x_1$ and $y < y_1$, and an element (p, q) in I which is not either endpoint, there is a subinterval J of I such that J has the same form as I and (p, q) is incomparable to each element of J. (There are several cases—all straightforward and left to the reader).

Let $C_0 = \{r\}$ where r is any red element incomparable to b_0. If C_n has been defined, set $C_{n+1} = C_n$ unless b_{n+1} is comparable to all elements of C_n. Suppose that (x, y) and (x_1, y_1) are consecutive elements of C_n with b_{n+1} in the interval I which they determine. Apply the observation to find a red element $\{r'\}$ of the subinterval J. Add this element to C_n to form C_{n+1}. (see Figure 1). ∎

The analogous proposition with $P_{\alpha\alpha}$ in place of $P = P_{00}$ is also true. The proof is essentially the same.

3. A natural example/counterexample?.

The examples of Section 2 were, for the most part, rich in covering pairs. A natural example of a structure without maximals, minimals and covers is obtained by taking the example $\mathbf{X}$ (Section 2) and factoring by the equivalence relation Δ. Let $\mathbf{X}_\Delta$ denote this ordered set. This structure has been studied for its applications in set-theoretic topology—for instance, see [Ju],[Ru]. We have not been able to decide if it possesses a good 2-coloring without appeal to extensions of ZFC. Our investigations led, however, to the following generalization of Theorem 1.

As usual, for an ordered set P, $\mathrm{cf}(P)$ is the *cofinality* of P, that is, the minimum cardinality of cofinal subsets of P. The *coinitiality* of P, $\mathrm{ci}(P)$, is defined dually.

THEOREM 3. *Let P be an ordered set of cardinality κ. Suppose that for each maximal chain C of P, $\mathrm{cf}(C), \mathrm{ci}(C) \in \{1, \kappa\}$. Then P admits a good 2-coloring with one color class an initial segment and the other, a final segment of P.*

PROOF. As in the proof of Theorem 1, we fix an enumeration of P—$\{x_\alpha \mid \alpha < \kappa\}$—and construct increasing sequences B_α and R_α for $\alpha < \kappa$. We begin the sequences

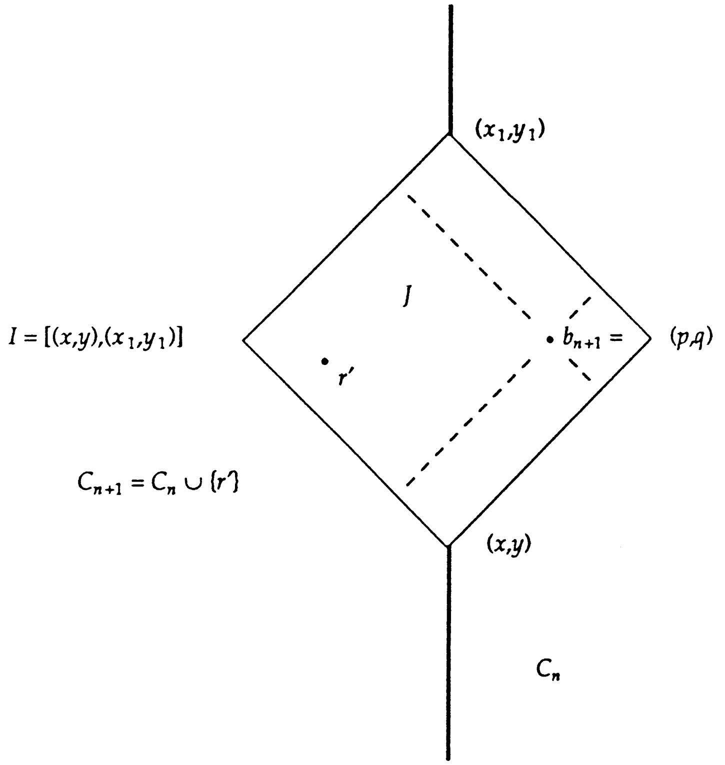

Figure 1. Construction of C_{n+1} in the proof of Proposition 5.

with $B_{-1} = \max(P)$ and $R_{-1} = \min(P)$. For $\alpha < \kappa$, let

$$B_\alpha = \bigcup_{\beta<\alpha} B_\beta \cup \Big([x_\alpha, \rightarrow) - \bigcup_{\beta<\alpha} R_\beta\Big),$$

$$R_\alpha = \bigcup_{\beta<\alpha} R_\beta \cup \Big((\leftarrow, x_\alpha) - \bigcup_{\beta\leq\alpha} B_\beta\Big).$$

We define B and R as the respective unions of these two sequences. The argument that there is no maximal chain C contained in B is essentially the same as that in the proof of Theorem 1. The only substantial modification is this: take α least such that $x_\alpha \in C$ and define C_β to be the set of elements of C which dominate x_β for $\beta < \alpha$. Since $\mathrm{ci}(C) = \kappa$ and $\alpha < \kappa$, the union of the final segments $C_\beta (\beta < \alpha)$ cannot be all of C. The initial segment of C not contained in this union is colored red at stage α. ∎

We wish to apply this result to the ordered set $\mathbf{X}_\Delta$ where $X = \omega$. It is known that no maximal chain in $\mathbf{X}_\Delta$ has countable cofinality (or coinitiality) [Ru]. For completeness, we furnish an argument now.

Consider a strictly increasing sequence of sets $(A_n \mid n < \omega)$ in ω such that for $n < m$,

(1) $A_m - A_n$ is infinite, and

(2) $A_n - A_m$ is finite.

(The sets A_n are representatives of a strictly increasing sequence in $\mathbf{X}_\Delta$). We construct a set A such that

(1) $\bar{A}$ is infinite, and

(2) for $n < \omega$, $A_n - A$ is finite.

Thus the Δ-equivalence class containing A is an upper bound of the sequence in $\mathbf{X}_\Delta$. We actually enumerate $\bar{A}$ as follows: let

$$x_n = \min \overline{(A_1 \cup A_2 \cup \cdots \cup A_n \cup \{x_m \mid m < n\})}.$$

The choice of x_n is always possible since only finitely many elements of A_i do not belong to A_n for $i < n$, while A_n is coinfinite. On the other hand, the elements of $A_n - A$ must be amongst $x_1, \dots, x_{n-1}$.

Theorem 3 can now be applied to $\mathbf{X}_\Delta$ whenever all maximal chains have cofinality and coinitiality $2^{\aleph_0}$. This is the case if the Continuum Hypothesis holds or if Martin's Axiom is invoked [Ru].

4. **The counterexamples.** The examples of this section provide a negative answer to Question 2. They depend upon results from the partition calculus, in particular the Erdős-Rado theorem, and properties of interval orders. Specifically, if the pairs from ω are finitely colored, Ramsey's theorem guarantees an infinite homogeneous subset. This can be used to easily construct a monochromatic chain in the interval order which is maximal in the segment above its least element. Igor Kriz suggested that this idea and a Ramsey cardinal could be used to produce an example. However, partition relations themselves are not sufficient to obtain monochromatic maximal chains (as we shall see). Some subtle adjustments are required to overcome this deficiency, and, as a result, the examples become rather large.

This notation will prove useful in the constructions to follow. Let $(S_i, <_i)(i = 0, 1)$ be given where $<_i$ is a transitive, antisymmetric relation on S_i (as when $<_i$ is a strict order or a reflexive one). Denote by $S_0 \otimes S_1$ the partially ordered set obtained as the reflexive closure of the relational product $S_0 \times S_1$. When S_0 is a reflexive order and S_1 is a strict order, this follows:

$$\text{(0)} \qquad \begin{aligned} &x \leq_0 y \text{ and } u \text{ is covered by } v \text{ in } S_1 \\ &\qquad \text{imply } (x, y) \text{ is covered by } (y, v) \text{ in } S_0 \otimes S_1. \end{aligned}$$

Given a chain C, the interval order $I(C)$ has underlying set $[C]^2$, the set of pairs $x < y$ of C, with relation $(x, y) < (z, w)$ if $y \leq z$ in C.

If κ is a cardinal, denote by κ^+ the successor cardinal. Also, define a sequence $\exp_\alpha = \exp_\alpha(\kappa)$ by $\exp_0 = \kappa$, $\exp_{\alpha+1} = 2^{\exp_\alpha}$, and $\exp_\delta = \sup(\exp_\alpha \mid \alpha < \delta)$, for δ a limit. With this notation, the form of the Erdős-Rado theorem (cf. [EH]) we apply is

$$\exp_1(\kappa)^+ \longrightarrow (\kappa)^2_\kappa.$$

This means that however the pairs of $\exp_1^+$ are κ-colored there is a subset of size κ all of whose pairs receive the same color.

Here is the observation that partition results are not sufficient to obtain a counterexample.

PROPOSITION 6. *Let κ be an uncountable cardinal. There is a 2-coloring of $I(\kappa)$ such that in every maximal chain of $I(\kappa)$ both colors occur cofinally.*

PROOF. Let $C = \{(x_\alpha, y_\alpha)\}$ be a chain in $I(\kappa)$. Note that C is maximal if and only if the set $\{x_\alpha\}$ of left endpoints of members of C contains 0 and for x_α different from 0, x_α is the supremum of $\{y_\beta \mid y_\beta \leq x_\alpha\}$. In particular, when C is maximal, the set of left endpoints forms a closed, unbounded set (club) in κ.

Since κ is uncountable, there exist disjoint *stationary* sets, that is, sets S and T, each of which has nonempty intersection with every club. A coloring f with the desired properties can be obtained as follows: $f(x, y) = 0$ or 1 according as $x \in S$ or is not. ∎

We have two objectives remaining: to provide an example answering Question 2 negatively, and to address μ-colorings for arbitrary cardinals μ. Although the lemmas required to do both are similar, we choose to construct the first example, for $\mu = 2$, separately. There are two reasons for this: the first example is simpler to understand, and it has smaller size than would be obtained by applying the general result in the special case that $\mu = 2$.

LEMMA 1. *Let κ be an infinite cardinal. There is an ordered set P of cardinality 2^κ such that for each κ-coloring of P there is a strictly increasing, unbounded sequence p_n which is monochromatic.*

PROOF. Let P be the set of functions from κ^+ to ω with support of cardinality at most κ. For $f \in P$, let supp(f) be the least ordinal $\alpha < \kappa^+$ such that $f(\beta) = 0$ for $\beta \geq \alpha$. Endow P with the coordinatewise order. It is easy to see that P has cardinality $\kappa^+ \cdot 2^\kappa = 2^\kappa$. To see that P has the desired property fix a κ-coloring c of P. Define an increasing sequence of elements of P, $g_\mu (\mu < \kappa)$, such that for $\mu < \nu$, $g_\nu = g_\mu$ on supp(g_μ). Let c_0 be the least color occurring among all elements of P. Suppose there is a sequence of function $p_n (n < \omega)$ satisfying

$$c(p_n) = c_0 \text{ for all } n, \text{ and}$$
$$\alpha < \operatorname{supp}(p_n) \text{ implies } p_n(\alpha) < p_{n+1}(\alpha).$$

Then the sequence p_n has the desired property. Thus, we may suppose that there is a function p_0 such that $c(p_0) = c_0$ and whenever $q \in P$ and $p_0(\alpha) < q(\alpha)$ for $\alpha <$ supp(p_0), $c(q)$ is not c_0. Fix such a p_0 and let g_0 be defined by supp(g_0) $=$ supp(p_0) and for $\alpha <$ supp(p_0), $g_0(\alpha) = p_0(\alpha) + 1$. Now, let $\lambda > 0$ be given and let

$$\hat{g}_\lambda = \sup(g_\beta \mid \beta < \lambda),$$

meaning that $\hat{g}_\lambda$ extends g_β on the common support. Let c_λ be the least color occurring for functions which agree with $\hat{g}_\lambda$ on its support. Notice that $c_\lambda \geq \lambda$. As with c_0, we may assume there is a function p_λ such that

(1) $c(p_0) = c_\lambda$,
(2) for $\alpha < \operatorname{supp}(\hat{g}_\lambda), \hat{g}_\lambda(\alpha) = p_\lambda(\alpha)$,
(3) for all q satisfying $q(\alpha) = p_\lambda(\alpha)$ for $\alpha < \operatorname{supp}(\hat{g}_\lambda)$ and $q(\alpha) > p_\lambda(\alpha)$ for $\operatorname{supp}(\hat{g}_\lambda) \le \alpha < \operatorname{supp}(p_\lambda)$, we have $c(q) \ne c_\lambda$.

We now define g_λ so that g_λ agrees with $\hat{g}_\lambda$ on the support of the latter, and so that $g_\lambda(\alpha) = p_\lambda(\alpha) + 1$ on the remainder of the support of p_λ.

Let $g = \sup(g_\lambda \mid \lambda < \kappa)$. Notice that $g \in P$ and that there is no color for g since $c(g) = \lambda$ contradicts the facts that $\lambda \le c_\lambda$ and g agrees with each g_β on the latter's support. This contradiction completes the proof. ■

LEMMA 2. *Let S be a set of cardinality κ and let P be as in Lemma 1. Let c be an arbitrary κ-coloring of the pairs in $P \times S$. Then there is $p \in P$ such that for each $s \in S$, there exists a strictly increasing, unbounded sequence p_n in P such that $p_0 = p$ and $\{(p_n, s) \mid n < \omega\}$ is monochromatic.*

PROOF. Suppose otherwise for some coloring c. For each $p \in P$, let $d(p) = (s, c(p, s))$ for a witness s to the nonexistence of the desired sequence. Then d induces a κ-coloring of P. Applying Lemma 1, we obtain a strictly increasing, unbounded d-monochromatic sequence p_n. This contradicts the choice of s for $p = p_0$. ■

We are now in a position to produce a counterexample for 2-colorings, indeed, for all finite μ.

EXAMPLE 1. Let P be as in Lemma 1 for $\kappa = \aleph_0$. Let

$$Q = \big(I(\omega^d) \times P\big) \otimes I\big(\exp_2(\aleph_0)^+\big).$$

This ordered set is depicted in Figure 2. The elements of Q are to be understood as rectangles labelled by pairs from $I(\omega^d)$ and $I\big(\exp_2(\aleph_0)^+\big)$, giving the horizontal and vertical boundaries, respectively with an element of P inserted in the rectangle.

THEOREM 4. *For finite μ, every μ-coloring of Q has a monochromatic maximal chain.*

PROOF. Let c be an arbitrary μ-coloring of Q and for brevity let $\exp_2(\aleph_0)^+ = \lambda$. For $\alpha < \beta < \lambda$, c induces a μ-coloring $c_{\alpha\beta}$ of the (α, β)-copy of $I(\omega^d) \times P$. Apply Lemma 2 with $S = I(\omega^d)$, selecting $p = p_{\alpha\beta}$. Now the triple (p, α, β) induces a μ-coloring of $I(\omega^d)$. Since μ is finite, we can apply Ramsey's Theorem to extract a sequence

$$\mathbf{n}_{p\alpha\beta} : n_0 > n_1 > \cdots$$

so that all pairs from this sequence receive the same color $c_{p\alpha\beta}$. With $s = (n_1, n_0)$, let $\mathbf{p}_{\alpha\beta} = (p_0, p_1, \ldots)$ be a strictly increasing, unbounded sequence in P such that $p_0 = p$ and the common color of $\big((n_1, n_0), p_n, (\alpha, \beta)\big)$ is $c_{p\alpha\beta}$ for all n. This effects a coloring of $I(\lambda)$ with the 4-tuples $(p_{\alpha\beta}, \mathbf{n}_{p\alpha\beta}, c_{p\alpha\beta}, \mathbf{p}_{\alpha\beta})$. Since the number of countable sequences from P is at most 2^ω, this is a 2^ω-coloring of the pairs of λ. The Erdős-Rado Theorem applies to guarantee a strictly increasing sequence α_n of ordinals so that for all n the pair

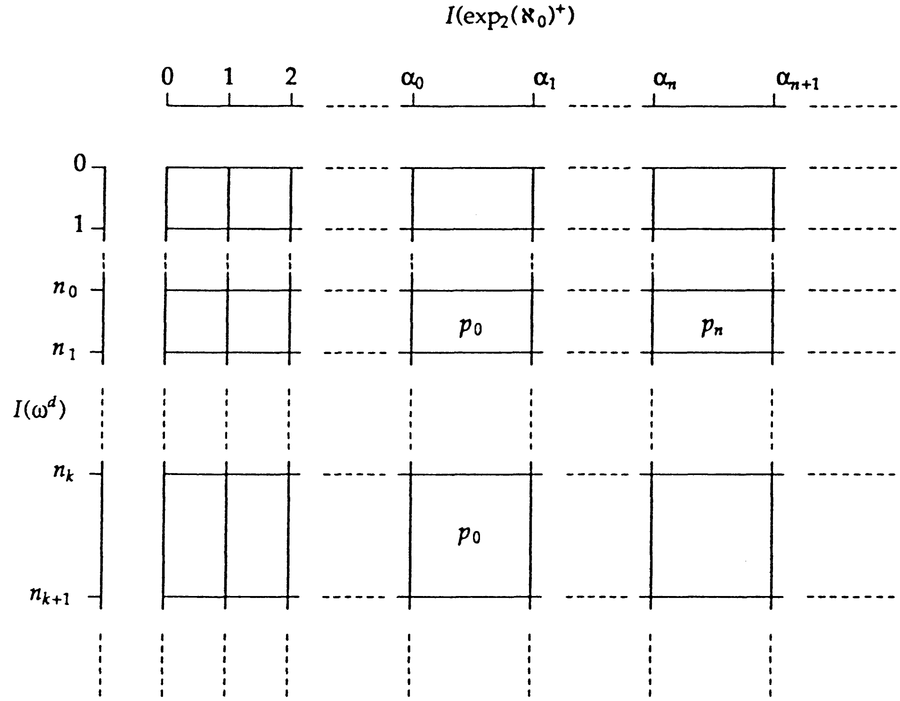

Figure 2. The ordered set Q of Example 1.

(α_n, α_{n+1}) is assigned the same 4-tuple, say $(p, \mathbf{n}, c, \mathbf{p})$. We describe a monochromatic maximal chain C by giving the final segment C^+ above $a = \big((n_1, n_0), p, (\alpha_0, \alpha_1)\big)$ and the initial segment C^- below a:

$$C^+ = \Big\{\big((n_1, n_0), p_k, (\alpha_k, \alpha_{k+1})\big) \mid k < \omega\Big\} \text{ and}$$
$$C^- = \Big\{\big((n_{k+1}, n_k), p, (\alpha_0, \alpha_1)\big) \mid k < \omega\Big\}.$$

When checking that C^+ is maximal above a, it is important to recall that the sequence p_k is not bounded above in P and that the elements $(\alpha_k, \alpha_{k+1}), (\alpha_{k+1}, \alpha_{k+2})$ are consecutive in the strict order on $I(\lambda)$ (see (0) of this section). The maximality of C^- is more obvious: it is contained in a copy of $I(\omega^d)$ (see Figure 2). ■

For the case of infinitely many colors μ, the first temptation is to replace $I(\omega^d)$ by $I(\lambda^d)$ for some sufficiently large λ so that the Erdős-Rado Theorem may be invoked instead of Ramsey's Theorem. After some reflection, it is obvious that the remark on the failure of $I(\kappa)$, contained in Lemma 4, applies in a dual form. This forces us to employ the techniques developed above to handle the initial as well as the final segments when building a monochromatic maximal chain.

EXAMPLE 2. Let μ be an infinite cardinal. Let P be as in Lemma 1 for $\kappa = \mu$ and set $\lambda = \exp_2(\mu)^+$. Let

$$R_1 = P^d \otimes I(\lambda^d).$$

Take Q as in Lemma 1 for $\kappa = \lambda$ and set $\nu = \exp_2(\lambda)^+$. Let

$$R_2 = Q \otimes I(\nu).$$

THEOREM 5. *Every μ-coloring of $R_1 \times R_2$ has a monochromatic maximal chain.*

PROOF. Let c be a μ-coloring of $R_1 \times R_2$. For $\alpha < \beta < \nu$, c induces a μ-coloring of the copy of $R_1 \times Q$ with at most μ colors. Now apply Lemma 2 with $S = R_1$ to select $q_{\alpha\beta} = q$ for this copy. Then $(q_{\alpha\beta}, \alpha, \beta)$ induces a μ-coloring of the copy of R_1.

For $\gamma < \delta < \lambda$, $\big((\delta, \gamma), q_{\alpha\beta}, (\alpha, \beta)\big)$ induces a μ-coloring of the copy of Q. Let

$$\mathbf{p}_{((\delta,\gamma),q_{\alpha\beta},(\alpha,\beta))} : p_0 < p_1 < \cdots$$

be a strictly increasing unbounded monochromatic sequence with common color $c_{((\delta,\gamma),q_{\alpha\beta},(\alpha,\beta))} = c$. An application of the Erdős-Rado Theorem to the resulting 2^μ-coloring of $I(\lambda^d)$ yields a sequence

$$\boldsymbol{\delta} : \delta_0 < \delta_1 < \cdots$$

so that for all $k < \omega$, c and sequence $\mathbf{p}$ are the same for all (δ_{k+1}, δ_k).

To complete the application of Lemma 2 with $s = \big(p_0, (\delta_1, \delta_0)\big)$, select a sequence

$$\mathbf{q} : q_0 < q_1 < \cdots$$

strictly increasing and unbounded in Q such that $\big\{\big(p_0, (\delta_1, \delta_0), q_k\big) \mid k < \omega\big\}$ is monochromatic with color c, in the (α, β)-copy of $R_1 \times Q$.

Assign the 5-tuple $(q_{\alpha\beta}, \mathbf{p}, \boldsymbol{\delta}, \mathbf{q}, c)$ to (α, β). Since $(2^\lambda)^\omega = 2^\lambda$, this is a 2^λ-coloring of $I(\nu)$ and the Erdős-Rado Theorem gives a sequence $\alpha_0 < \alpha_1 < \cdots$ such that for all $l < \omega, (\alpha_l, \alpha_{l+1})$ is assigned the same 5-tuple $(q, \mathbf{p}, \boldsymbol{\delta}, \mathbf{q}, c)$.

A maximal chain C is determined by giving its final segment C^+ above $a = \big(p_0, (\delta_1, \delta_0), q_0, (\alpha_0, \alpha_1)\big)$ and the initial segment below a:

$$C^+ = \big\{\big(p_0, (\delta_1, \delta_0), q_l, (\alpha_l, \alpha_{l+1})\big) \mid l < \omega\big\},$$
$$C^- = \big\{\big(p_k, (\delta_{k+1}, \delta_k), q_0, (\alpha_0, \alpha_1)\big) \mid k < \omega\big\}.$$

The verification of C's maximality is as in the result for finite μ. ∎

With $R_2 = Q \otimes I(\nu)$ as in Example 2, it is easy to see that each maximal chain in R_2 must have a minimum element of the form $\big(q, (0, \alpha)\big)$. In fact, R_2 has no infinite decreasing chains as this is true of the irreflexive order on $I(\nu)$. Dually, every maximal chain of R_1 contains a maximal element of R_1. Therefore, each of R_1 and R_2 admits a good 2-coloring by initial and final segments. Moreover, each is a scattered order, and, thus $R_1 \times R_2$ is a scattered order with no good 2-coloring.

Finally, observe that with the Generalized Continuum Hypothesis and $\mu \leq \aleph_0$, Example 1 is of size $\aleph_3$, while the second is of size $\aleph_6$.

PROBLEM 2. Are there any counterexamples of size $\aleph_1$?

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