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Note

Fibres and Ordered Set Coloring

DWIGHT DUFFUS*

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322

H. A. KIERSTEAD† AND W. T. TROTTER‡

Department of Mathematics, Arizona State University, Tempe, Arizona 85287

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A *fibre* F of a partially ordered set P is a subset which intersects each nontrivial maximal antichain of P. Let λ be the smallest constant such that each finite partially ordered set P contains a fibre of size at most $\lambda \cdot |P|$. We show that $\lambda \leqslant \frac{2}{3}$ by finding a good 3-coloring of the nontrivial antichains of P. Some decision problems involving fibres are also considered. For instance, it is shown that the problem of deciding if a partially ordered set has a fibre of size at most k is NP-hard.

1. INTRODUCTION

When considering relational structures such as graphs and (partially) ordered sets, subsets which induce maximal complete or maximal empty substructures are often of interest. For graphs, these are maximal complete subgraphs (maximal *cliques*) and maximal *independent* sets; for ordered sets, we have maximal *chains* (totally or linearly ordered subsets) and maximal *antichains*. With these structures, comparability graphs provide a direct link—the maximal chains (respectively, antichains) of an ordered set are exactly the maximal cliques (independent sets) of its comparability graph.

We are interested in subsets, often minimal, with the property that they contain at least one element from each instance of such a set. For a

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graph G, a *transversal* of G is a subset of vertices which intersects each maximal clique of G. A *cutset* of an ordered set X is a subset which meets each maximal chain; a *fibre* of X is a subset which meets each maximal antichain. An element of a graph or an ordered set is *isolated* if it is related (adjacent or comparable) to no distinct element; call an element of an ordered set a *splitting* element if it is comparable to all elements.

According to Erdős (cf. [AA]) in prehistory Gallai asked whether a triangulated graph on n vertices with no isolated points has a subset of size $n/2$ which intersects every maximal clique. In settling this question, Aigner and Andreae [AA] noted that this was easily seen for ordered sets—if X has no isolated elements then either $\max(X)$, the set of all maximal elements of X, or $\min(X)$ is a cutset of size at most $|X|/2$. They also raised the dual question. Has an ordered set X with no splitting elements a fibre of size (at most) $|X|/2$? Lonc and Rival [LR] conjectured something stronger: for a finite ordered set X without splitting element there exists $F \subseteq X$ such that both F and $X \backslash F$ are fibres. Although they verified this for ordered sets with chains of cardinality at most 4, it is false in general. In [DW] there is a 17-element ordered set whose smallest fibre has size 9. This is strengthened by Maltby [Ma]: for all n there is an ordered set (without splitting elements) of size $15n+2$ with smallest fibre of size $8n+1$.

We are left with two attractive problems:

(1) determine the least λ such that for any finite ordered set X without splitting elements there is a fibre of size at most $\lambda \cdot |X|$;

(2) determine the least c such that every finite ordered set has a coloring of the elements with at most c colors such that every nontrivial maximal antichain receives more than one color.

(We can lift the prohibition on splitting elements if we only insist that *nontrivial* antichains, those with more than one element, are not monochromatic. Also, see [DW] for more on both these problems.)

It turns out that the next best thing to the Lonc–Rival conjecture is true. Here we prove.

THEOREM 1. *The elements of any finite ordered set can be* 3*-colored so that all nontrivial maximal antichains receive at least two colors.*

As a consequence, we get an upper bound for λ. Combined with the result of Maltby [Ma] noted above we obtain

COROLLARY 2. $\frac{8}{15} \leqslant \lambda \leqslant \frac{2}{3}$.

The proof of Theorem 1 allows us to obtain a good 3-coloring in polynomial time. We have a couple of other observations concerning the complexity of problems involving fibres.

THEOREM 3. *The problem of determining whether a given subset of an ordered set P is a fibre of P is co-*NP*-complete.*

THEOREM 4. *It is* NP*-hard to determine whether, for a given ordered set P and integer k, P has a fibre of size k.*

Concerning the organization of the rest of the paper, Section 2 is devoted to a proof of Theorem 1. In Sections 3 we prove Theorems 3 and 4.

2. COLORING

Let X be a finite ordered set and let $x \in X$. Here are several subsets of X which we shall need

$$U(x) = \{z \in X \mid z > x\}, \qquad D(x) = \{z \in X \mid z < x\},$$
$$C[x] = U(x) \cup \{x\} \cup D(x),$$
$$I(x) = X \setminus C[x].$$

These are the up-set, down-set, cone, and incomparables of x. Also, let $m(x) = \min(I(x))$. We also use $U[x] = U(x) \cup \{x\}$ and, for $A \subseteq X$, $U[A] = \{z \in X \mid z \geqslant a \text{ for some } a \in A\}$.

Obviously, no maximal antichain can be confined to $I(x)$. Thus, $C(x)$ has half a chance to be a fibre whose complement is as well. This is often the case—see [LR]. In fact, cones are exploited in both [LR] and [DS] to construct fibres, whose complementary sets are too, in special circumstances. The 3-coloring we give below builds from this notion as well.

Proceed by induction on $|X|$ to define a 3-coloring of the elements of X. This allows us to assume that X has no splitting elements and, more, that X has no linear decomposition $X = Y \oplus Z$. That is, if $X = Y \cup Z$, $Y \cap Z = \varnothing$, $y < z$ for all $y \in Y$, $z \in Z$ then $Y = \varnothing$ or $Z = \varnothing$.

We construct a sequence $\{x_1, x_2, ...\}$ of elements of X (cf. Theorem 3 in [KT] for a similar construction).

(1) Choose $x_1 \in X$ minimal such that $I(x_1) \cap \max(X) \neq \varnothing$. By induction, $|\max(X)| > 1$, so this can be done; moreover, for the same reason, $x_1 \notin \max(X)$. Let $M_0 = \max(X) \cap I(x_1)$ and $M_1 = m(x_1)$. If $M_1 \cap \min(X) \neq \varnothing$ then we 2-color X by coloring the elements of $C[x]$ red and those of $I(x)$, blue. Any antichain of $C[x]$ can be extended by either an element of M_0 or M_1, while x can be adjoined to an antichain of $I(x)$. Thus we may assume $M_1 \cap \min(X) = \varnothing$.

(2) Choose $x_2 \in D(x_1) = X - (U[x_1] \cup U[M_1])$ to be minimal such that $I(x_2) \cap M_1 \neq \varnothing$. We may assume that such an x_2 exists as if not

$$X = D(x_1) \oplus Y, \qquad Y = U[x_1] \cup U[M_1],$$

where $D(x_1) \neq \varnothing$ ($x_1 \in \min(X)$ implies $M_1 \cap \min(X) \neq \varnothing$ as in (1)), and $Y \neq \varnothing$. With $M_2 = m(x_2)$, if $M_2 \cap \min(X) \neq \varnothing$ then stop, otherwise continue to (3).

(**k**) Choose $x_k \in D(x_{k-1}) = X - (U[x_{k-1}] \cup U[M_{k-1}])$ minimal such that $I(x_k) \cap M_{k-1} \neq \varnothing$. By induction, such an x_k exists as if not

$$X = D(x_{k-1}) \oplus Y, \qquad Y = U[x_{k-1}] \cup U[M_{k-1}],$$

where $D(x_{k-1}) \neq \varnothing$ ($x_{k-1} \in \min(X)$ implies $M_{k-1} \cap \min(X) \neq \varnothing$, as in (**k** − **1**)), and $Y \neq \varnothing$. With $M_k = m(x_k)$, if $M_k \cap \min(X) \neq \varnothing$ then stop, otherwise continue to (**k** + **1**). (See Fig. 1.) Since $x_1 > x_2 > \cdots > x_{k-1} > x_k > \cdots$ we stop at some k_0.

We can now color the elements of X. If y is comparable to all the x_i's, color y red. Otherwise there is a least s, $1 \leqslant s \leqslant k_0$, such that y is incomparable to x_s. If s is odd then color y blue, if s is even then color y yellow. Let us argue that this colors each maximal antichain with at least two colors. (See Fig. 2.)

(**r**) Suppose A is a nontrivial maximal antichain with all elements red. If $A \subseteq U(x_1)$ then $M_0 \neq \varnothing$ yields a contradiction. If $j \geqslant 2$ is least with

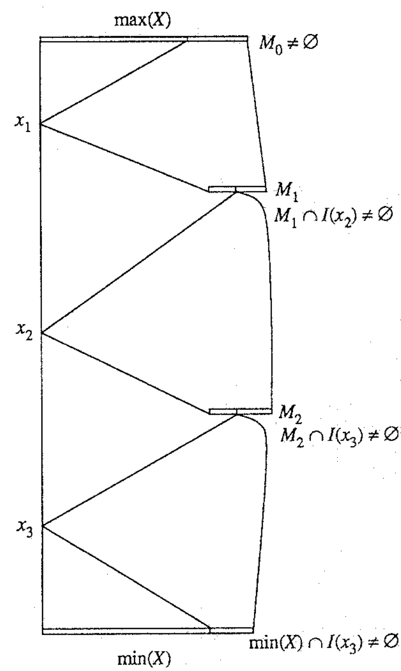

FIG. 1. An example illustrating the sequence $(x_i \mid i = 1, 2, \ldots, k_0)$ with $k_0 = 3$.

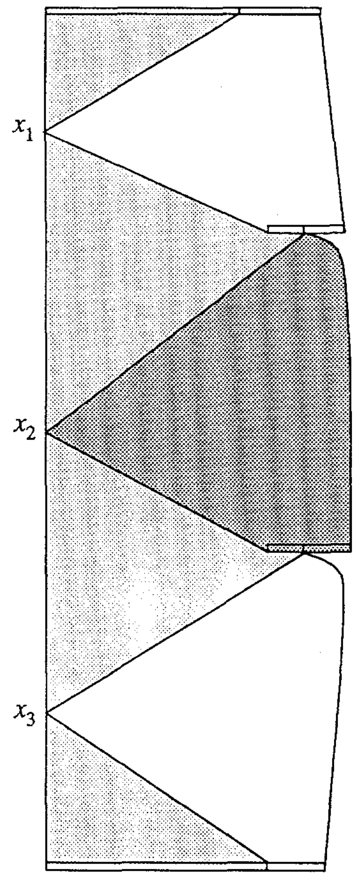

FIG. 2. The resulting 3-coloring of X.

$A \subseteq U(x_j)$ then $A \subseteq D(x_{j-1}) \cap U(x_j)$. Choose $z \in M_{j-1} \cap I(x_j)$. If $a \in A$ then $z > a > x_j$ contradicts $z \in I(x_j)$. If $z < a < x_{j-1}$ then we contradict $z \in M_{j-1} = \min(I(x_{j-1}))$. Thus, the blue or yellow z takes care of A. Finally, if $A \subseteq D(x_{k_0})$ then $M_{k_0} \cap \min(X) \neq \emptyset$ yields the element required to contradict the maximality of A.

(b) Suppose A is a maximal antichain with all elements blue. Since A cannot be contained in $I(x_i)$ for a single index i, we can choose $m < n$ such that $A \cap I(x_{2m-1}) \neq \emptyset$ and $A \cap I(x_{2n-1}) \neq \emptyset$.

Let $a \in A \cap I(x_{2m-1})$ and $b \in A \cap I(x_{2n-1})$. We know that $b < x_{2n-2} \leqslant x_{2m}$ so, by the choice of x_{2m} among the elements of $D(x_{2m-1})$, b is less then every element of M_{2m-1}. As a is greater than or equal to at least one of $M_{2m-1} = \min(I(x_{2m-1}))$, $b < a$.

(y) An argument similar to that in (**b**) shows there is no yellow maximal antichain.

This completes the proof of Theorem 1. ▮

The proof of Theorem 1 provides an algorithm, polynomial in the size of the ordered set, for 3-coloring the elements so that all nontrivial maximal antichains receive more than one color. The next section contains two related results on complexity.

3. Fibres and Complexity

As noted in the introduction, it is a simple matter to locate minimal cutsets; in any finite ordered set, the set of minimal elements and set of maximal elements are both cutsets. In fact, minimum-sized cutsets can be found in polynomial time using network flow algorithms. In contrast, by proving Theorem 4 we establish that it is NP-hard to determine, for arbitrary k, whether an ordered set contains a fibre of size at most k. Let us begin by showing that the problem of determining whether a given subset of an ordered set is a fibre is co-NP-complete.

Proof of Theorem 3. To show that a subset of an ordered set is not a fibre it is sufficient to exhibit an antichain disjoint from the subset such that each element of the subset is comparable to some element of the antichain. Thus our problem is in co-NP.

Next we reduce the NP-complete decision problem MONOTONE SATISFIABILITY (MSAT) [GJ] to the problem of deciding whether a given subset of an ordered set is a fibre.

Recall that an instance of MSAT is a set U of variables and a set $\mathscr{C}$ of clauses over U in which each clause contains only negated variables or only unnegated variables. Let $P = P(\mathscr{C})$ have elements $\mathscr{C} \cup U \cup \{\bar{u} \mid u \in U\}$ with these comparabilities:

$$\begin{aligned} \bar{u} &< u && \text{for all } u \in U, \\ \bar{u} &< C && \text{whenever } \bar{u} \in C \in \mathscr{C}, \\ C &< u && \text{whenever } u \in C \in \mathscr{C}. \end{aligned}$$

Thus, P is an ordered set of height 2 with $\min(P)$ all negated variables and all those clauses containing only unnegated variables and $\max(P)$ all unnegated variables and clauses comprised of negated variables. (See Fig. 3.)

We now consider P with specified subset $F = \mathscr{C}$. A satisfying assignment T yields a maximal antichain $\{u \mid T(u) = 1\} \cup \{\bar{u} \mid T(u) = 0\}$ disjoint from F.

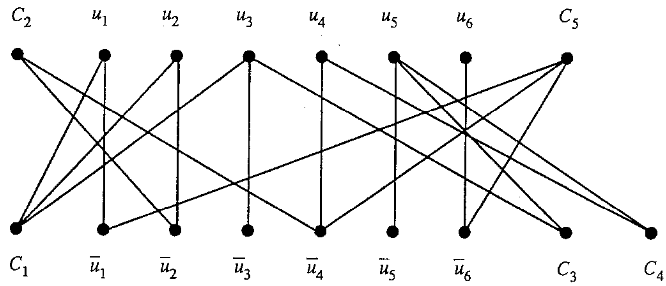

FIG. 3. $P(\mathscr{C})$, where $C = \{C_1, C_2, ..., C_5\}$, $C_1 = \{u_1, u_2, u_3\}$, $C_2 = \{\bar{u}_2, \bar{u}_4\}$, $C_3 = \{u_3, u_5\}$, $C_4 = \{u_4, u_5\}$, $C_5 = \{\bar{u}_1, \bar{u}_4, \bar{u}_6\}$.

Conversely, a maximal antichain A disjoint from F contains exactly one of the literals u, $\bar{u}$ for each $u \in U$. This defines a truth assignment T, with $T(u) = 1$ iff $u \in A$, which satisfies each $C \in \mathscr{C}$. ∎

Proof of Theorem 4. Let $\mathscr{C} = \{C_1, C_2, ..., C_k\}$ over $U = \{u_1, u_2, ..., u_n\}$ be an instance of MSAT. Let P' be the ordered set obtained from P (as in the proof of Theorem 3) by replacing each u_i and each $\bar{u}_i$ $(i = 1, 2, ..., n)$ with a chain of length $k+1$. Since each of these new chains is an autonomous set, any minimal fibre which contains one of them contains all of them. Thus F (as in the proof of Theorem 3) is the only candidate for a k-element fibre. One can verify, much as above, that $\mathscr{C}$ has a satisfying assignment if and only if P' has no fibre of size at most k. ∎

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