ELSEVIER

Discrete Mathematics 250 (2002) 79–91

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Some progress on the Aharoni–Korman conjecture ☆

D. Duffus*, T. Goddard

Department of Mathematics and CS, Emory University, Atlanta GA 30322, USA

Received 18 February 2000; revised 17 January 2001; accepted 5 February 2001

Abstract

Aharoni and Korman (Order 9 (1992) 245) have conjectured that every ordered set without infinite antichains possesses a chain and a partition into antichains so that each part intersects the chain. Related to both Aharoni's extension of the König duality theorem and Dilworth's theorem, this is an important conjecture in the theory of infinite orders. It is verified for ordered sets of the form $C \times P$, where C is a chain and P is finite, and for ordered sets with no infinite antichains and no infinite intervals.

MSC: 06A10; 05C70

Keywords: (Partially) ordered set; Maximal chain; König duality theorem

1. Introduction

The investigations presented here revolve around the following generalization of Hall's matching theorem. (For the source reference see [4]. Recall that a *cover* in a graph is a collection of vertices that intersects every edge and a *matching* is a set of edges, no two of which have a vertex in common.)

Theorem (König duality theorem).
Every bipartite graph Γ contains a cover D and a matching F so that D consists of precisely one vertex from each edge of F.

König proved this theorem in the finite case, while Aharoni [1] applied a substantial body of infinite matching theory to obtain the full generality. Note that Hall's theorem, in its usual formulation, is false for infinite graphs, whereas the duality theorem holds

☆ Research supported in part by ONR Grant N00014-91-J-1150.

* Corresponding author. Tel.: +1-404-7277957; fax: +1-404-7275611.
E-mail addresses: dwight@mathcs.emory.edu (D. Duffus), goddard@mathcs.emory.edu (T. Goddard).

0012-365X/02/$ - see front matter
PII: S0012-365X(01)00185-6

true for bipartite graphs of any cardinality. The simplicity of the statement of the theorem in the infinite case stands in sharp contrast to the great effort required to prove it. As a demonstration of its utility, let us consider a generalization of Dilworth's theorem on chain partitions of (partially) ordered sets.

Dilworth's theorem was originally proved for finite ordered sets, and can be extended to infinite ordered sets of bounded width via logical compactness. Below, we give another result capturing the spirit of Dilworth's theorem, but this result has the requirement that every chain is finite.

Theorem 1.1 (Aharoni and Korman [2]). *An ordered set with no infinite chains may be partitioned into disjoint chains, each of which intersects some common antichain.*

Proof. To begin, we form a bipartite graph Γ based on P as follows. Let $P' = \{x' \colon x \in P\}$ and $P'' = \{x'' \colon x \in P\}$. Then define Γ on the vertex set $P' \cup P''$ with the edge set $\{\{x', y''\} \colon x < y\}$. Notice that a matching F in Γ results in a chain partition $\mathscr{C}_F$ of P, since a chain partition can easily be reconstructed from a set of covering comparabilities (in an ordered set with no infinite chains). For instance, the chain $x < y < z$ is given by the two edges $\{x', y''\}$ and $\{y', z''\}$.

By the König–Aharoni duality theorem, Γ contains a matching F and a cover D such that D consists of exactly one vertex from each edge in F. Let $A = \{x \in P \colon \{x', x''\} \cap D = \emptyset\}$. If $x, y \in A$ then $x', y'' \notin D$; since D is a cover, $\{x', y''\}$ cannot be an edge of Γ. Thus, A is an antichain.

Now observe that each chain in the chain partition $\mathscr{C}_F$ intersects A. Let $C \in \mathscr{C}_F$ be a chain with $n+1$ elements. Each edge $\{x', y''\}$ contains just one vertex of D. Since C has n covering comparabilities, $\{x', x'' \colon x \in C\} \cap D$ must contain n elements. But C has $n+1$ elements, so some element $x \in C$ satisfies $\{x', x''\} \cap D = \emptyset$; that is, $x \in A$. $\square$

Dualizing the above, one obtains a conjecture of Aharoni and Korman [2].

Conjecture. Given an ordered set P with no infinite antichains, there is a partition of P into antichains and there is a chain of P that intersects every member of the partition.

In fact, their ordered set conjecture is more general: for an ordered set P with no infinite antichains and any positive integer k, there are k chains $C_1, \dots, C_k$ and a partition of P into antichains $(A_i \colon i \in I)$ such that each A_i intersects $\min(|A_i|, k)$ chains C_j. This conjecture is dual to a theorem which Aharoni and Korman refer to as the "'correct' infinite version" of the well-known theorem of Greene and Kleitman on Sperner k-families [3].

This conjecture is but the weakest of several which Aharoni and Korman offer in the language of hypergraphs. If true, verification of the strongest of these might require work whose scope exceeds the developments in infinite matching theory leading up to the full graph duality theorem. In light of this, it may not be surprising that our results hold only in special cases.

2. Well-founded ordered sets

Well-founded ordered sets are just those that contain no infinite descending chains. This allows a straightforward inductive definition of a height function, as follows. For a well-founded ordered set P, let $h(x)=0$ for all minimal elements x of P, and inductively set

$$h(x)=\sup(h(y)+1 : y<x), \text{ for all } x\in P.$$

Consequently, we have the usual partition into levels:

$$P=\bigcup_{\alpha<\kappa} P_\alpha, \quad P_\alpha=\{x\in P\colon h(x)=\alpha\}.$$

Just as every finite ordered set has some chain that intersects every level, every well-founded ordered set with every level finite also has some chain that meets every level. Credit for this observation goes to Aharoni and Korman [2], where they note that it follows easily by logical compactness. Compactness proofs generally leave an aftertaste of insubstantiality. We present an alternate proof based on transfinite induction. As usual, given an ordered set X and a pair $a\leqslant b$ in X, the interval $[a,b]$ is the set $\{x\in X\colon a\leqslant x\leqslant b\}$. We shall also use

$$(\leftarrow,b]=\{x\in X : x\leqslant b\}, \quad [a,\rightarrow)=\{x\in X : a\leqslant x\},$$
$$(\leftarrow,b)=\{x\in X : x<b\}, \quad (a,\rightarrow)=\{x\in X : a<x\}.$$

Theorem 2.1 (Aharoni and Korman [2]). *If P is a well-founded ordered set with every level finite then there is a chain in P that intersects every level of P.*

Proof. We proceed by induction using the ordinal-valued height function of well-founded ordered sets. Assume that for all ordinals λ less than κ all well-founded ordered sets of height λ, with no infinite levels, have a chain that intersects all levels. Let P be a well-founded ordered set with all levels finite and with $h(P)=\kappa$. If κ is a successor, say $\kappa=\kappa'+1$, then the ordered set $(\leftarrow,x)$ is of height κ' for any $x\in P_{\kappa'}$. By induction, $(\leftarrow,x)$ has a level-meeting chain, and this chain may be extended to a level-meeting chain of P simply by the addition of the element x. We may suppose, then, that κ is a limit ordinal.

For κ a limit ordinal, let $\lambda=\operatorname{cf}\kappa$ and let the sequence $\{\gamma_\beta\colon \beta<\lambda\}$ be cofinal in κ. By induction, for each $\beta<\lambda$ we have a chain C_{γ_β} of height γ_β that meets every level of P below level γ_β. From these chains, we will inductively construct a chain C that meets every level of P with induction hypotheses as follows. Suppose that $\{c_\delta\colon \delta<\alpha\}$ has been constructed so that

(i) $c_\delta\in P_\delta$,
(ii) $|\{\gamma_\beta\colon c_\delta\in C_{\gamma_\beta}\}|=\lambda$,
(iii) $\{c_\rho\colon \rho\leqslant\delta\}$ is a chain (Fig. 1).

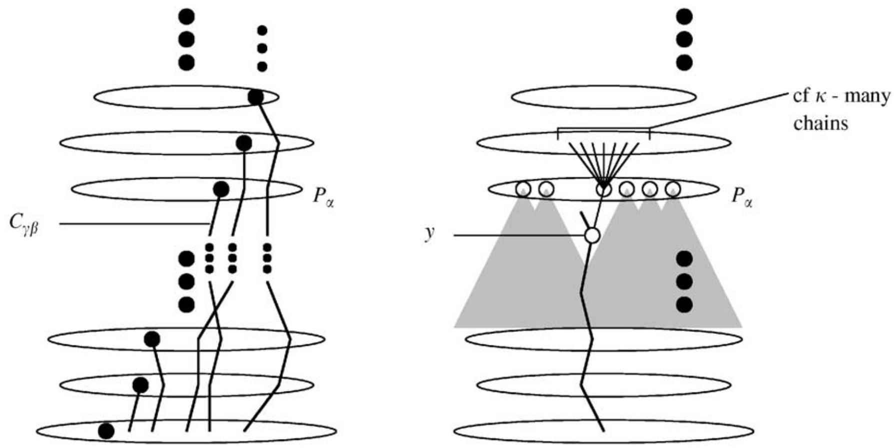

Fig. 1. The construction of a chain meeting every level.

We wish to choose c_α in accordance with the induction hypotheses. If this is not possible then each element of $\{x_1,\ldots,x_n\}=P_\alpha$ fails for some reason. If none can extend the chain $\{c_\delta\colon \delta<\alpha\}$, select $y_1,\ldots,y_n\in\{c_\delta\colon \delta<\alpha\}$ so that x_i is incomparable to y_i (for each $i=1,\ldots,n$). Let $y=\max\{y_1,\ldots,y_n\}$. By the induction hypotheses, λ-many chains C_{γ_β} pass through y. Thus, λ-many of these chains also must pass through P_α, so y cannot be incomparable to every member of P_α. So, some member of P_α extends the chain $\{c_\delta\colon \delta<\alpha\}$. Moreover, since P_α is finite, we may select a member of P_α that meets λ-many of the chains C_{γ_β}. This completes the induction. □

3. Products of chains and finite length orders

In addition to their observations about well-founded ordered sets, Aharoni and Korman [2] use the duality theorem to show that every ordered set of width two (that is, without a three-element antichain) can be partitioned into antichains, each of which meets a common chain. We apply this to obtain our result for the direct product of any chain C and any finite ordered set P. For the sake of completeness, and the fact that it is a nice application of the duality theorem, we begin with their argument for the width-two case.

Let P be an ordered set of width two. By Dilworth's theorem, P is the union of two chains, say $P=X\cup Y$. The incomparability graph of P is then bipartite with parts X and Y. By the König–Aharoni duality theorem, there is a matching F and a cover D so that each edge of F consists of exactly one vertex from D. Denote the complement of the cover by $M_0=P\backslash D$. Since M_0 cannot contain both endpoints of an edge of the incomparability graph, it is a chain and may be extended to a maximal chain M. We claim that every element of P belongs to M or to an element of F. This is because every element that does not belong to M belongs to D, and every element of D belongs to some edge in F. It is clear then that the antichains comprised of the elements of the edges in F, together with the elements of $M\setminus\bigcup F$ as singleton antichains, constitute a partition of P into antichains. Each of these antichains intersects the chain M, and we are done.

We apply this width-two observation in the proof of the following result concerning a collection of ordered sets somewhat more general than products of chains and finite orders. Recall that an ordered set P has *finite length* if there is an integer bound on the cardinality of its chains. In this case, the *length* of P, $l(P)$, is the maximum chain cardinality less one. Also recall that the *direct product* of ordered sets X and Y, denoted by $X \times Y$, is the ordered set on the cartesian product with ordering $(x, y) \leqslant (x', y')$ if and only if $x \leqslant x'$ in X and $y \leqslant y'$ in Y.

Theorem 3.1. *Let C be a chain and let P be an ordered set of finite length. Then there is a partition of the direct product $C \times P$ into antichains and there is a chain of $C \times P$ that intersects every member of the partition.*

Proof. First consider $C \times \mathbf{2}$, where $\mathbf{2} = \{0, 1\}$ denotes the two-element chain. By the König–Aharoni duality theorem, there is a maximal chain M and an antichain partition $\bigcup_{\alpha<\kappa} A_\alpha$ of $C \times \mathbf{2}$ so that $M \cap A_\alpha \neq \emptyset$ for each $\alpha < \kappa$. We may express $M = (I \times \{0\}) \cup (X \times \mathbf{2}) \cup (I' \times \{1\})$ as a disjoint union of (possibly empty) sets based on the following partition of C:

I is an initial segment of C,
$X \subseteq \{\hat{x}\}$, and
I' is a final segment of C.

The antichains $\{A_\alpha\}$ yield maps $f : I \to I \cup X$ and $g : I' \to I' \cup X$ obtained by considering the family of non-singleton antichains $\mathscr{A} = \{A_\alpha : |A_\alpha| = 2\}$ as

$$\mathscr{A} = \{\{\langle f(x), 0\rangle, \langle x, 1\rangle\} : x \in I\} \cup \{\{\langle x', 0\rangle, \langle g(x'), 1\rangle\} : x' \in I'\}.$$

The domains and ranges of f and g are as claimed because the antichains A_α partition $C \times \mathbf{2}$ and each member of $\mathscr{A}$ must meet M. Since $\mathscr{A}$ is composed of antichains, for $x \in I$ and $x' \in I'$, $x < f(x)$ and $x' > g(x')$. Furthermore, f and g are one-to-one, as otherwise the sets in $\mathscr{A}$ would not be pairwise disjoint. The singleton antichains on the chain M give rise to elements not in the image of f or g.

Now, let us turn our attention to $C \times P$. We continue to use the partition $C = I \cup X \cup I'$ arising from $C \times \mathbf{2}$. Denote the levels of P by $L_0, \ldots, L_k$. Let $M_0 = \{m_0, \ldots, m_k\}$ be a chain for which $m_i \in L_i$ $(i = 0, \ldots, k)$; such a chain is easy to find as P has finite length. Then in $C \times P$ let M be the chain

$$M = (I \times \{m_0\}) \cup (X \times M_0) \cup (I' \times \{m_k\}).$$

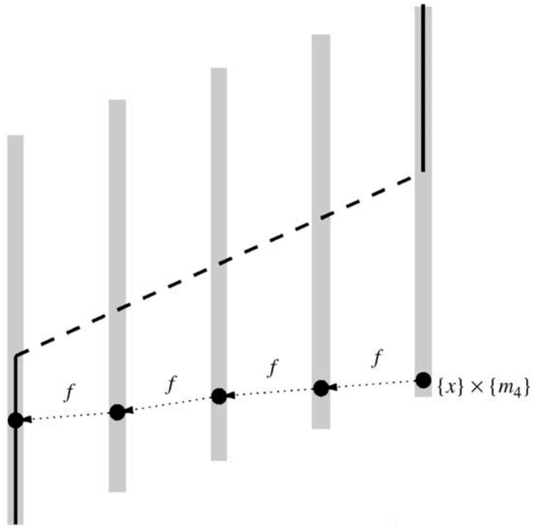

Fig. 2. An antichain in $C \times P$.

Now, using the maps f and g we are ready to define several (but similar) antichains as illustrated in Fig. 2. Each is based on an element x or x' of the chain C and a level k of the ordered set P. For the most part, these antichains are disjoint. The only exception occurs when $\hat{x}$ is in the range of both f and g, in which case we will take the union of two antichains to produce a single antichain.

For each $x \in I$ we have the antichain

$$F_{x,k} = (\{x\} \times L_k) \cup (\{f(x)\} \times L_{k-1}) \cup \cdots \cup (\{f^k(x)\} \times L_0).$$

It intersects M in the element $\langle f^k(x), m_0 \rangle$. Keep in mind (for this antichain and those that follow) that f is not defined on X. Thus, if $f^i(x) = \hat{x}$ (for some $i < k$) $\{f^{i+1}(x)\} \times L_{k-i-1}$ is empty and the above antichain is simply

$$(\{x\} \times L_k) \cup (\{f(x)\} \times L_{k-1}) \cup \cdots \cup (\{f^i(x)\} \times L_{k-i}).$$

This antichain meets M at $\langle f^i(x), m_{k-i} \rangle = \langle \hat{x}, m_{k-i} \rangle$. Dually, for each $x' \in I'$, we have the antichain

$$G_{x',0} = (\{x'\} \times L_0) \cup (\{g(x')\} \times L_1) \cup \cdots \cup (\{g^k(x')\} \times L_k).$$

This antichain meets M at $\langle g^k(x'), m_k \rangle$. As above, we can handle the case when $g^i(x') \in X$.

Additionally, for each $x \in I$ not in the range of f and each i $(0 \leqslant i < k)$, we take the antichain

$$F_{x,i} = (\{x\} \times L_i) \cup (\{f(x)\} \times L_{i-1}) \cup \cdots \cup (\{f^i(x)\} \times L_0)$$

(which meets M at $\langle f^i(x), m_0\rangle$) and for each $x' \in I'$ not in the range of g and each i ($0 \leqslant i < k$), we take the antichain

$$G_{x',i} = (\{x'\} \times L_{k-i}) \cup (\{g(x')\} \times L_{k-i+1}) \cup \cdots \cup (\{g^i(x')\} \times L_k)$$

(which meets M at $\langle g^i(x'), m_k\rangle$).

The actual antichain partition that we take depends on the incidence of $\hat{x}$ in the ranges of f and g, so for brevity let us illustrate the case where $\hat{x}$ is not in the range of f or g, then the case where $\hat{x}$ is in the range of both f and g.

If $\hat{x}$ is not in the range of f or g, we use this set of antichains:

$$\begin{aligned}&\{F_{x,k} : x \in I\} \cup \{F_{x,i} : x \in I \backslash f[I],\ 0 \leqslant i < k\}\\ &\quad \cup \{X \times L_i : i = 0, \ldots, k\}\\ &\quad \cup \{G_{x',0} \colon x' \in I'\} \cup \{G_{x',i} \colon x' \in I' \backslash g[I],\ 0 \leqslant i < k\}.\end{aligned}$$

If $\hat{x}$ is in the range of both f and g, we use this set of antichains:

$$\begin{aligned}&\{F_{x,k} : x \in I,\ f^k(x) \in I\} \cup \{F_{x,i} : x \in I \backslash f[I], f^i(x) \in I,\ 0 \leqslant i < k\}\\ &\quad \cup \{F_{x,i} \cup G_{x',i'} : x \in I,\ x' \in I',\ F_{x,i} \cap G_{x',i'} \neq \emptyset\}\\ &\quad \cup \{G_{x',0} : x' \in I',\ g^k(x') \in I'\} \cup \{G_{x',i} : x' \in I' \backslash g[I],\ g^i(x') \in I',\ 0 \leqslant i < k\}.\end{aligned}$$

Notice that the case where we take a union of two antichains really does produce an antichain because of the way that $F_{x,i}$ and $G_{x',i'}$ are constructed.

Each of the antichains in the sets above intersects M, and the antichains are pairwise disjoint and cover $C \times P$, so the proof is complete. □

The consequences of applying the duality theorem to $C \times \mathbf{2}$ seem interesting for their own sake, so let us state them more succinctly. Every chain C can be partitioned into $I \cup X \cup J$ where I is an initial segment, J is a final segment, and X is either empty or a singleton set in such a way that there exist injective maps $f : I \to I \cup X$ and $g : J \to X \cup J$ that satisfy $f(x) > x$ (for all $x \in I$) and $g(x) < x$ (for all $x \in J$). This is not difficult to prove directly for countable chains but requires some argument for the general case.

4. Finite intervals

Partially well ordered sets—well-founded ordered sets with finite antichains—are examples of infinite orders with a strong finiteness or discreteness condition. As another class of manageable infinite orders, we consider ordered sets without infinite antichains and for which every interval is finite. As examples of such orders we offer these:

1. any suborder inherited by a connected component of the covering graph of an ordered set with no infinite antichains;
2. any linear sum of finite ordered sets, indexed by any subchain of the integers;

3. any linear sum of the form $X \oplus Y$ where both X^* (the dual of X) and Y are partially well ordered with height at most ω;

4. unordered sums of finitely many ordered sets defined in 1, 2, or 3.

Our analysis of chains and antichain partitions of ordered sets with finite intervals and finite antichains uses some special types of chains.

Let P be a finite ordered set with the level partition arising from the height function, say $P = \bigcup_{k=0}^{n} P_k$ (as in the proof of Theorem 2.1). A chain C of P is called *level-hitting* if $C \cap P_k \neq \emptyset$ for $k = 0, \ldots, n$. Now let P be any ordered set with finite intervals. We say that a chain C is a *long chain* in P if

(i) for all $a < b$ in C, the subchain $C(a,b) = \{x \in C\colon a \leqslant x \leqslant b\}$ is level-hitting in the interval $[a,b]$ of P, and
(ii) any bound of C is a member of C.

Notice that a long chain is necessarily maximal—(i) prevents any element from being inserted between elements of C and (ii) disallows an element below or above all of C.

Following standard notation (see, for instance, [5]), a chain C has type ζ (respectively, type ω, type ω^*) if C is isomorphic to the integers (respectively, isomorphic to the natural numbers, dually isomorphic to the natural numbers).

We recall that a *cutset* of an ordered set is a subset that intersects every maximal chain of the ordered set. Also, for elements x, y of an ordered set, we say that x *covers* y, and write $x \succ y$ or $y \prec x$, if $x > y$ and there is no z such that $x > z > y$. Finally, for a subset S of an ordered set P, the *convex hull* of S in P is the set $\mathrm{con}(S) = \{t \in P\colon s \leqslant t \leqslant s' \text{ for } s, s' \in S\}$.

Theorem 4.1. *If P is an ordered set with no infinite intervals and no infinite antichains then there is a partition of P into antichains and there is a chain of P that intersects every member of the partition.*

Proof. First observe that if all chains of P are finite, then P is finite and there is nothing to prove. So, we assume that P has infinite chains.

Our proof proceeds through four steps. We establish these facts.

A. P has a finite cutset.

B. P has a long chain M with additional properties (iii)–(v) below.

C. The convex hull X of M has a partition into antichains each of which intersects M.

D. The antichains in the partition of X can be extended to obtain an antichain partition of P.

A. Let A be a maximal antichain in P. Let $\mathscr{M} = \{M_i \mid i \in I\}$ be the collection of those maximal chains in P that are disjoint from A. Let $M_i \in \mathscr{M}$. Since A is a maximal antichain, $M_i = L_i \cup U_i$ where

$$L_i = \{x \in M_i: x < a \text{ for some } a \in A\}, \quad \text{and} \quad U_i = \{x \in M_i: x > a \text{ for some } a \in A\}.$$

Because A is an antichain, $L_i \cap M_i = \emptyset$, L_i is an initial segment of M_i, and U_i is a final segment. Since A is finite there is a pair $x_i, y_i \in A$ such that $x_i < u$ for all $u \in U_i$ and $y_i > l$ for all $l \in L_i$. Because M_i is maximal, $x_i \neq y_i$ and $U_i, L_i \neq \emptyset$. All intervals of P are finite, so there exist $u_i, l_i \in M_i$ such that $l_i \prec u_i$ in P,

$$L_i = (\leftarrow, l_i], \quad \text{and} \quad U_i = [u_i, \rightarrow).$$

Let $U = \{u_i \mid i \in I\}$ and $L = \{l_i \mid i \in I\}$. If one of these sets is finite, say U, then $A \cup U$ is a finite cutset of P. So, we assume that both U and L are infinite and, as P contains no infinite antichains, both sets contain infinite chains. Let C_U denote an infinite chain contained in U and let $L' = \{l_i \mid u_i \in C_U\}$. Then L' must be infinite, for were it finite, one of its elements would have infinitely many $u_i's$ from C_U as upper covers, impossible as P has no infinite antichains. Let C_L denote an infinite chain in L'.

Suppose that the chain C_U contains an infinite descending subset $u_0 > u_1 > \dots$. Since each $u_k > x_k$ and all x_k are in the finite set A, some x_{k_0} is less that all u_k. Then the interval $[x_{k_0}, u_0]$ is infinite, a contradiction. So, C_U is isomorphic to ω and, dually, C_L is isomorphic to ω^*. Now, choose $l_r \in C_L$. Since the number of elements of C_U less than or equal to u_r is finite, there exists $l_s \in C_L$ such that $l_s < l_r$ and $u_s > u_r$. This contradicts the fact that $l_s \prec u_s$ in P. Thus, one of U or L is finite and we have a finite cutset in P.

This completes the proof of A.

B. Let K be a finite cutset in P guaranteed by A. Define a nested sequence $\{F_i : i < \omega\}$ of subsets of P as follows:

$$F_0 = \mathrm{con}(K), \quad \text{and} \quad F_{i+1} = \mathrm{con}(F_i \cup \{x \in P: x \prec y \text{ or } x \succ y,\ y \in F_i\}).$$

First, let us see that $\bigcup_{i<\omega} F_i = P$. Given any $x \in P$, let C be a maximal chain containing x and let $z \in F_0 \cap C$. The interval determined by the comparable pair x and z is finite, so we can reach x from z by finitely many covering relations. Thus, $x \in F_i$ for some i. Second, F_0 is finite because K is finite and the hypothesis that all intervals in P are finite guarantees that the convex hull of a finite set is finite. So, each F_i is finite because F_0 is finite, each element of P can have only finitely many covers (P has no infinite antichains), and convex hulls of finite sets are finite.

Now, for each $i < \omega$ let $C_i \subseteq F_i$ be a level-hitting chain of F_i. We construct a chain M in P as follows. Since F_0 is finite, there exist an infinite subset I_0 of ω and $M_0 \subseteq F_0$ such that for all $i \in I_0$, $C_i \cap F_0 = M_0$. We establish that $M_0 \neq \emptyset$ as follows. For any $i \in I_0$, extend C_i to a maximal chain D_i. Since F_i is convex and C_i is level-hitting in F_i, $D_i \cap F_i = C_i$. Since F_0 is a cutset of P, there exists some $x \in D_i \cap F_0$. This shows that $x \in F_i$. Thus $x \in C_i$, so $x \in M_0$.

Suppose that I_k and M_k have been defined with

$$I_k \text{ an infinite subset of } \omega, \quad \text{and} \quad C_i \cap F_k = M_k, \quad \text{for all } i \in I_k.$$

We first ensure that our choice of M_{k+1} properly contains M_k. Since P is infinite and its antichains are finite, P contains an infinite chain, say C, which we take to be a maximal chain. Then $|C \cap K| \geqslant 1$ because K is a cutset of P. By definition of F_i, $|C \cap F_i| \geqslant i$ and by the choice of the level hitting chain C_i, $|C_i| \geqslant i$. Let M_k have minimum element u and maximum v. For an infinite subset I of I_k, $|C_i| > |M_k|$ for all $i \in I$. Since M_k is a convex subset of each C_i, we may assume that there is an infinite index set I' such that for all $i \in I'$ there is $v_i \in C_i$ with $v_i > v$. Since C_i is level-hitting in F_i and the latter is convex, for all $i \in I'$ there is $w_i \in C_i$ such that $v_i \geqslant w_i \succ v$. Since v has only finitely many upper covers, there is an infinite $I'' \subseteq I'$ and $w \in P$ such that $w \succ v$ in P and $w \in C_i$ for all $i \in I''$.

If the minimum element u of M_k is not minimal in P then u is not minimal in F_{k+1}. For all $i \in I''$ with $i > k$, C_i is level-hitting in F_i, $u \in C_i$, and $C_i \cap F_k = M_k$, so C_i contains an element less than u. Reasoning as above, we find an infinite $J_k \subseteq I''$ and a lower cover $x \prec u$ such that $x \in C_i$ for all $i \in J_k$.

Since F_{k+1} is finite and J_k is infinite, there is an infinite subset I_{k+1} of J_k and $M_{k+1} \subseteq F_{k+1}$ such that for all $i \in I_{k+1}$, $C_i \cap F_{k+1} = M_{k+1}$. Note that $M_k \subseteq M_{k+1}$ and $w \in M_{k+1} \backslash M_k$. In case M_k does not contain a minimal element of P, $x \in M_{k+1} \backslash M_k$ as well.

Let $M = \bigcup_{k<\omega} M_k$. We claim that M is a long chain in P. To establish (i) of the definition, let $a, b \in M$. Since $[a, b]$ is finite there exists some m such that $[a, b] \subseteq F_m$. Choose $n \in I_m$ with $m < n$. Then $C_n \cap F_m = M_m$. Recall that C_n is a long chain in F_n, $[a, b] \subseteq F_m \subseteq F_n$, each set is a convex subset of the larger, and $a, b \in C_n$. So, $M(a, b) = C_n(a, b)$ and since $C_n(a, b)$ is level-hitting in $[a, b]$, so is $M(a, b)$.

Now turn to (ii). Suppose that b is an upper bound of M. Were b not an element of M, then the fact that P has finite intervals guarantees that M has a maximum element, say v. Then v would be the maximum element of M_k, say, and the argument above would produce an element of M_{k+1} larger than v, a contradiction. Similarly, any lower bound of M is a member of M.

Here are the additional properties required.

(iii) M is infinite.

This is immediate from the construction.

(iv) We may assume that M has type ω or type ζ.

Obviously, M cannot have both a maximum and minimum as these would bound an infinite interval. Because all intervals of P are finite and M is infinite, M must have one of the types ζ, ω or ω^*. By duality, we can assume that the type of M is one of ζ or ω.

(v) If M has type ω then $D = \bigcup_{a \in M} (\leftarrow, a]$ is well-founded.

Suppose that $a \in M$, that $M' = M \cap (\leftarrow, a]$, and that C is any chain in $(\leftarrow, a]$. Let us take a_0 to be the minimum element of M, so $M(a_0, a) = M'$; also, set $|M'| = k$. By (ii), a_0 is minimal in P. Suppose that $C' \subseteq C$ and that C' has $k+1$ elements. Since $M' \cup C'$ is finite there some index m such that $M' \cup C' \subseteq F_m$. Choose $n \in I_m$ with $n > m$; we know that

$$C_n \cap F_m = M_m = M \cap F_m.$$

Thus $a_0, a \in C_n$. In verifying (i) we showed that $C_n(a_0, a) = M(a_0, a) = M'$. Thus we have that C_n is a level-hitting chain in F_n, it has k elements in the subchain $C_n(a_0, a)$ where a_0 is minimal in P, and $C' \subseteq F_n \cap (\leftarrow, a]$ is a $k+1$-element chain. This is a contradiction. We conclude that $|C| \leqslant k$. Since any infinite descending chain in D would have to be contained in some $(\leftarrow, a]$, we have proved (v).

This completes the proof of B.

C. Let $X = \mathrm{con}(M)$. In the case that M has type ω, with minimum element a_0, X has minimum a_0 as well, and X is a well-founded ordered set. In fact, (i) guarantees that M intersects every level of X defined by the height function, so we can take these levels as the partition of X.

In case M has type ζ, we apply logical compactness to the ordered set X, using one predicate P. For all $x,\ y \in X$,

Pxy: x is in an antichain with the element y and $y \in M$.

The axioms to support this predicate are as follows.

Since every interval is finite, for each $x \in X$, we have the finite disjunction

$$\mathrm{P}xm_1 \vee \mathrm{P}xm_2 \vee \cdots \vee \mathrm{P}xm_k$$

where $m_1, m_2, \ldots, m_k \in M$ are the elements of M incomparable with x. To ensure that P denotes antichains, we also include

$$\neg(\mathrm{P}xm \wedge \mathrm{P}zm)$$

for every comparable pair $x < z$ in X and $m \in M$. For convenience, we ensure that each element $x \in X$ belongs to at most one antichain with

$$\neg(\mathrm{P}xm_1 \wedge \mathrm{P}xm_2)$$

for every $m_1 < m_2 \in M$.

Clearly any finite subset of these axioms can be satisfied, as the level-hitting property of the long chain M guarantees a compatible antichain partition in the interval containing the elements referenced in that finite set of axioms.

A model of these axioms yields a mapping from X into the chain M. The domain is all of X because of the finite disjunction. The pre-image of an element $m \in M$ is an antichain because of the second axiom scheme. The third axiom guarantees that our model defines a partition.

This completes the proof of C.

Since all antichains and intervals in P are finite, P is countably infinite. So, for the remainder of the proof, let $M = \{m_n \mid n < \omega\}$, let $X = \{X_n \mid n \in \omega\}$ be a partition of X into antichains, and let $X_n \cap M = \{m_n\}$. Also, let $P \backslash X = \{p_n \mid n < \omega\}$.

D. We prove that there is an antichain partition $\{Y_n \mid n < \omega\}$ of P such that $X_n \subseteq Y_n$ and $Y_n \cap M = \{m_n\}$ for all $n < \omega$.

Note that an element of $P\backslash X$ cannot both be larger and smaller than elements of X. Partition $P\backslash X$ into two subsets: as in (v), let D be the set of those elements of $P\backslash X$ less than some element of X, and let U be the rest, those elements of $P\backslash X$ less than no element of X.

Suppose that some $p = p_n \in U$ is larger than elements $x_{n_k} \in X_{n_k}$ where m_{n_k} is a cofinal set in M. Since $\{x_{n_k} \mid k < \omega\}$ is infinite, it contains an infinite ascending chain or an infinite descending chain. If x is any element of the former, $[x, p]$ is infinite. In the latter case, for convenience take $x_{n_1} > x_{n_2} > \cdots$ to be an infinite descending chain. Then for some k we have $m_{n_k} > x_{n_1} > x_{n_k}$, a contradiction.

By (iv), M has type ζ or ω.

Case 1. M has type ζ.

In this case, we know that for each element p_n of U, there are infinitely many indices k such that all elements of the corresponding X_k's are incomparable to p_n. We also know that the same is true for the elements of D, using a dual argument and the infinite coinitiality of M. We construct the partition $\{Y_n | n < \omega\}$ as follows:

let $X_n \subseteq Y_n$ for all $n < \omega$;
place p_0 in Y_{n_0} where n_0 is least such that p_0 is comparable to no element in X_{n_0};
place p_k in Y_{n_k} where n_k is least such that p_k is comparable to no element yet in Y_{n_k}.

This is possible, as only finitely many elements comparable to p_k could have yet been placed in Y_n's. Thus, we obtain the desired partition of P into antichains Y_n, each with $Y_n \cap M = \{m_n\}$.

Case 2. $M = \omega$.

Let us assume that $m_n < m_{n+1}$ for all $n < \omega$. Since X has m_0 as a minimum element, X is well-founded. We may take the partition $X = \{X_n \mid n \in \omega\}$ to be that given by the height function. Indeed, the proof of (v) shows that D is well-founded as well, and that the height of any element of M is the same in $X \cup D$ as in X. So, if we take $\{Z_n \mid n < \omega\}$ to be the antichain partition defined according to the height function of $X \cup D$ then $Z_n \cap M = \{m_n\}$ for all $n < \omega$.

For $p \in U$, we argue just as in *Case* 1, with Z_n in place of X_n.

This completes the proof of D. $\square$

5. Conclusion

As a possible generalization of the above result (not necessarily in accordance with the prohibition against infinite antichains) one could ask:

If P and Q both have chains with associated antichain partitions, does $P \times Q$?

As the standard Sierpinski example shows, the answer, however, is 'no'. To see this, let $2^{\aleph_0}$ be a well-ordering of the reals, let $\mathbb{R}$ denote the reals with the usual order, and consider $S = 2^{\aleph_0} \times \mathbb{R}$. Recall that S has no uncountable chains or antichains. Clearly, S is the product of two ordered sets that may be partitioned into antichains (singletons) so that some (the) chain meets each. However, one cannot possibly partition the uncountable set S into antichains each of which meets a common chain, as this would imply that S is a countable union of countable sets. Perhaps some more restrictive conditions would help. One could assume that one or both of P or Q is well-founded, or that Q is finite. A first step would be to determine the status for $Q = \mathbf{2}$.

There are certainly other interesting questions that may be more tractable and relevant to the conjecture than those involving the direct product construction. For instance, it would be interesting to determine whether ordered sets of width three (even countable ones) satisfy the conjecture.

One might also wonder about the relative 'strengths' of the König–Aharoni duality theorem and of the consequences for ordered sets presented here. For instance, given the fact that every ordered set with no three-element antichain can be partitioned into antichains, each of which meets a common chain, can one give a short proof of the König duality theorem?

References

[1] R. Aharoni, König's duality theorem for infinite bipartite graphs, J. London Math. Soc 29 (1984) 1–12.
[2] R. Aharoni, V. Korman, Greene–Kleitman's theorem for infinite posets, Order 9 (1992) 245–253.
[3] C. Greene, D.J. Kleitman, The structure of Sperner k-families, J. Combin. Theory Ser. A 20 (1976) 69–79.
[4] D. König, Theorie der endlichen und unendliechen Graphen, Chelsea, New York, 1950.
[5] J. Rosenstein, Linear Orderings, Academic Press, New York, 1982.