Extremal Problems for Boolean Lattices and their Quotients

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Dedicated to Professor Maurice Pouzet on the occasion of his 75th birthday

Many beautiful results and conjectures in the 1970's and 1980's concern extremal and symmetry properties of finite set systems and, in particular, the finite Boolean lattice $\mathcal{B}(n)$. This presentation concerns two types of questions, the second of which Maurice Pouzet and his collaborators studied extensively.

First, in the 1970's, Peter Frankl raised a number of questions about families of subsets of a finite set. Here is one posed by Daykin and Frankl [3] concerning the minimum width $w(\mathcal{C})$ of a convex subset of $\mathcal{B}(n)$.

Conjecture 1. *For any nonempty convex subset $\mathcal{C}$ of $\mathcal{B}(n)$,* $\dfrac{w(\mathcal{C})}{|\mathcal{C}|} \geq \dfrac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}$.

(Recall that $\mathcal{C}$ is *convex* in $\mathcal{B}(n)$ if whenever $X, Y \in \mathcal{C}$ with $X \subseteq Z \subseteq Y$ then also $Z \in \mathcal{C}$.) We made little progress on the general conjecture beyond small cases but D. Howard, I. Leader and D. Duffus were able to show that Conjecture 1 holds for *binary downsets* in $\mathcal{B}(n)$ [4]. (For $A, B \subseteq [n]$, the *binary order* is given by $A <_b B$ if $\max(A \Delta B) \in B$. A downset of $\mathcal{B}(n)$ whose elements constitute an initial segment of $\mathcal{B}(n)$ with the binary order is a *binary downset*.). Also, it is not difficult to show that among all d-element downsets in $\mathcal{B}(n)$, the binary ones maximize the number of comparabilities and, so, minimize the incomparabilities. Since we are inclined to support Conjecture 1, it is natural to conjecture the following [4] (also proposed by J. Goldwasser [7]).

Conjecture 2. *Among all d-element downsets in $\mathcal{B}(n)$, the binary d-element downset has minimum width.*

We considered properties of convex families in $\mathcal{B}(n)$ that might help us to understand the relationship between width and size, in particular, partitions of convex families by width-many chains. Given elements $x < y$ in a partially ordered set X, write $x \prec y$ (and say y *covers* x) if y is an immediate successor of x. A chain C in X is *skipless* if $x \prec y$ in C implies $x \prec y$ in X. A partition of X into a family of chains is a *Dilworth partition* if there are $w(X)$ chains in the family. For brevity, call a Dilworth partition of X into skipless chains an SD-partition of X.

Theorem 1. *[4] Every convex subset of $\mathcal{B}(n)$ has an SD-partition.*

Still thinking about convex sets, B. Sands and D. Duffus became interested in covers of convex subsets of $\mathcal{B}(n)$ by small families of intervals: given $X \subseteq Y \subseteq [n]$, $[X, Y] = \{Z | X \subseteq Z \subseteq Y\}$ is the interval determined by X and Y. It is obvious that if a convex family $\mathcal{C}$ in $\mathcal{B}(n)$ has r minimals and s maximals, it can be covered by $r \cdot s$ intervals. One can construct a convex family in a Boolean lattice that requires that many, as Frankl observed. What happens if we focus attention on convex subsets of $\mathcal{B}(n)$ defined by levels? For $1 \leq k \leq l \leq n$, let $\mathcal{B}(n; k, l) = \{S \subseteq [n] \mid k \leq |S| \leq l\}$.

We rediscovered a result of Voigt and Wegener [16]: $\mathcal{B}(n; k, l)$ can be covered by the minimum possible number, $\max(\binom{n}{k}, \binom{n}{l})$, of intervals. We investigated finite distributive lattices D and found that an analogous result holds for convex subsets determined by the atoms and coatoms of D and have a conjecture for the smallest covering families of intervals determined by any two levels of D [5].

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The second type of problem concerns preservation of the Sperner property, the strong Sperner property, rank unimodality, rank symmetry and the existence of symmetric chain decompositions (SCDs) in quotients of $\mathcal{B}(n)$. (See [1] for definitions.) Stanley [13, 14, 15] is concerned with quotients of a partially ordered set P defined by subgroups of the automorphism group of P. Pouzet [10] and Pouzet and Rosenberg [11] study more general quotients on P defined by *hereditary equivalences* with ordering induced from that on P. The new ordered set is called the *age* of the equivalence on P. The main tools in Pouzet's work are linear algebraic as is the case with Stanley's papers though he also employs methods from algebraic geometry.

These papers settled many of the questions about preserving symmetry properties with a couple of notable exceptions. First, the existence of SCDs does not follow from their general theorems – both Stanley and Pouzet have raised several specific questions and it is possible that all quotients $\mathcal{B}(n)/G$ by subgroups G of S_n have SCDs [2]. Progress has been slow (see, for instance, [17] and [6]). (Stanley gives an interesting explanation why linear algebra does not provide us with SCDs – see Section 7 in [13]).

Another question that has survived is whether the collection of all downsets of $\mathcal{B}(n)$, again ordered by containment, has the Sperner property. This partially ordered set is a distributive lattice, in fact, is the free distributive lattice $FD(n)$ on n generators and is a sublattice of $\mathcal{B}(2^n)$. It does not appear that $FD(n)$ can be obtained from $\mathcal{B}(2^n)$ as a quotient defined by a subgroup of S_{2^n} or as the age of a hereditary equivalence. However, one can show that $FD(n)$ can be obtained by a series of n "compressions" applied to the subsets of $\mathcal{B}(n)$, where $\mathcal{B}(n)$ is labelled by $[2^n]$ using the binary order. We want to see if the linear algebraic approach used by Pouzet and Rosenberg [11] can be applied in conjunction with these compressions to find families of disjoint chains in $FD(n)$, which would imply that $FD(n)$ has the Sperner property.

It is interesting to note that the maximum size of an antichain in $FD(n)$ determines the chromatic number of an iterated arc graph (see Theorem 3 in [12]). Were $FD(n)$ known to be Sperner and rank unimodal, its width would be the number of downsets in $\mathcal{B}(n)$ of size 2^{n-1}. This has been verified for $n \leq 6$ and has been conjectured for all n [9]. It would be (more than) enough to show that $FD(n)$ has a symmetric chain decomposition.

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