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The Minimum Number of Edges in Uniform Hypergraphs with Property O

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An oriented k-uniform hypergraph (a family of ordered k-sets) has the ordering property (or Property O) if, for every linear order of the vertex set, there is some edge oriented consistently with the linear order. We find bounds on the minimum number of edges in a hypergraph with Property O.

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1. Introduction

This note is motivated by two types of problem concerning hypergraphs. The first is well known and regards 2-colourable hypergraphs, also said to possess Property B. Several papers have presented bounds on $m(k)$, the minimum number of edges in a k-uniform hypergraph that does not have Property B (see [1], [2], [6] and [7]). The second comes from Ramsey theory, where appropriate properties of graphs containing a given graph with a fixed order can be used to prove negative partition relations for unordered graphs (see [4] and [5] for early papers on this topic).

We would like to determine the minimum number of edges in an oriented uniform hypergraph needed to ensure that, for every ordering of the vertex set, some edge is ordered in the same way. Here are the required definitions followed by our results and a conjecture.

Fix a positive integer $k \geqslant 2$ and a finite set V. An *ordered k-set* $\overline{E}$ is a k-tuple $(x_1, x_2, \ldots, x_k)$ of distinct elements of V; we use E to denote the unordered set $\{x_1, x_2, \ldots, x_k\}$. Given a family of ordered k-sets $\mathcal{E} \subseteq V^k$ with no two k-tuples on the same k-element set, call $\mathcal{H} = (V, \mathcal{E})$ an *oriented k-uniform hypergraph*, or, more briefly, an

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oriented k-graph. In the case that $\mathcal{E}$ contains an ordered k-set for each k-element subset of V, call $\mathcal{H}$ a *k-tournament*. So, a k-tournament is obtained from the complete k-uniform hypergraph $K_n^{(k)}$ by giving each k-set an orientation. For $E \subseteq V$ and a linear order $<$ on V, an ordered k-set $\overline{E} = (x_1, x_2, \ldots, x_k)$ is *consistent* with $<$ if $x_1 < x_2 < \cdots < x_k$.

Here is the property that interests us.

Definition. Given an oriented k-graph $\mathcal{H} = (V, \mathcal{E})$ we say that $\mathcal{H}$ has the *ordering property*, or *Property O*, if for every (linear) order $<$ of V there exists $\overline{E} \in \mathcal{E}$ that is consistent with $<$. For an integer $k \geqslant 2$, let

$f(k)$ be the minimum number of edges in an oriented k-graph with Property O.

Here is what we know about bounds for $f(k)$.

Theorem 1.1. *The function $f(k)$ satisfies $k! \leqslant f(k) \leqslant (k^2 \ln k)k!$, where the lower bound holds for all k and the upper bound holds for sufficiently large k.*

The upper bound for $f(k)$ is proved in Section 2. The lower bound $k! \leqslant f(k)$ follows from this simple random argument. Let $\mathcal{H} = ([n], \mathcal{E})$ be an oriented k-graph, and consider a random linear order $<$ on $[n]$. The expected number of $\overline{E} \in \mathcal{E}$ consistent with $<$ is $|\mathcal{E}|/k!$. If $\mathcal{H}$ has Property O then $|\mathcal{E}|/k! \geqslant 1$.

We would like to decide if $f(k)$ is bounded away from $k!$, in analogy with Property B.

Problem 1. *Determine whether $f(k)/k! \to \infty$ as $k \to \infty$.*

We are unable to improve the simple lower bound for $f(k)$ at this point, but we can show that for appropriately chosen k and $n = n(k)$, almost all k-tournaments on n fail to have Property O. This is made precise in Theorem 1.2. Let $\mathcal{T}_{n,k}$ denote the set of all k-tournaments on $[n]$.

Theorem 1.2. *Let*

$$0 < \alpha < 1, \quad c = \frac{2\pi}{3e} e^{e^2/2} \quad \text{and} \quad n = (c\alpha)^{1/k} \left(\frac{k}{e}\right)^2 k^{3/2k}.$$

Then for k sufficiently large at least $(1-\alpha)|\mathcal{T}_{n,k}|$ members of $\mathcal{T}_{n,k}$ do not have Property O.

In the next section we prove the upper bound of $f(k)$ given in Theorem 1.1. In Section 3 we prove Theorem 1.2. In Section 4 we provide a construction of k-graphs with Property O, investigate the situation for small values of n and k, and pose a few problems.

We close this section with an observation used in Sections 2 and 3:

$$\text{if } n = \left(\frac{k}{e}\right)^2 (1 + o(1)) \quad \text{then } \binom{n}{k} = (e^{-e^2/2}) \frac{n^k}{k!} (1 + o(1)). \tag{1.1}$$

The proof is an elementary calculation.

2. Proof of Theorem 1.1

We verify the upper bound in Theorem 1.1 by showing that, for k large enough, there exists a k-tournament with $(k^2 \ln k)k!$ edges which has Property O. Indeed, we show that for an appropriate choice of n, a randomly selected member $\mathcal{H}$ of $\mathcal{T}_{n,k}$ has Property O with positive probability.

Let $\mathcal{H} = ([n], \mathcal{E}) \in \mathcal{T}_{n,k}$. For a fixed order $<$ on $[n]$ and a fixed $\overline{E} \in \mathcal{E}$, the probability that $\overline{E}$ is not consistent with $<$ is $1 - 1/k!$. Since the edges of $\mathcal{H}$ are oriented independently, the probability that no edge of $\mathcal{H}$ is consistent with $<$ is $(1 - 1/k!)^{\binom{n}{k}}$. Taking the union bound over all orders on V, we see that the probability that there exists an order $<$ on V so that no edge of $\mathcal{H}$ is consistent with $<$ is at most $n!(1 - 1/k!)^{\binom{n}{k}}$.

The upper bound follows once we verify (2.1) and (2.2), below, for k sufficiently large. Let $n = (k/e)^2(\pi \cdot \exp(e^2/2) \cdot k^3 \ln k)^{1/k}$. Then

$$\binom{n}{k}\frac{1}{k!} \leqslant k^2 \ln k, \quad \text{and} \tag{2.1}$$

$$n!\left(1 - \frac{1}{k!}\right)^{\binom{n}{k}} < 1. \tag{2.2}$$

To prove (2.1), we apply (1.1) and the Stirling approximation $k! = (k/e)^k\sqrt{2\pi k}(1 + o(1))$, that is,

$$\binom{n}{k}\frac{1}{k!} = e^{-e^2/2}\left(\frac{(k/e)^{2k}\pi \cdot e^{e^2/2} \cdot k^3 \ln k}{(k!)^2}\right)(1 + o(1)) = \frac{1}{2}(1 + o(1))k^2 \ln k. \tag{2.3}$$

Hence (2.1) holds for k sufficiently large.

Turning to inequality (2.2), we use the choice of n and (2.3) to infer that

$$n \ln n = 2(1 + o(1))\left(\frac{k}{e}\right)^2 \ln k < \binom{n}{k}\frac{1}{k!}, \tag{2.4}$$

for k sufficiently large. We have

$$\begin{aligned} n!\left(1 - \frac{1}{k!}\right)^{\binom{n}{k}} &\leqslant n^n\left(1 - \frac{1}{k!}\right)^{\binom{n}{k}} \\ &\leqslant \exp(n \ln n) \cdot \exp\left(-\binom{n}{k}\frac{1}{k!}\right) \\ &= \exp\left(n \ln n - \binom{n}{k}\frac{1}{k!}\right) \\ &< 1, \end{aligned}$$

where the last inequality follows from (2.4). This proves (2.2) and completes the proof of the upper bound.

3. Proof of Theorem 1.2

Let α, c and n be as in the statement of Theorem 1.2. We first obtain an expression for $\binom{n}{k}$, the number of edges in a tournament in $\mathcal{T}_{n,k}$, which we use in the proof of the theorem.

Apply Stirling's formula for $k!$:

$$\begin{aligned} n^k &= c\alpha\left(\frac{k}{e}\right)^{2k} k^{3/2} \\ &= \frac{e^{e^2/2}}{3e}\alpha(k!)^2 k^{1/2}(1+o(1)). \end{aligned} \tag{3.1}$$

On the other hand, by (1.1),

$$n^k = e^{e^2/2}(n)_k(1+o(1)). \tag{3.2}$$

Equate the right-hand sides of (3.1) and (3.2) and, for brevity, set $\omega = (\alpha/3e)k^{1/2}(1+o(1))$:

$$\binom{n}{k} = \frac{\alpha}{3e}k^{1/2}k!(1+o(1)) = \omega k!, \tag{3.3}$$

the estimate we require.

We will show that if T is sampled from $\mathcal{T}_{n,k}$, the set of all k-tournaments on $[n]$, according to the uniform distribution, the probability that T has Property O is at most α. It will follow that at least $(1-\alpha)|\mathcal{T}_{n,k}|$ members of $\mathcal{T}_{n,k}$ fail to have Property O.

The random sampling of $T = ([n], \mathcal{E})$ from $\mathcal{T}_{n,k}$ is done in two phases. In the first phase we will select k-tuples which are consistent with the natural order $<$ on $[n]$ and in the second phase we will assign to the remaining k-tuples one of the $k!-1$ remaining orientations.

Phase 1. Reveal the set

$$C(T) = \left\{K \in \binom{[n]}{k} \mid \overline{K} \text{ is oriented consistently with } < \right\}.$$

For any $K \in \binom{[n]}{k}$, $\mathbb{P}(K \in C(T)) = 1/k!$ and thus, by (3.3),

$$\mathbb{E}(|C(T)|) = \binom{n}{k}\frac{1}{k!} = \omega.$$

Let A_ω be the event that $|C(T)| \leqslant (2/\alpha)\omega$. By Markov's inequality we have

$$\mathbb{P}\left(|C(T)| > \frac{2}{\alpha}\omega\right) < \frac{\alpha}{2} \quad \text{and so} \quad \mathbb{P}(A_\omega) > 1 - \frac{\alpha}{2}.$$

Assume that A_ω occurs. For each $K \in \binom{[n]}{k}$, let $\min K$ be the $<$-least element of K. Let

$$M = \{\min K \mid K \in C(T)\}.$$

Note that $|M| \leqslant |C(T)| \leqslant (2/\alpha)\omega < k-1$.

Thus, for each $K \in C(T)$, $K \setminus M \neq \emptyset$. Let $W \subseteq [n]$ be obtained by selecting one element from each $K \setminus M$. Then

$$|W| \leqslant |C(T)| \leqslant \frac{2}{\alpha}\omega. \tag{3.4}$$

We now define $<'$ to be the natural order $<$ on each of W and $[n] \setminus W$, and let $u <' v$ for $u \in W$, $v \in [n] \setminus W$.

We claim that no $K \in C(T)$ has $\overline{K}$ consistent with $<'$. To see this, let $v \in K \cap W$. On the one hand, $\min K \notin W$ by the way that we selected W. On the other hand, $v <' \min K$ by the definition of $<'$. However, $K \in C(T)$ means precisely that $\overline{K}$ is consistent with $<$, and so $v <' \min K$ is a contradiction.

Phase 2. Reveal the orientation of $\overline{K}$ for each $K \notin C(T)$.

For each $K \notin C(T)$ there are $k! - 1$ possible orientations $\overline{K}$ in T – any one except that given by the natural order $<$. Since T is chosen according to the uniform distribution, each orientation is equally likely and at most one of these is consistent with $<'$. Thus,

$$\mathbb{P}(\overline{K} \text{ is consistent with } <') \leqslant 1/(k! - 1).$$

Also, if $K \cap W = \emptyset$, then $<$ and $<'$ coincide on K, so $\overline{K}$ cannot be consistent with $<'$.

Since the only k-tuples which may become consistent with $<'$ are those with non-empty intersection with W, in view of (3.3),

$$\mathbb{P}(\exists K \notin C(T),\ \overline{K} \text{ consistent with } <' \mid A_\omega) \leqslant \frac{1}{k!-1}\left|\left\{K \in \binom{[n]}{k} \mid K \cap W \neq \emptyset\right\}\right|. \tag{3.5}$$

In order to bound the right-hand side of (3.5), we first bound the number of k-sets intersecting W. To this end, we observe that the expected size of the intersection of a uniformly chosen k-set with W is $k|W|/n$. Hence, the probability of a uniformly chosen k-set intersecting W is bounded above by $k|W|/n$. Consequently, the number of such k-sets is at most

$$\frac{k|W|}{n}\binom{n}{k}.$$

This fact, and (3.5), (3.4), and (3.3), allow us to infer that

$$\begin{aligned}\mathbb{P}(\exists K \notin C(T),\ \overline{K} \text{ consistent with } <' \mid A_\omega) &< \frac{1}{k!-1}\cdot\frac{k|W|}{n}\binom{n}{k}\\ &\leqslant \frac{1}{k!-1}\cdot\frac{2\omega k}{\alpha n}\binom{n}{k}\\ &= \frac{2k\omega^2}{\alpha n}(1+o(1)).\end{aligned} \tag{3.6}$$

We apply

$$\omega = (\alpha/3e)k^{1/2}(1+o(1)) \quad \text{and} \quad n = \left(\frac{k}{e}\right)^2(1+o(1))$$

to the expression in (3.6) to obtain

$$\frac{2k\omega^2}{\alpha n}(1+o(1)) = \frac{2\alpha}{9}\frac{(k/e)^2}{n}(1+o(1)) = \frac{2\alpha}{9}(1+o(1)) < \frac{\alpha}{2}$$

for k sufficiently large. Thus,

$$\mathbb{P}(\exists\ K \notin C(T),\ \overline{K} \text{ consistent with } <' \mid A_\omega) < \alpha/2.$$

If T has Property O then either A_ω does not occur, or A_ω does occur and some $K \notin C(T)$ has $\overline{K}$ consistent with $<'$. Consequently, we have

$$\mathbb{P}(T \text{ has Property O}) \leqslant \mathbb{P}(A_\omega^c) + \mathbb{P}(A_\omega) \cdot \mathbb{P}(\exists K \notin C(T),\ \overline{K} \text{ consistent with } <' \mid A_\omega) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Hence, the probability that T fails to have Property O is at least $(1-\alpha)$. Since T is a uniform selection from $\mathcal{T}_{n,k}$, this is equivalent to saying at least $(1-\alpha)|\mathcal{T}_{n,k}|$ members of $\mathcal{T}_{n,k}$ fail to have Property O.

This completes the proof of Theorem 1.2.

4. A construction, small values of *n*, and problems

We have an upper bound for $f(k)$, the minimum number of edges in k-graphs with Property O, in Theorem 1.1: for k sufficiently large, $f(k) \leqslant (k^2 \ln k)k!$. We now construct k-graphs with Property O, for all $k \geqslant 2$. While these k-graphs have edge sets that are larger than the upper bound obtained by the probabilistic proof in Section 2, the hypergraphs are not unreasonably large.

For each $k \geqslant 2$ we construct an oriented k-graph $\mathcal{G}_k = (V_k, \mathcal{E}_k)$ that has Property O, where

$$|V_k| = 3^{k-1} \quad \text{and} \quad |\mathcal{E}_k| = 3^{\binom{k}{2}}. \tag{4.1}$$

To begin, let $\mathcal{G}_2 = (V_2, \mathcal{E}_2)$ be an oriented 3-cycle. It is clear that $\mathcal{G}_2$ has Property O and its vertex and edge sets have the sizes given in (4.1).

Here is the induction hypothesis: $\mathcal{G}_k = (V_k, \mathcal{E}_k)$ is an oriented k-graph with Property O and satisfies the conditions in (4.1). Let X, Y and Z be three disjoint copies of V_k and let $\mathcal{G}_X = (X, \mathcal{E}_X)$, $\mathcal{G}_Y = (Y, \mathcal{E}_Y)$ and $\mathcal{G}_Z = (Z, \mathcal{E}_Z)$ each be isomorphic to $\mathcal{G}_k$. Define $\mathcal{G}_{k+1} = (V_{k+1}, \mathcal{E}_{k+1})$ as follows (see Figure 1):

- let $V_{k+1} = X \cup Y \cup Z$; and
- let $\mathcal{E}_{k+1} = T_1 \cup T_2 \cup T_3$ with

$$T_1 = \{(\overline{x}, y) : \overline{x} \in \mathcal{E}_X \text{ and } y \in Y\},$$
$$T_2 = \{(\overline{y}, z) : \overline{y} \in \mathcal{E}_Y \text{ and } z \in Z\}, \quad \text{and}$$
$$T_3 = \{(\overline{z}, x) : \overline{z} \in \mathcal{E}_Z \text{ and } x \in X\}.$$

To see that $\mathcal{G}_{k+1}$ has Property O, let $<$ be any linear order on V_{k+1}. Since each of $\mathcal{G}_X$, $\mathcal{G}_Y$ and $\mathcal{G}_Z$ has Property O, there exist $\overline{x} \in \mathcal{E}_X$, $\overline{y} \in \mathcal{E}_Y$ and $\overline{z} \in \mathcal{E}_Z$ consistent with $<$. If $\max Y > \max X$, then $(\overline{x}, \max Y) \in T_1$ is consistent with $<$, so we may assume that $\max X > \max Y$. Similarly, since $(\overline{y}, \max Z) \in T_2$, we may assume that $\max Y > \max Z$, and, as $(\overline{z}, \max X) \in T_3$, we may assume that $\max Z > \max X$. Therefore $\max Z > \max X > \max Y > \max Z$, a contradiction. Therefore, $\mathcal{G}_{k+1}$ has Property O.

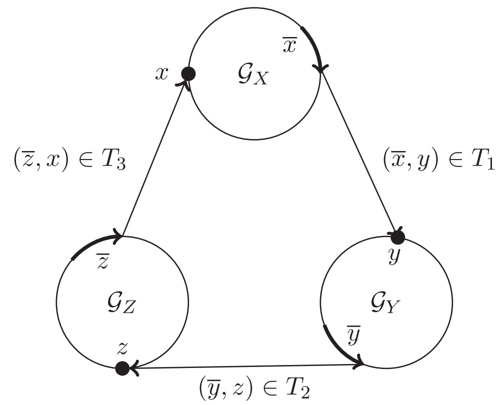

Figure 1. Constructing $\mathcal{G}_{k+1}$ from $\mathcal{G}_k$.

Let us see that $\mathcal{G}_{k+1} = (V_{k+1}, \mathcal{E}_{k+1})$ satisfies the conditions in (4.1). First, $|V_{k+1}| = 3|V_k| = 3 \cdot 3^{k-1} = 3^k$. Second,

$$|\mathcal{E}_{k+1}| = 3 \cdot |V_k| \cdot |\mathcal{E}_k| = 3 \cdot 3^{k-1} \cdot 3^{\binom{k}{2}} = 3^{\binom{k+1}{2}}.$$

Let $n(k)$ be the minimum number of vertices in a k-tournament with Property O. We have already seen that for any oriented k-graph to have Property O, it must have at least $k!$ edges. Since $\binom{n}{3} \geqslant 3!$ forces $n \geqslant 5$, we have $n(3) \geqslant 5$. An exhaustive computer search shows that there are no 3-tournaments on 5 vertices with Property O. However, the case where $n = 6$ is already much more time-consuming. On the other hand, from the construction above, we have an oriented 3-graph on 9 vertices which has Property O. Thus $n(3) \leqslant 9$. It remains an open question as to whether there exists a 3-tournament with Property O on $n = 6$, 7 or 8 vertices, hence the following problem.

Problem 2. *Find the minimum number of vertices $n(3)$ in a 3-tournament with Property O.*

Returning to the function $f(k)$, we would like to determine $f(3)$, the minimum number of edges in an oriented 3-graph with Property O. It is easily seen that $f(3) > 6$. For k in general, it would be interesting to find a construction that improves the upper bound in Theorem 1.1. Finally, we would like to settle Problem 1, that is, to determine whether $f(k)/k! \to \infty$. However, at present we are not even able to decide whether there is some $c > 1$ such that $f(k) > ck!$ holds for all k sufficiently large.

Remark. Recently, we have learned that the upper bound on $f(k)$ in Theorem 1.1 has been improved by a factor of $k \ln k$: for all $k \geqslant 3$, $f(k) \leqslant (\lfloor k/2 \rceil)k! - \lfloor k/2 \rfloor (k-1)!$; see [3]. This arXiv article also offers a solution to Problem 2: $n(3) = 6$.

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