CONVEX SUBLATTICES OF A LATTICE AND A FIXED POINT PROPERTY

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ABSTRACT. The collection $\mathcal{C}_L(T)$ of nonempty convex sublattices of a lattice T ordered by bi-domination is a lattice. We say that T has the *fixed point property for convex sublattices* (CLFPP for short) if every order preserving map $f : T \to \mathcal{C}_L(T)$ has a fixed point, that is $x \in f(x)$ for some $x \in T$. We examine which lattices may have CLFPP. We introduce the *selection property for convex sublattices* (CLSP); we observe that a complete lattice with CLSP must have CLFPP, and that this property implies that $\mathcal{C}_L(T)$ is complete. We show that for a lattice T, the fact that $\mathcal{C}_L(T)$ is complete is equivalent to the fact that T is complete and the lattice $\wp(\omega)$ of all subsets of a countable set, ordered by containment, is not order embeddable into T. We show that for the lattice $T := \mathcal{I}(P)$ of initial segments of a poset P, the implications above are equivalences and that these properties are equivalent to the fact that P has no infinite antichain. A crucial part of this proof is a straightforward application of a wonderful Hausdorff type result due to Abraham, Bonnet, Cummings, Džamondja and Thompson 2010 [1].

1. Introduction

Let E be a set. A *multivalued map* defined on E is a map f from E into $\wp(E)$, the power set of E, and a *fixed point* of f is an element $x \in E$ such that $x \in f(x)$. The consideration of the existence of fixed points for various kind of multivalued maps originates in analysis, with Kakutani's proof of von Neumann's minmax theorem. Investigation of fixed points of multi-valued maps occurs also in the study of partially ordered sets ([23], see [21], [25]), and that is our goal in this paper.

Given a poset $P := (E, \leq)$, we shall reserve the notation $\wp(E)$ for the power set of E equipped with the usual subset inclusion. We shall use instead $\mathfrak{P}(E) := \wp(E) \setminus \{\varnothing\}$ to denote the nonempty subsets of E endowed with the following *bi-dominating* preorder: for $A, B \subseteq E$, $A \leq B$ if every $x \in A$ is below some $y \in B$ and every $y \in B$ is above some $x \in A$. Note that we removed the empty

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set for convenience only, as that subset would be incomparable to any other subset in the bi-dominating ordering. A map $f : E \to \mathfrak{P}(E)$ such that $x \leq y$ implies $f(x) \leq f(y)$ is said to be *order preserving* (despite the fact that the bi-dominating preorder is not an order in general).

The poset P has the *relational fixed point* (RFPP for short) if every order preserving map $f : P \to \mathfrak{P}(P)$ has a fixed point. This property and the bi-dominating preorder were introduced by Walker in [25] (who used the phrase "isotone maps" for what we call "order preserving maps" and gave no name to the bi-dominating preorder), in connection with the fixed point property for the product of posets. He proved that a finite poset P has RFPP if and only if it is dismantlable. In this paper, we are primarily interested by a special case of this notion: P is a lattice T and multivalued maps take values in the set $\mathcal{C}_L(T)$ of nonempty convex sublattices of T. We say that T has the *fixed point property for convex sublattices* (CLFPP for short) if every order preserving map from T into $\mathcal{C}_L(T)$ has a fixed point. We concentrate on the following general question:

Question 1.1. *Which lattices have CLFPP?*

This property extends the fixed point property for order preserving maps of a lattice into itself. As is well known, *a lattice T has the fixed property if and only if T is complete* [24], [4]. Since CLFPP extends FPP, a lattice satisfying CLFPP must be complete. More is true: as we will see, $\mathcal{C}_L(T)$ must be complete too (see Proposition 4.16). It is a tempting conjecture that the converse holds; we have not yet succeeded in proving or disproving this. However, a main result of this paper characterizes lattices T such that $\mathcal{C}_L(T)$ is complete. This characterization is based on the nonembeddability of $\wp(\omega)$, the set of all subsets of ω, the set of nonnegative integers, ordered by containment.

Main Theorem 1. *Let T be a lattice. Then the following properties are equivalent:*

(i) $\mathcal{C}_L(T)$ is a complete lattice.
(ii) Every lattice quotient of every retract of T is complete.
(iii) T is a complete lattice and $\wp(\omega)$ is not order embeddable in T.

The implication $(iii) \Rightarrow (ii)$ is, under a seemingly weaker form, in [17].

Complete lattices in which $\wp(\omega)$ is not order embeddable arise naturally. A quite familiar example allows us to sharpen the preceding characterization for a particular family of complete lattices. (For more sophisticated examples, see [15, 16].) First, we introduce another property that is closely related to CLFPP and completeness of $C_L(T)$.

We say that a lattice T has the *selection property for convex sublattices* (CLSP) if there is an order preserving map φ from $\mathcal{C}_L(T)$ into T such that $\varphi(X) \in X$ for all $X \in \mathcal{C}_L(T)$. For complete lattices, CLSP implies CLFPP (Proposition 4.16).

Let us recall that a *closure* on a set E is a function φ defined on the subsets of E such that $X \subseteq \varphi(X) \subseteq \varphi(Y) = \varphi(\varphi(Y))$ whenever $X \subseteq Y \subseteq E$. A subset

X of E is *closed* if $\varphi(X) = X$; it is *independent* if $x \notin \varphi(X \setminus \{x\})$ for every $x \in X$. The set $\mathcal{C}_\varphi$ of closed sets is a complete lattice. Here is a well-known fact (see [3] Theorem 1.2 or [15]).

Fact 1.2. *The lattice $\wp(\omega)$ is not order embeddable in the complete lattice $\mathcal{C}_\varphi$ if and only if E contains no infinite independent set.*

If P is an ordered set, the map which associates $\downarrow X$, the initial segment generated by X, to each $X \subseteq P$ is a closure and the lattice of closed sets is the set $\mathcal{I}(P)$ of initial segments of P, ordered by containment. A subset S is independent with respect to this closure if and only if S is an antichain in P. Hence from Fact 1.2, we have that $\wp(\omega)$ *is not order embeddable in $\mathcal{I}(P)$ if and only if P contains no infinite antichain.*

In general we have the following.

Main Theorem 2. *Let P be a poset and $T := \mathcal{I}(P)$ be the lattice of initial segments of P. Then the following properties are equivalent:*

(i) T has CLSP;
(ii) T has CLFPP;
(iii) $\mathcal{C}_L(T)$ is a complete lattice;
(iv) P has no infinite antichain.

The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$, included in Proposition 4.16, are easy. The implication $(iii) \Rightarrow (iv)$ mixes the easy part of Theorem 1, namely implication $(i) \Rightarrow (iii)$, with Fact 1.2. The implication $(iv) \Rightarrow (i)$, the content of Lemma 4.32, is the cornerstone of the theorem. It follows from a Hausdorff type result due to Abraham, Bonnet, Cummings, Džamondja and Thompson 2010 [1].

From previous results in [8] and [18], we know that CLFPP holds for countable complete lattices and finite dimensional complete lattices. In Section 4 we look at the relationship between various properties which may hold for the class of lattices with CLSP, the class of lattices with CLFPP, as well as the class of lattices such that $\mathcal{C}_L(T)$ is complete. Results and problems are reviewed at the end of Section 5.

In Section 3, we consider also maps defined on a poset P and whose values are convex subsets of P. We establish the exact relationship between FPP for those maps (CFPP) and RFPP, which yields the following:

Main Theorem 3. *A poset has RFPP if and only if it has CFPP and contains no chain of the same type as the integers. In particular RFPP and CFPP coincide for finite posets.*

We illustrate the differences with CLFPP, the main illustration being Theorem 3.49.

In the following section, we present the required notation, terminology and preliminary results.

2. Preliminaries

Our notation is standard apart from one instance. Let f be a map with domain a set E. For $X \subseteq E$, let $f[X] = \{f(x) : x \in X\}$, rather than the more common $f(X)$. We denote by ω the order type of the set $\mathbb{N}$ of non negative integers and use $n \in \mathbb{N}$ as well as $n \in \omega$ and $n < \omega$. Similarly we denote by θ the order type of the chain $\mathbb{Z}$ of integers. We recall below some basic notions of the theory of ordered sets, but we refer to [13] for other undefined set theoretical notation.

Let $P := (E, \leq)$ be a poset. In the sequel we will denote by P the set E; that is we write $A \subseteq P$ to mean $A \subseteq E$. Let A be a subset of P. We say that A is an *initial segment* of P if $x \in P$, $y \in A$ and $x \leq y$ imply $x \in A$. We say that A is *up-directed* if every pair of elements of A has a common upper bound in A. An *ideal* of P is a nonempty up-directed initial segment of P. A *final segment*, resp. a down directed subset, resp. *filter*, of P is any initial segment, resp. up-directed subset, resp. ideal, of P^*, the dual of P.

We use this notation and terminology:

- $U(A) := \{x \in P : y \leq x \text{ for all } y \in A\}$ is the set of upper bounds of A and $L(A)$, the set of lower bounds, is defined dually;
- $\downarrow A := \{x \in P : x \leq y \text{ for some } y \in A\}$ is the *initial segment generated by* A and $\uparrow A$ is defined dually and called the *final segment generated by* A.

For a singleton $x \in P$, we use $\downarrow x$ instead of $\downarrow\{x\}$. If reference to P is needed, particularly in case of several orders on the same ground set, we may use the notation $\downarrow_P A$.

A subset A of P is *cofinal*, resp. *coinitial*, in P if $\downarrow A = P$, resp. $\uparrow A = P$. The *cofinality* of P, denoted by $cf(P)$ is the least cardinal κ such that P contains a cofinal subset of size κ.

We now review the basic notions associated with gaps. Let (A, B) be a pair of subsets of P.

- (A, B) is a *pregap* of P if $A \subseteq L(B)$ or, equivalently, if $B \subseteq U(A)$;
- (A, B) is *totally ordered* if $A \cup B$ is totally ordered via the order on P;
- If (A, B) is a pregap, set $S(A, B) := U(A) \cap L(B)$;
- A pregap (A, B) is called *separable* if $S(A, B) \neq \varnothing$;
- A pregap (A, B) is called a *gap* if $S(A, B) = \varnothing$.

Concerning the separation of pregaps, we recall that a poset is a complete lattice iff every pregap is separated. As it is easy to see a lattice is complete iff every pregap where A is up-directed and B down-directed is separated. It is known that this later condition is equivalent to the fact that every totally ordered pregap is separated, see [9].

We denote by $\mathcal{I}(P)$, resp. $\mathcal{I}d(P)$, the set of initial segments, resp. ideals of P ordered by inclusion. We denote by $\mathcal{F}(P)$, resp. $\mathcal{F}i(P)$, the set of final segments, resp. filters, of P ordered by inclusion.

Let A be subset of P. We say that A is *convex* if $x, y \in A$, $z \in P$ and $x \leq z \leq y$ imply $z \in A$. Such sets are also called *intervals*. If $a, b \in P$ we set $[a, b] := \{x \in P : a \leq x \leq b\}$. This set is a *closed interval.* A nonempty set is of this form if and only if it is convex with a least and largest element. A subset A of P is convex if and only if $A := I \cap F$ for some pair $(I, F) \in \mathcal{I}(P) \times \mathcal{F}(P)$. In particular, the set $Conv_P(A) := \downarrow A \cap \uparrow A$ $(=\{z \in P : x \leq z \leq y$ for some $(x, y) \in A^2\})$ is convex and called the *convex envelope* of A. The set of convex subsets of P is a closure system, and the convex envelope of A is nothing else than the closure of A. Ordered by inclusion, this set is a complete lattice (in fact an algebraic lattice) which is the object of several studies eg [2], [22]. We denote the set on nonempty convex subsets of P by $\mathcal{C}(P)$. In this paper we are concerned with $\mathcal{C}(P)$ under a different ordering, which is the subject of the next section.

3. The poset of convex subsets of a poset

Let P be a poset. As noted in the introduction, we use $\mathfrak{P}(P) := \wp(P) \setminus \{\varnothing\}$ to denote the nonempty subsets of (the domain of) P endowed with the following *bi-dominating* preorder. For $X, Y \in \mathfrak{P}(P)$ define:

$$X \leq Y \text{ if } X \subseteq \downarrow Y \text{ and } Y \subseteq \uparrow X. \tag{1}$$

We also set:

$$X \equiv Y \text{ if } X \leq Y \text{ and } Y \leq X. \tag{2}$$

Lemma 3.1. *The relation $\leq$ on $\mathfrak{P}(P)$ is a preorder. It induces an order on $\mathcal{C}(P)$ which is isomorphic to the quotient of this preorder by the equivalence associated with the preorder.*

Proof. The first part of this lemma is obvious. The second part relies on the following facts both of which are straightforward to prove:

$$Conv(X) \equiv X, \text{ and}$$

$$X \equiv Y \Leftrightarrow Conv(X) = Conv(Y)$$

for all $X, Y \in \mathfrak{P}(P.)$ □

Note that we could have defined this preorder on the collection of all subsets, but then the empty set would have been incomparable to every nonempty subset. Therefore it is avoided.

The poset $\mathcal{C}(P)$ has a simple representation:

Fact 3.2. *The map ϕ from $\mathcal{C}(P)$ to the product $\mathcal{I}(P) \times \mathcal{F}(P)^*$, defined by $\phi(A) := (\downarrow A, \uparrow A)$ is an embedding.*

Remark 3.3. *Let $X, Y \in \mathcal{C}(P)$. If $Y \subseteq X$ then $X \leq Y$ if and only if $X \subseteq \downarrow Y$, that is Y cofinal in X. Consequently,* a convex subset of P is above P w.r.t. the bi-domination order if and only if this is a final segment of P which is cofinal in P. *Subsets of a poset with these properties are called* open dense sets *and are well known in the literature in particular regarding Baire spaces.*

Definition 3.4. *Let P be a poset. We say that P has the* convex fixed point property *(CFPP for short) if every order preserving map $f : P \to \mathcal{C}(P)$ has a fixed point.*

The relationship between RFPP and CFPP is quite simple and was given in Main Theorem 3.

type θ. In particular RFPP and CFPP coincide for finite posets.

The proof is related to ideas due to Walker. The first involves the notion of retraction, where a poset Q is called a *retract* of another poset P if there are order preserving maps $s : Q \to P$ and $r : P \to Q$ such that $r \circ s = 1_Q$. The maps r and s are called a *retraction* and a *coretraction*.

Lemma 3.5. *RFPP is preserved under retracts.*

Proof. Let P be a poset satisfying RFPP, Q an order retract of P, and $g : Q \to \mathfrak{P}(Q)$ an order preserving map. Let $s : Q \to P$ and $r : P \to Q$ be two order preserving maps such that $r \circ s = 1_Q$. The map $h : P \to \mathfrak{P}(P)$ defined by $h(x) := s[g(r(x)]$ is order preserving, hence it has a fixed point, say u. Since $u \in h(u)$ there is some $y \in g(r(u))$ such that $u = s(y)$. We have $r(u) = r(s(y)) = y$, thus $y \in g(y)$ and hence y is a fixed point of g. □

The second idea involves the notion of well foundedness.

Definition 3.6. *We recall that a poset P is* well founded *if every nonempty subset contains some minimal element; equivalently, P contains no infinite chain of type ω^*.*

The next lemma is essentially Proposition 5.2 of [25].

Lemma 3.7. *Le P be a poset and $f : P \to \mathfrak{P}(P)$ be a preorder preserving map. If there exists some $x \in P$ such that $\downarrow x$ is well founded and meets $f(x)$ then f has a fixed point.*

Proof. Define by induction a sequence $(x_n)_{n<\omega}$ of elements of P. Let $m < \omega$ and suppose $(x_n)_{n<m}$ be defined. If $m = 0$, set $x_m := x$. Otherwise, choose $x_m \in f(x_{m-1}) \cap \downarrow x_{m-1}$. This sequence is well defined. It is descending, thus stationary, hence it yields a fixed point.

□

Proof of Theorem 3. Let P be a poset. Suppose that P has RFPP. Trivially, it has CFPP. Suppose that it contains a chain Z of type θ. Extend this chain to a maximal chain C. As a maximal chain of P, this is a retract of P by [8]. Since RFPP is preserved under retraction (by Lemma 3.5), C has RFPP. In particular, C has FPP, hence C is complete. Hence Z has an infimum a and a supremum b. As it is easy to see, the chain $D := \{a\} \cup Z \cup \{b\}$ is a retract of C, thus it has RFPP. But this is trivially false. Indeed, let $f : D \to \mathfrak{P}(D)$ defined by $f(x) := Z \setminus \{x\}$ if $x \in Z$ and $f(x) := Z$ if $x \in \{a, b\}$. This map is order preserving but has no fixed point. Thus P cannot contain a chain of type θ.

Conversely, suppose that P has CFPP and contains no chain of type θ. Let $f : P \to \mathfrak{P}(P)$ be an order preserving map. Let $\overline{f} : P \to \mathcal{C}(P)$ defined by setting $\overline{f}(x) := Conv_P(f(x))$. This map is order preserving too. Thus it has a fixed point x. Since $x \in \overline{f}(x) := Conv_P(f(x))$ there are $u, v \in f(x)$ such that $u \le x \le v$. Since P contains no chain of type θ, either $\downarrow x$ is well founded or $\uparrow x$ is dually well founded. W.l.o.g we may suppose that $\downarrow x$ is well founded (otherwise consider P^*). Apply Lemma 3.7 and we obtain that f has a fixed point. □

3.1. **FPP for the poset of convex subsets of a poset.** Since the set $\mathcal{C}(P)$ of nonempty convex subsets of a poset P is a poset, a straightforward question emerges:

Question 3.8. *How can we relate FPP for $\mathcal{C}(P)$ and CFPP for P?*

This simple minded question is at the root of this paper. As we will see, there is no relation in general. There are posets P without CFPP for which $\mathcal{C}(P)$ has FPP; a straightforward example is the ordinal sum of two 2-element chains (see Example 3.31). And, on an other hand, there are posets P with CFPP but such that $\mathcal{C}(P)$ does not have FPP (see Lemma 3.22). According to Theorem 3.32 below these posets are infinite.

The example we give in Lemma 3.22 relies on the well-known fact that a poset containing a totally ordered gap does not have FPP. With that in hand, we construct a complete lattice $\overline{Q}$ with CFPP such that $\mathcal{C}(\overline{Q})$ contains a totally ordered gap.

We start our discussion with some simple facts about pregaps, namely a necessary condition for separation and the fact that each gap yields a gap having a special form.

Let P be a poset. If $\mathcal{A} \subseteq \mathcal{C}(P)$, we set $I_{\mathcal{A}} := \bigcap\{\downarrow A : A \in \mathcal{A}\}$ and $F_{\mathcal{A}} := \bigcap\{\uparrow A : A \in \mathcal{A}\}$. If $(\mathcal{A}, \mathcal{B})$ is a pregap of $\mathcal{C}(P)$ we set $\mathcal{A}_{\mathcal{B}} := \{I_{\mathcal{B}} \cap \uparrow A : A \in \mathcal{A}\}$ and $\mathcal{B}_{\mathcal{A}} := \{F_{\mathcal{A}} \cap \downarrow B : B \in \mathcal{B}\}$.

Lemma 3.9. *Let $(\mathcal{A}, \mathcal{B})$ be a pregap of $\mathcal{C}(P)$.*

- *(i) If $(\mathcal{A}, \mathcal{B})$ is separable then $F_{\mathcal{A}} \cap I_{\mathcal{B}}$ separates it; furthermore it contains every separator. In particular $F_{\mathcal{A}} \cap I_{\mathcal{B}}$ is nonempty.*
- *(ii) $(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}})$ is a pregap of $\mathcal{C}(P)$ such that $F_{\mathcal{A}_{\mathcal{B}}} = F_{\mathcal{A}}$, $I_{\mathcal{B}_{\mathcal{A}}} = I_{\mathcal{B}}$ and $S(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}}) \subseteq S(\mathcal{A}, \mathcal{B})$. In particular $(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}})$ is a gap whenever $(\mathcal{A}, \mathcal{B})$ is a gap.*

Proof. Item (i). Set $Z := F_{\mathcal{A}} \cap I_{\mathcal{B}}$. Let $C \in S(\mathcal{A}, \mathcal{B})$. For every $A \in \mathcal{A}$, $B \in \mathcal{B}$ we have $C \subseteq \uparrow A$ and $C \subseteq \downarrow B$ hence $C \subseteq Z$. Let $A \in \mathcal{A}$. We have trivially $Z \subseteq \uparrow A$. We have $A \subseteq \downarrow C$ and $C \subseteq Z$ hence $A \subseteq \downarrow Z$ thus $A \le Z$. By the same token, we have $Z \le B$ if $B \in \mathcal{B}$ hence $Z \in S(\mathcal{A}, \mathcal{B})$.

Item (ii). We first prove two basic claims. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Claim 3.10. $\uparrow(I_{\mathcal{B}} \cap \uparrow A) = \uparrow A$ *and* $\downarrow(F_{\mathcal{A}} \cap \downarrow B) = \downarrow B$.

Proof. We prove only the first equality. We have $A \subseteq I_{\mathcal{B}}$ since $(\mathcal{A}, \mathcal{B})$ is a pregap. This yields $A \subseteq \uparrow(I_{\mathcal{B}} \cap \uparrow A)$. Thus $\uparrow A \subseteq \uparrow(I_{\mathcal{B}} \cap \uparrow A)$. The reverse inclusion holds since $I_{\mathcal{B}} \cap \uparrow A$ is a subset of $\uparrow A$. □

Claim 3.11. $A \le I_{\mathcal{B}} \cap \uparrow A \le F_{\mathcal{A}} \cap \downarrow B \le B$.

Proof. We have trivially $I_{\mathcal{B}} \cap \uparrow A \subseteq \uparrow A$. Since, as seen in Claim 3.10 above, $A \subseteq I_{\mathcal{B}}$ we have $A \subseteq \downarrow (I_{\mathcal{B}} \cap \uparrow A)$. The inequality $A \leq I_{\mathcal{B}} \cap \uparrow A$ follows. Similarly, we have $F_{\mathcal{A}} \cap \downarrow B \leq B$. Finally, from $\mathcal{I}_{\mathcal{B}} \subseteq \downarrow B$ and Claim 3.11 above, we have $I_{\mathcal{B}} \cap \uparrow A \subseteq \downarrow B = \downarrow (F_{\mathcal{A}} \cap \downarrow B)$. Similarly, we have $(F_{\mathcal{A}} \cap \downarrow B) \subseteq \uparrow (I_{\mathcal{B}} \cap \uparrow A)$, proving $I_{\mathcal{B}} \cap \uparrow A \leq F_{\mathcal{A}} \cap \downarrow B$. □

Now back to the proof of (ii), the equality $F_{\mathcal{A}_{\mathcal{B}}} = F_{\mathcal{A}}$ follows from the first part of Claim 3.10; the equality $I_{\mathcal{B}_{\mathcal{A}}} = I_{\mathcal{B}}$ follows from the second part. From Claim 3.11, we have $S(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}}) \subseteq S(\mathcal{A}, \mathcal{B})$. □

These particular gaps will play an important role and thus deserve a dedicated nomenclature.

Definition 3.12. *A pregap $(\mathcal{A}', \mathcal{B}')$ of the form $(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}})$ will be called* special.

The following result on special pregaps relies on the simple fact that a nonempty final segment of an up-directed poset is cofinal in that poset.

Lemma 3.13. *Let $(\mathcal{A}', \mathcal{B}')$ be a special pregap of $\mathcal{C}(P)$ and let $(\mathcal{A}, \mathcal{B})$ such that $(\mathcal{A}', \mathcal{B}') = (\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}})$. If $I_{\mathcal{B}}$ and $F_{\mathcal{A}}$ are respectively up and down directed, then*

(i) The order on $\mathcal{A}'$ coincides with the reverse of the inclusion and the order on $\mathcal{B}'$ coincides with the inclusion.

(ii) The map $A \hookrightarrow I_{\mathcal{B}} \cap \uparrow A$ from $\mathcal{A}$ onto $\mathcal{A}'$ and the map $B \hookrightarrow F_{\mathcal{A}} \cap \downarrow B$ from $\mathcal{B}$ onto $\mathcal{B}'$ are two order preserving maps.

(iii) $\downarrow A' = I_{\mathcal{B}'}$ and $\uparrow B' = F_{\mathcal{A}'}$ for all $A' \in \mathcal{A}'$, $B' \in \mathcal{B}'$;

(iv) The following properties are equivalent:

(a) $(\mathcal{A}', \mathcal{B}')$ is a gap;

(b) $F_{\mathcal{A}'} \cap I_{\mathcal{B}'} = F_{\mathcal{A}} \cap I_{\mathcal{B}} = \varnothing$;

(c) $(\mathcal{A}, \mathcal{B})$ is a gap.

Proof. We begin with a claim.

Claim 3.14. *If $I_{\mathcal{B}}$ is up-directed then $\downarrow (I_{\mathcal{B}} \cap \uparrow A) = I_{\mathcal{B}}$ for every $A \in \mathcal{A}$.*

Proof. Indeed, let $A \in \mathcal{A}$. Since $(\mathcal{A}, \mathcal{B})$ is a pregap we have $A \subseteq I_{\mathcal{B}}$. Hence, the final segment $I_{\mathcal{B}} \cap \uparrow A$ of $I_{\mathcal{B}}$ is nonempty. Since $I_{\mathcal{B}}$ is up directed, $I_{\mathcal{B}} \cap \uparrow A$ is cofinal in $I_{\mathcal{B}}$ that is $\downarrow (I_{\mathcal{B}} \cap \uparrow A) = I_{\mathcal{B}}$, proving our claim. □

Item (i). Let $A, A' \in \mathcal{A}$. Suppose $I_{\mathcal{B}} \cap \uparrow A \leq I_{\mathcal{B}} \cap \uparrow A'$. Necessarily, we have $I_{\mathcal{B}} \cap \uparrow A' \subseteq \uparrow (I_{\mathcal{B}} \cap \uparrow A)$. It follows that $I_{\mathcal{B}} \cap \uparrow A' \subseteq I_{\mathcal{B}} \cap \uparrow A$. Conversely, suppose $I_{\mathcal{B}} \cap \uparrow A' \subseteq I_{\mathcal{B}} \cap \uparrow A$. Then trivially, $I_{\mathcal{B}} \cap \uparrow A' \subseteq \uparrow (I_{\mathcal{B}} \cap \uparrow A)$. With Claim 3.14 we have $(I_{\mathcal{B}} \cap \uparrow A) \subseteq I_{\mathcal{B}} = I_{\mathcal{B}} \cap \uparrow A'$, hence $I_{\mathcal{B}} \cap \uparrow A \leq I_{\mathcal{B}} \cap \uparrow A'$ as required.

Item (ii). Now, let $A, A' \in \mathcal{A}$. According to Claim 3.14, $\downarrow (I_{\mathcal{B}} \cap \uparrow A) = I_{\mathcal{B}} = \downarrow (I_{\mathcal{B}} \cap \uparrow A')$. Suppose $A \leq A'$. Then, in particular $\uparrow A' \subseteq \uparrow A$ hence $I_{\mathcal{B}} \cap \uparrow A' \subseteq \uparrow (I_{\mathcal{B}} \cap \uparrow A)$. The inequality $I_{\mathcal{B}} \cap \uparrow A \leq I_{\mathcal{B}} \cap \uparrow A'$ follows. Hence, the map $A \hookrightarrow I_{\mathcal{B}} \cap A$ is order preserving as claimed. Since $F_{\mathcal{A}}$ is down directed, the same property holds for the map $B \hookrightarrow F_{\mathcal{A}} \cap \downarrow B$.

Item (iii) According to Claim 3.14, $\downarrow A' = I_{\mathcal{B}}$; since $I_{\mathcal{B}} = I_{\mathcal{B}'}$ we have $\downarrow A' = I_{\mathcal{B}'}$. Similarly, $\uparrow B' = F_{\mathcal{A}'}$ for every $B' \in \mathcal{B}$ and thus (iii) holds.

Item (iv). We prove the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. Suppose that (b) does not hold. Since $I_{\mathcal{B}'} = I_{\mathcal{B}}$ and $F_{\mathcal{A}'} = F_{\mathcal{A}}$, this amounts to $F_{\mathcal{A}} \cap I_{\mathcal{B}} \neq \varnothing$. We claim that $F_{\mathcal{A}} \cap I_{\mathcal{B}}$ separates $(\mathcal{A}', \mathcal{B}')$, that is $I_{\mathcal{B}} \cap \uparrow A \leq F_{\mathcal{A}} \cap I_{\mathcal{B}} \leq F_{\mathcal{A}} \cap \downarrow B$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, hence (a) does not hold. Indeed, let $A \in \mathcal{A}$. With Claim 3.10, we have $F_{\mathcal{A}} \cap I_{\mathcal{B}} \subseteq \uparrow A = \uparrow (I_{\mathcal{B}} \cap \uparrow A)$. Also, $F_{\mathcal{A}} \cap I_{\mathcal{B}}$ is a nonempty final segment of $I_{\mathcal{B}}$. This later set being up-directed, $\downarrow (F_{\mathcal{A}} \cap I_{\mathcal{B}}) = I_{\mathcal{B}}$. Since $\downarrow (I_{\mathcal{B}} \cap \uparrow A) = I_{\mathcal{B}}$ from Claim 3.14 it follows that $I_{\mathcal{B}} \cap \uparrow A \subseteq \downarrow (F_{\mathcal{A}} \cap I_{\mathcal{B}})$. Thus $I_{\mathcal{B}} \cap \uparrow A \leq F_{\mathcal{A}} \cap I_{\mathcal{B}}$. The proof that $F_{\mathcal{A}} \cap I_{\mathcal{B}} \leq F_{\mathcal{A}} \cap \downarrow B$ for $B \in \mathcal{B}$ is similar. Implication $(b) \Rightarrow (c)$ is the contraposition of Item (i) of Lemma 3.9. Implication $(c) \Rightarrow (a)$ is contained in Item (ii) of Lemma 3.9. □

Lemma 3.15. *If $\mathcal{C}(P)$ contains a totally ordered pregap $(\mathcal{A}, \mathcal{B})$ such that $F_{\mathcal{A}} \cap I_{\mathcal{B}} = \varnothing$, then P does not have CFPP.*

Proof. W.l.o.g we may suppose that $\mathcal{A} := \{A_\alpha : \alpha < \mu\}$ and $\mathcal{B} := \{B_\beta : \beta < \lambda\}$ are a well ordered chain and a dually well ordered chain in $\mathcal{C}(P)$ satisfying: $A_\alpha < A_\gamma$ if and only if $\alpha < \gamma < \mu$ and $B_\delta < B_\beta$ if and only if $\beta < \delta < \lambda$. Define $f : P \to \mathcal{C}(P)$ as follows. Let $x \in P$. If $x \notin I_{\mathcal{B}}$, set $f(x) := B_\beta$ where β is minimum such that $x \notin \downarrow B_\delta$. If $x \in I_{\mathcal{B}}$ then $x \notin F_{\mathcal{A}}$ and we set $f(x) := A_\alpha$ where α is minimum such that $x \notin \uparrow A_\alpha$. It is easy to see that this map is order preserving. By construction it has no fixed point, hence P does not have CFPP. □

Remark 3.16. *The set $\mathcal{C}(P)$ of (nonempty) convex subsets of a poset P may contain a gap $(\mathcal{A}, \mathcal{B})$ which is totally ordered and such that $F_{\mathcal{A}} \cap I_{\mathcal{B}}$ is nonempty. In this case, $\mathcal{C}(P)$ does not have FPP. But, one cannot use this gap to construct a fixed point free map from P to $\mathcal{C}(P)$ as in Lemma 3.15 above (See Example 3.21).*

The following fact follows from Lemma 3.7:

Fact 3.17. *A well founded poset with a largest element has CFPP.*

Definition 3.18. *We recall that a poset P is a* well-quasi-order *(wqo for short) if it contains no infinite antichain and no infinite descending chain.*

A well-known result of Higman [11] shows that if P is well-quasi-ordered, then $\mathcal{I}(P)$ is well founded. Here we have the following.

Fact 3.19. *If P is wqo then $\mathcal{C}(P)$ is well founded.*

Proof. According to Fact 3.2, $\mathcal{C}(P)$ is embeddable into the direct product $\mathcal{I}(P) \times \mathcal{F}(P)^*$. Since $\mathcal{F}(P)^*$ is isomorphic to $\mathcal{I}(P)$, each factor of this product is well founded. It turns out that the product is well founded, and that its subsets are well founded too. □

Lemma 3.20. *If P is wqo with a largest element, then $\mathcal{C}(P)$ has FPP.*

Proof. Let $Q := \mathcal{C}(P)$. Then Q has a largest element (namely $\{a\}$ where a is the largest element of P) and is well founded (Fact 3.19). Thus it has FPP. □

As the following example shows, the well foundedness of P is not enough in Lemma 3.20.

Example 3.21. *Let $F := \{a, b, c\}$ be a three element set and $Q := F \times \mathbb{N}$. For $x \in F$ set $x_n := (x, n)$. Order Q in such a way that $a_n < b_n$, $c_m < b_n$ and $c_m \le c_n$ for all $m \le n$. With a top 0 and bottom 1 added to Q, the resulting poset $\overline{Q}$ is a lattice, in fact a complete lattice.*

Claim 3.22. *The complete lattice $\overline{Q}$ has CFPP but $\mathcal{C}(\overline{Q})$ does not have FPP.*

Proof. $\overline{Q}$ has a largest element and is well founded, thus it has CFPP (Fact 3.17). $\mathcal{C}(\overline{Q})$ does not have FPP because it contains an (ω, ω^*)-gap.

Indeed let:

$$\begin{array}{ll} A = \{a_n : n \in \mathbb{N}\}, & A_{\ge n} = \{a_m : m \ge n\} \\ B = \{b_n : n \in \mathbb{N}\}, & B_{\ge n} = \{b_m : m \ge n\} \\ C = \{c_n : n \in \mathbb{N}\}, & C_{\ge n} = \{c_m : m \ge n\} \\ F_n = \uparrow(A \cup \{c_n\}), & F = \bigcap F_n \\ I_n = \downarrow(A \cup B_{\ge n}), & I = \bigcap I_n \end{array}$$

Then a straightforward calculation yields:

$$\begin{array}{ll} F_n = A \cup B \cup C_{\ge n} \cup \{1\} & F = A \cup B \cup \{1\} \\ I_n = A \cup B_{\ge n} \cup C \cup \{0\} & I = A \cup C \cup \{0\} \end{array}$$

Now let $A_n = F_n \cap I = A \cup C_{\ge n}$ and $B_n = I_n \cap F = A \cup B_{\ge n}$, and define $\mathcal{G} := (\mathcal{A}, \mathcal{B})$ where $\mathcal{A} := \{A_n : n \in \mathbb{N}\}$, $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$. We have $A_n < A_{n+1} \le B_{m+1} < B_m$ for all $n, m \in N$, hence $\mathcal{G}$ is a totally ordered pregap; we also have $\downarrow B_n = I_n$, $\uparrow A_n = F_n$, hence $F_{\mathcal{A}} = F$, $I_{\mathcal{B}} = I$ and $F_{\mathcal{A}} \cap I_{\mathcal{B}} = A$. If $\mathcal{G}$ was a separable pregap then $F_{\mathcal{A}} \cap I_{\mathcal{B}}$ would separate it (Lemma 3.9); since $A_n \nleq A$ this is not the case. Thus $\mathcal{G}$ is a gap.

□

Fact 3.17 and Lemma 3.20 immediately yield that *a finite poset P with a largest element has CFPP and $\mathcal{C}(P)$ has FPP.* As we will see in Theorem 3.32 this property extends to dismantlable posets. For that, we need some properties of retracts.

Lemma 3.23. *If Q is an order retract of P via the maps $s : Q \to P$ and $r : P \to Q$ then $\mathcal{C}(Q)$ is an order retract of $\mathcal{C}(P)$ via the maps $\overline{s} : \mathcal{C}(Q) \to \mathcal{C}(P)$ and $\overline{r} : \mathcal{C}(P) \to \mathcal{C}(Q)$ defined by $\overline{s}(Y) := Conv_P(s[Y])$ and $\overline{r}(X) := Conv_Q(r[X])$ for all $Y \in \mathcal{C}(Q)$ and $X \in \mathcal{C}(P)$.*

Proof. As it is easy to check, the maps $\overline{s}$ and $\overline{r}$ are order preserving. To conclude it suffices to prove that $\overline{r} \circ \overline{s}$ is the identity on $\mathcal{C}(Q)$. Let $Y \in \mathcal{C}(Q)$. Since $s[Y] \subseteq Conv_P(s[Y])$ we have $Y = r[s[Y]] \subseteq r[Conv_P(s[Y])] \subseteq Conv_Q(r[Conv_P(s[Y])]) = \overline{r}(\overline{s}(Y))$. Observing that $r[Conv_P(Z)] \subseteq conv_Q(r[Z])$ for every subset Z of P, we have $r[Conv_P(s[Y])] \subseteq Conv_Q(r[s[Y]]) = Conv_Q[Y] =$

Y and since Y is convex $\overline{r}(\overline{s}(Y)) = Conv_Q(r[Conv_P(s[Y])]) \subseteq Y$. Thus $\overline{r}(\overline{s}(Y)) = Y$. □

Since FPP is preserved under retraction, we have immediately the following:

Corollary 3.24. *If Q is an order retract of P and $\mathcal{C}(P)$ has FPP then $\mathcal{C}(Q)$ has FPP.*

Corollary 3.25. *CFPP is preserved under retraction.*

Proof. Let P be a poset satisfying CFPP. Suppose that Q is an order retract of P and let $g : Q \to \mathcal{C}(Q)$ be an order preserving map. Let $s : Q \to P$ and $r : P \to Q$ be two order preserving maps such that $r \circ s = 1_Q$ and let $\overline{s} : \mathcal{C}(Q) \to \mathcal{C}(P)$ and $\overline{r} : \mathcal{C}(P) \to \mathcal{C}(Q)$ be given by Lemma 3.23 above. The map $h : P \to \mathcal{C}(P)$ defined by $h := \overline{s} \circ g \circ r$ is order preserving, hence it has a fixed point, say x. We claim that $y := r(x)$ is a fixed point of g. Indeed, since $x \in h(x)$, $r(x) \in \overline{r}(h(x)) = \overline{r} \circ \overline{s}(g(r(x))) = g(r(x))$, proving our claim. □

Definition 3.26. *An element x of a poset P is an* irreducible *of P if either $\{y \in P : y < x\}$ has a largest element or else or $\{y \in P : y > x\}$ has a least element.*

We note that if x is an irreducible of P then P_{-x}, the poset obtained from P by deleting x, is a retract of P.

Lemma 3.27. *Let P be a finite poset and x be an irreducible of P. Then $\mathcal{C}(P)$ has FPP if and only if $\mathcal{C}(P_{-x})$ has FPP.*

Proof. Since P_{-x} is a retract of P, then $\mathcal{C}(P_{-x})$ is a retract of $\mathcal{C}(P)$ (by Lemma 3.23). Since FPP is preserved under retraction, if $\mathcal{C}(P)$ has FPP, $\mathcal{C}(P_{-x})$ has FPP too.

Conversely, suppose that $\mathcal{C}(P_{-x})$ has FPP, and let $f : \mathcal{C}(P) \to \mathcal{C}(P)$ be an order preserving map. We prove that f has a fixed point. For that, we will set $Q := P_{-x}$, denote by s the identity map from Q to P, suppose that $\{y \in P : y < x\}$ has a largest element x^-, and denote by r the retraction map defined on P by $r(x) := x^-$ and $r(y) = y$ for all $y \neq x$. Let $\overline{s}$ and $\overline{r}$ the maps defined in Lemma 3.23 (that is $\overline{s}(Y) := Conv_P(Y)$ and $\overline{r}(X) := Conv_Q(r[X])$ if $Y \in \mathcal{C}(Q)$ and $X \in \mathcal{C}(Q)$). The map $g := \overline{r} \circ f \circ \overline{s}$ has a fixed point Y. Set $X := \overline{s}(Y)$.

Claim 3.28. $X \leq f(X)$.

Proof. Note that $Y \subseteq X \subseteq Y \cup \{x\}$ and $Y \subseteq f(X) \subseteq Y \cup \{x\}$.
Case 1. $X = Y$, that is $x \notin X$.

Subcase 1. $x \notin f(X)$. In this case, $\overline{r}(f(X)) = f(X)$ thus $Y = f(X)$ and since $X = Y$, $X = f(X)$ proving our claim.

Subcase 2. $x \in f(X)$. In this case $f(X) = X \cup \{x\}$ and $x^- \in X$. Since $x^- \leq x$ this yields $X \leq f(X)$.
Case 2. $x \in X$. In this case $X = Y \cup \{x\}$ and $f(X) = X$. The first equality amounts to $x \in X$. For the second note that there are $a, b \in Y$ such that

$a \le x \le b$. Since $Y \subseteq f(X)$ this yields $Y \cup \{x\} \subseteq f(X)$. Since $f(X) \subseteq Y \cup \{x\}$, this gives $f(X) = Y \cup \{x\}$. □

Now, set $X_0 := X$ and $X_{n+1} := f(X_n)$ for every $n \in \mathbb{N}$. We have $X_n \le X_{n+1}$. Since P is finite, $\mathcal{C}(P)$ is finite too, hence the sequence is stationary. Its largest element is a fixed point of f.

Now, suppose that $\{y \in P : y > x\}$ has a least element. Since $\mathcal{C}(Q^*)$ is the dual of $\mathcal{C}(Q)$ it has FPP, thus the proof above tells us that $\mathcal{C}(P^*)$ has FPP, hence by the same token $\mathcal{C}(P)$ has FPP. □

We mention here that we do not know if the the finiteness assumption in Lemma 3.27 can be removed.

Definition 3.29. *We recall that a finite poset P is* dismantlable *if there is an enumeration $x_0, \dots, x_{n-1}$ of its elements such that x_i is irreducible in $P \setminus \{x_j : j < i\}$ for every $i < n-1$.*

For example every finite poset with a least, or a largest, element is dismantlable.

Corollary 3.30. *If a finite poset P is dismantlable then $\mathcal{C}(P)$ has FPP.*

Proof. We argue by induction on the cardinality n of P. If $n \le 1$, P has FPP. If $n \ge 2$ then P contains an irreducible element x such that P_{-x} is dismantlable. By induction $\mathcal{C}(P_{-x})$ has FPP. According to Lemma 3.27 $\mathcal{C}(P)$ has FPP. □

The following example shows that the converse does not holds.

Example 3.31. *Let P be the ordinal sum of two antichains $\{a, b\}$ and $\{c, d\}$. Since P has no irreducible it is not dismantlable. On the other hand $\mathcal{C}(P)$ has FPP. Indeed, let $f : \mathcal{C}(P) \to \mathcal{C}(P)$ be an order preserving map. If there is some $X \in \mathcal{C}(P)$ such that $f(X)$ is comparable to X then, since $\mathcal{C}(P)$ is finite, f has a fixed point. Let $X := \{c, d\}$ and $Y := \{a, b\}$. We may suppose that $f(X)$ is incomparable to X and $f(Y)$ is incomparable to Y. The elements incomparable to X are $\{c\}$ and $\{d\}$, whereas the elements incomparable to Y are $\{a\}$ and $\{b\}$. With no loss of generality, we may suppose that $f(X) =\{c\}$andf(Y)=$\{a\}$. We have $Y \le \{a, c\} \le X$, hence $\{a\} \le f(\{a, c\}) \le \{c\}$. But then $f(\{a, c\})$ is comparable to $\{a, c\}$. Thus f has a fixed point.*

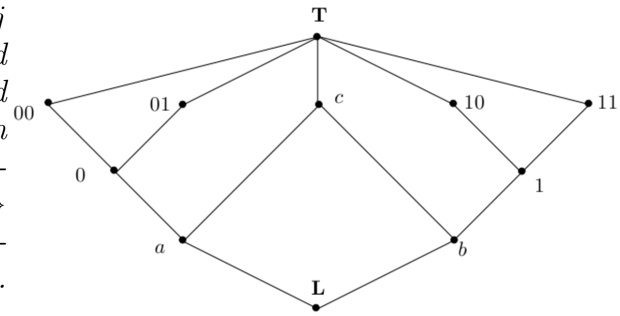
Walker's characterization of finite posets with RFPP leads to:

Theorem 3.32. *If a finite poset P has CFPP then $\mathcal{C}(P)$ has FPP.*

Indeed, if P finite has CFPP then it has RFPP (Theorem 3). According to Walker it is dismantlable, thus from Corollary 3.30, $\mathcal{C}(P)$ has FPP.

3.2. **Lattices properties and CFPP.** Lattice properties do not easily transfer from a poset P to the poset $\mathcal{C}(P)$. We give in Example 3.33 a finite lattice P such that $\mathcal{C}(P)$ is not a lattice (for an example of infinite and complete P see Example 3.34). Note that since P and $\mathcal{C}(P)$ are finite with a largest element, both have CFPP (Fact 3.17).

Example 3.33. *Let* $Q := \{a, b, 0, 1, c\} \cup \{ij : i < 2, j < 2\}$ *be the* 9*-element poset whose covering pairs are* $i < ij$ *for* $i, j < 2$, $a < 0$, $a < c$, $b < c$, $b < 1$, *and let* P *be obtained by adding a least and a largest element to* Q. *Then* P, *shown here, is a lattice. Furthermore, the subsets* $X := \{00, c, 11\}$ *and* $Y := \{01, 10\}$ *are two convex subsets (they are antichains of* P*) which have no infimum. Indeed, let* $Z := (\downarrow X) \cap (\downarrow Y) = \downarrow\{0, 1\}$. *This is a lower bound of* X *and* Y. *Inside,* $Z_0 := \{0, a, 1\}$ *and* $Z_1 := \{0, b, 1\}$ *are two maximal lower bounds.*



On the other hand, we give in Example 3.34 below an example of a complete lattice T without CFPP and such that $\mathcal{C}(T)$ does not have FPP. This lattice contains no chain of type θ.

We recall that *a poset* P *contains no chain of type* θ *if and only if it is the union of a well founded initial segment and a dually well founded final segment*. Indeed, set $P^- := \{x \in P : \downarrow x \text{ is well founded}\}$ and $P^+ := \{x \in P : \uparrow x \text{ is dually well founded}\}$. These two sets are an initial segment and a final segment respectively. As it is easy to see, their union is P iff P contains no chain of type θ.

Example 3.34. *Let* $2^{<\omega}$, *resp.* 2^ω, *be the set of finite sequences, resp. of* ω*-sequences of* 0 *and* 1. *Let* $s \in 2^{<\omega} \cup 2^\omega$; *we denote by* $dom(s)$ *the domain of* s; *the* length *of* s, $l(s)$, *is the cardinality of* $dom(s)$; *hence the empty sequence has length* 0. *Let* $s, s' \in 2^{<\omega} \cup 2^\omega$, *we set* $s \leq s'$ *if* $s = s'_{\restriction dom(s')}$. *Set* $D := 2^{<\omega} \times \{0\}$, $U := 2^{<\omega} \times \{1\}$, $A := 2^\omega$. *Let* $T := D \cup U \cup A$ *and* $S : T \to T$ *define by* $S(x) := x$ *if* $x \in A$ *and* $S((s, i)) := (x, i \dot{+} 1)$ *where the sum* $i \dot{+} 1$ *is* 1 *if* i *is* 0 *and* 0 *otherwise. Let* $x, y \in T$, *we set* $x < y$ *in the following cases:*

- $x = (s, i), y = (s', i)$ *and either* $i = 0$ *and* $s < s'$ *or* $i = 1$ *and* $s' < s$;
- $x = (s, 0), y = (s', 1)$ *and either* $s < s'$ *or* $s' < s$;
- $x = (s, 0), y = s'$ *and* $s < s'$;
- $x = s, y = (s', 1)$ *and* $s' < s$.

With this in mind we can prove the following claim.

Claim 3.35. *The set* T *with the relation* $<$ *is a complete lattice containing no chain of type* θ. *In particular,* $\wp(\omega)$ *is not embeddable into* T. *The lattice* T *does not have CFPP, the poset* $\mathcal{C}(T)$ *is not a lattice and does not have FPP.*

Proof. The map S is a self dual map which sends D onto U, fixes A and reverses the relation $<$. The relation $<$ defined on $D \cup A$ yields the binary tree with ends hence yields an ordering and in fact a meet semilattice. On

$A \cup U$, this relation is the dual of $<$, hence it yields an ordering too which is a join semilattice. To see that this is an ordering on the union, note that since D is a tree, $(s,0) < (s',1)$ if and only if there is some $t \in A$ such that $(s,0) < t < (s',1)$. Since T is self dual, to prove that this is a lattice it suffices to prove that this is a join semilattice. Let x, y be two arbitrary elements of T. If x and y are in $A \cup U$ they have a join since $A \cup U$ is a join semilattice. We may suppose that $x \in D$. If y is comparable to x, the join is the largest of the two. If y is incomparable to x then, as it is easy to see, we have $x \vee y = x \vee S(y) = S(x) \vee y = S(x) \vee S(y)$. Thus T is a lattice. Each maximal chain of T has order type $\omega + 1 + \omega^*$ hence is a complete chain. Since T is a lattice, this ensures that it is complete lattice. The set $D \cup A$ is a well founded initial segment of T and the set $A \cup U$ is a dually well founded final segment of T; since their union covers T, no chain of type θ is contained in it. Let $f : T \to \mathcal{C}(T)$ defined by setting $f(x) := \{(s',0) : l(s') > l(s)\}$ if $x = (s,0)$, $f(x) := \{(s',1) : l(s') > l(s)\}$ if $x = (s,1)$ and $f(x) := A \setminus \{x\}$ if $x \in A$. This map is order preserving. Since it does have a fixed point, CFPP fails.

It remains to show that FPP fails for $\mathcal{C}(T)$. For that, we will apply the following result of Rutkowski ([21] Lemma 1):

Fact 3.36. *If a poset P contains a totally ordered pregap (A,B) such that $S(A,B)$ does not have FPP then P does not have FPP.*

To do so, let $\mathcal{A} := \{A_n : n \in \mathbb{N}\}$ where $A_n := \{(s,0) \in D : l(s) \leq n\}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$ where $B_n := \{(s,1) \in U : l(s) \leq n\}$. As it is easy to see, $(\mathcal{A},\mathcal{B})$ is a totally ordered pregap. Furthermore, $S(\mathcal{A},\mathcal{B}) = \{X \subseteq A : X \text{ is topologically dense in } A\}$ (note that A being the set of branches of the binary tree is homeomorphic to the Cantor space). Hence, $S(\mathcal{A},\mathcal{B})$ is an infinite antichain of $\mathcal{C}(T)$, thus it does not have FPP. Fact 3.36 now ensures that $\mathcal{C}(T)$ does not have FPP. In addition, we get from this construction that $\mathcal{C}(T)$ is not a lattice. Indeed, otherwise $S(\mathcal{A},\mathcal{B})$ would be a lattice. This is impossible since it is an infinite antichain. □

As a consequence of this last example we obtain the following.

Corollary 3.37. *CFPP is not preserved under finite product. Indeed, CFPP holds for complete chains, but this property fails for the direct product $[0,1] \times [0,1]$.*

Proof. Let T be the lattice defined defined in Example 3.34. This lattice is an order retract of the direct product $[0,1] \times [0,1]$ because it is complete and can be embedded into this direct product. Now, CFPP is preserved under order retraction (Corollary 3.25). Thus, if $[0,1] \times [0,1]$ had CFPP, T would have CFPP, which is not the case. □

3.3. **Selection properties.**

Definition 3.38. *A poset P has* selection property for convex subsets *(CSP for short) if there is an order preserving map $s : \mathcal{C}(P) \to P$ such that $s(S) \in S$ for every $S \in \mathcal{C}(P)$.*

It is a simple exercise to prove that if P has CSP, then P has FPP if and only if it has CFPP.

CSP is a strong condition, in fact too strong w.r.t. CFPP. Indeed, CSP is preserved by convex subsets while CFPP is not. In addition, note that every finite poset with a least element has CFPP, while we will see that the finite lattices in Corollary 3.41 do not have CSP.

Definition 3.39. *Let us recall that a poset P is* bipartite *if its comparability graph $G(P)$ is bipartite; equivalently P is the union of the set* $\min(P)$ *of minimal elements and the set* $\max(P)$ *of maximal elements.*

A *crown* is a poset whose comparability graph is a cycle – a crown is bipartite and every vertex has degree two.

Lemma 3.40. *A bipartite poset where every vertex has degree at least two does not have CSP.*

Proof. Let P be such a poset. Suppose that there is some selection $s : \mathcal{C}(P) \to P$. Let $P_0 := \min(P)$, $a := s(P_0)$, $P_1 := \max(P)$, and $b_0 := s(P_1)$. Due to our assumption on P, we have $P_0 \leq P_1$ hence $a < b_0$. Since every vertex of P has degree at least two, there is some $b_1 \in P_1$ with $b_1 \neq b_0$ and $a < b_1$. For $i \in \{0, 1\}$ we set $X_i := (P_0 \setminus \{a\}) \cup \{b_i\}$. As it is easy to check, we have $P_0 \leq X_i \leq P_1$ hence $a \leq s(X_i) \leq b_0$. Since $s(X_i) = b_i$, this yields a contradiction for $i = 1$. □

Since CSP is preserved by convex subsets we immediately have:

Corollary 3.41. *The lattices made of a crown with a top and bottom element added do not have CSP.*

Definition 3.42. *We can weaken CSP by simply supposing that for every chain $\mathcal{C}$ in $\mathcal{C}(P)$ there is an order preserving map $s : \mathcal{C} \to P$ such that $s(S) \in S$ for every $S \in \mathcal{C}$. We will call this property CCSP.*

It is easy to prove that every finite poset satisfies CCSP. There are infinite posets with CFPP and not CCSP. We give examples below, but first recall the notion of *cofinality* of a chain.

Definition 3.43. *The* cofinality *of a chain C, written $cf(\mathcal{C})$, is the least ordinal κ such that C contains a cofinal subset with order type κ.*

Note that the image C' of C by an order preserving map has either a largest element or the same cofinality than C.

In our setting, this yields the following:

Fact 3.44. *Let P be a poset with CCSP. Let $\mathcal{C}$ be a chain in $\mathcal{C}(P)$ and $A(\mathcal{C}) := (\cap_{C \in \mathcal{C}} \uparrow C) \cap (\cup_{C \in \mathcal{C}} \downarrow C)$. If $A(\mathcal{C}) = \varnothing$ then P contains a chain of type $cf(\mathcal{C})$.*

Proof. Let $\mathcal{C}'$ be the image of $\mathcal{C}$ by a selection map. Since $A(\mathcal{C}) = \varnothing$, $\mathcal{C}'$ does not have a largest element, hence $cf(C') = cf(C)$. □

Before we produce the example we need the following lemma.

Lemma 3.45. *Let Q be a well founded poset such that $cf(Q) = cf(\uparrow x) = \omega_1$ for every $x \in Q$. If Q contains no chain of type ω_1 then the poset $P := Q + 1$ obtained from Q by adding a largest element has CFPP and not CCSP.*

Proof. Since P is well founded with a largest element, it has CFPP from Fact 3.17. Next, let $(x_\alpha)_{\alpha<\omega_1}$ be a sequence of elements of Q which is cofinal in Q. Let $C_\alpha := Q \setminus \downarrow\{x_\beta : \beta < \alpha\}$ for each $\alpha < \omega_1$. Set $\mathcal{C} := \{C_\alpha : \alpha < \omega_1\}$. Let $\alpha \leq \beta < \omega_1$. The set C_β is a final segment of C_α and it is cofinal in Q. Indeed, let $x \in Q$, since $cf(\uparrow x) = \omega_1$, $\uparrow x \setminus \downarrow\{x_\gamma : \gamma < \beta\} \neq \emptyset$. Thus C_β is cofinal in C_α, hence, $C_\alpha \leq C_\beta$ in $\mathcal{C}(P)$. Consequently, $\mathcal{C}$ is a chain. Furthermore $A(\mathcal{C}) = Q \cap \bigcap_{\alpha<\omega_1} C_\alpha$ hence $A(\mathcal{C}) = \emptyset$. According to Fact 3.44, if P had CCSP, it would contain an uncountable chain. Hence, CCSP fails. □

Definition 3.46. *For κ be a cardinal, let $[\kappa]^{<\omega}$ be the poset of finite subsets of κ ordered by inclusion. We further denote by $[\kappa]^{<\omega} + 1$ the complete lattice obtained by simply adding a largest element to the previous poset.*

Example 3.47. *If a poset Q is up-directed then $cf(Q) = cf(\uparrow x)$ for every $x \in Q$, thus a well founded up-directed poset of cofinality ω_1 with no chain of type ω_1 will satisfy the conditions of the Lemma. A natural example is $[\omega_1]^{<\omega}$. A second example is $\mathcal{I}_{<\omega}(Q')$, the set of finitely generated initial segments of $Q' := \oplus_{\alpha<\omega_1} L_\alpha$, the direct sum of $\aleph_1$ copies of well ordered chains L_α having order type α (the fact that this poset is well-founded follows from a result of Birkhoff). For an example of non directed poset, take a regular Aronszajn tree (see* [13]*).*

Remark 3.48. *As for CFPP, CSP is not preserved under finite product. Indeed, CSP holds for chains (see Lemma 4.26) but this property fails for the direct product $[0,1] \times [0,1]$. Otherwise, since this direct product is a complete lattice, it would have CFPP, which is not the case according to Lemma 3.37.*

The following result uses a typical example to illustrate the relationship between fixed point properties and selection.

Theorem 3.49. *Le κ be a cardinal. And let $P := [\kappa]^{<\omega} + 1$ be the complete lattice defined above. Then the following hold:*

(i) P has CFPP;

(ii) P has CSP if and only if $\kappa \leq 2$;

(iii) P does not have CCSP if κ is uncountable; and,

(iv) $\mathcal{C}(P)$ has FPP if and only if $\kappa < \omega$.

Proof. (i). P is well founded with a largest element. Apply Fact 3.17.

(ii). If $\kappa \leq 2$ a simple inspection proves that CSP holds. If $\kappa \geq 3$ then P embeds the lattice L made of the 6-element crown with top and bottom added. Since every complete lattice is a retract of any poset in which it can be embedded, L is a retract of P. Since CSP is preserved under retract, if P had CSP, L would have CSP. According to Corollary 3.41, this is not the case.

(iii). If κ is uncountable, a surjective map from κ onto ω_1 induces a retraction of P onto $[\omega_1]^{<\omega}+1$. Since CCSP is preserved under retract, if P had CCSP, $[\omega_1]^{<\omega}+1$ would have CCSP. According to Lemma 3.45, this is not the case.

(iv). If $\kappa<\omega$, P is finite. Since it has a largest element, $\mathcal{C}(P)$ has FPP by Lemma 3.20. If $\kappa \geq \omega$, the poset Q defined in Example 3.21 is embeddable in $[\kappa]^{<\omega}$ because for every $x \in Q$, $\downarrow x$ is finite. Thus $\overline{Q}$ is embeddable in P. Since $\overline{Q}$ is a complete lattice, this is a retract of P. According to Lemma 3.23, $\mathcal{C}(\overline{Q})$ is a retract of $\mathcal{C}(P)$. Since FPP is preserved under retraction, and $\mathcal{C}(\overline{Q})$ does not have FPP, $\mathcal{C}(P)$ does not have FPP. □

4. The lattice of convex sublattices of a lattice

Let T be a lattice. The join and meet of two elements $x, y \in T$ are, as usual, denoted respectively by $x \vee y$ and $x \wedge y$ and a sublattice is a nonempty subset closed under these operations. The set $\mathcal{I}d(T)$ of ideals of T, ordered by inclusion, is a lattice (a complete lattice provided that T has a least element), the join, and meet, of two ideals A and B being

$$A \vee B = \downarrow\{a \vee b : a \in A, b \in B\} \text{ and } A \wedge B = A \cap B .$$

Similarly, $\mathcal{F}i(P)$, the set of filters of T is a lattice (a complete lattice provided that T has a largest element): for every $A, B \in \mathcal{F}i(T)$

$$A \vee B = \uparrow\{a \wedge b : a \in A, b \in B\} \text{ and } A \wedge B = A \cap B.$$

It is it immediate that an ideal of a lattice is a nonempty initial segment closed under pairwise joins, and a filter of a lattice is a final segment closed under pairwise meets. Hence, the up and down directed convex subsets of T are simply the convex sublattices of T.

We denote by $\mathcal{C}_L(T)$ the set of nonempty convex sublattices of T, ordered with the bi-domination preorder.

Proposition 4.1. *Let T be a lattice. Then:*

(a) $\mathcal{C}_L(T)$ is a lattice and the map $\vartheta : \mathcal{C}_L(T) \to \mathcal{I}d(T) \times \mathcal{F}i(T)^*$, defined by $\vartheta(S) := (\downarrow S, \uparrow S)$ for $S \in \mathcal{C}_L(T)$, is a one to one lattice homomorphism. The image is the subset

$$\mathcal{K}(T) := \{(I, F) \in Id(T) \times \mathcal{F}i(T)^* : I \cap F \neq \varnothing\}$$

of $\mathcal{I}d(T) \times \mathcal{F}i(T)^*$.

(b) In particular, if $A, B \in \mathcal{C}_L(T)$, then:

(i) $A \leq B$ *if and only if* $a \vee b \in B$ *and* $a \wedge b \in A$ *for every* $a \in A, b \in B$*;*

(ii) $A \vee B = ((\downarrow A) \vee (\downarrow B)) \cap ((\uparrow A) \cap (\uparrow B))$*; and*

(iii) $A \wedge B = ((\downarrow A) \cap (\downarrow B)) \cap ((\uparrow A) \wedge (\uparrow B))$.

Proof. *(a).* From Fact 3.2 the map ϑ is an order isomorphism from $\mathcal{C}_L(T)$ on its image. Thus, to prove that $\mathcal{C}_L(T)$ is a lattice, it suffices to prove that its image is a lattice. This image is clearly included in $\mathcal{K}(T)$. The reverse inclusion is due to the fact that if $(I, F) \in \mathcal{K}(T)$ then $\downarrow(I \cap F) = I$ and $\uparrow(I \cap F) = F$. In fact, the first equality holds whenever I is an ideal and F is a final segment which meets I. With the product order, $\mathcal{I}d(T) \times \mathcal{F}i(T)^*$ is a lattice in which $(I, F) \vee (I', F') = (I \vee I', F \cap F')$ for every $(I, F), (I', F') \in \mathcal{I}d(T) \times \mathcal{F}i(T)^*$. If $(I, F), (I', F') \in \mathcal{K}(T)$ then $(I, F) \vee (I', F') \in \mathcal{K}(T)$ since

$$\{x \vee x' : x \in I \cap F, x' \in I' \cap F'\} \subseteq (I \vee I') \cap (F \cap F').$$

The same property holds with meet instead of join. Hence $\mathcal{K}(T)$ is a sublattice of the lattice $\mathcal{I}d(T) \times \mathcal{F}i(T)^*$.
(b). This follows from *(a)* and the form of joins and meets in $\mathcal{I}d(T) \times \mathcal{F}i(T)^*$.
□

Lemma 4.2. *Let* $(\mathcal{A}, \mathcal{B})$ *be a pregap of* $\mathcal{C}_L(T)$*. Then the following properties are equivalent:*
(a) $(\mathcal{A}, \mathcal{B})$ *is a gap of* $\mathcal{C}_L(T)$*;*
(b) $I_{\mathcal{B}} \cap F_{\mathcal{A}} = \varnothing$*;*
(c) $(\mathcal{A}, \mathcal{B})$ *is a gap of* $\mathcal{C}(T)$*.*

Proof. $(b) \Rightarrow (c)$: This is the contrapositive of *(i)* of Lemma 3.9.
$(c) \Rightarrow (a)$: This follows from the fact that $\mathcal{C}_L(T)$ is a subposet of $\mathcal{C}(T)$.
$(a) \Rightarrow (b)$: Suppose that (b) does not hold. Then observe that $I_{\mathcal{B}}$ and $F_{\mathcal{A}}$ are up and down directed, respectively. Thus Lemma 3.13 applies and from implication $(c) \Rightarrow (b)$ of (ii) of this Lemma, $(\mathcal{A}, \mathcal{B})$ is not a gap in $\mathcal{C}(T)$. According to Lemma 3.9*(i)*, $I_{\mathcal{B}} \cap F_{\mathcal{A}}$ separates $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}(T)$. But $I_{\mathcal{B}} \cap F_{\mathcal{A}} \in \mathcal{C}_L(T)$, hence it separates $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}_L(T)$, hence (a) does not hold. □

The proof of the following lemma is straightforward.

Lemma 4.3. *Let* T *be a lattice, the map* $i : T \to \mathcal{C}_L(T)$ *defined by* $i(x) := \{x\}$ *is a lattice homomorphism, it preserves all infinite joins and meets in* T *and all gaps of* T*.*

Now since a complete lattice has no gap, we deduce immediately that:

Corollary 4.4. *If* $\mathcal{C}_L(T)$ *is a complete lattice then* T *is a complete lattice.*

Lemma 4.5. *Let* T *be a complete lattice. There is an order embedding from* $\wp(\omega)$ *into* T *if and only if there are sequences* $(x_n)_{n\in\omega}$ *and* $(y_n)_{n\in\omega}$ *in* T *such that:*

(i) $x_0 > x_1 > x_2 > \dots x_n > \dots$*;*

(ii) $y_0 \nleq x_0,\ y_1 \nleq x_1 \vee y_0,\ y_2 \nleq x_2 \vee y_0 \vee y_1,\ \dots\ ,\ y_n \nleq x_n \vee \bigvee_{j<n} y_j, \dots$;

(iii) $y_1 \leq x_0,\ y_2 \leq x_1,\ \dots\ ,\ y_{n+1} \leq x_n, \dots$.

Proof. Suppose that f is an order embedding from $\wp(\omega)$ into T. Set $x_n := f(\omega \setminus \{j : j \leq n\})$ and $y_n := f(\{n\})$ for each $n \in \omega$. Then (i) and (iii) are trivially satisfied. Concerning (ii), note first that since $\{n\} \nsubseteq \omega \setminus \{n\}$ we have $y_n = f(\{n\}) \nleq f(\omega \setminus \{n\})$ and next, that since $\omega \setminus \{j : j \leq n\} \cup \bigcup_{j<n}\{j\} \subseteq \omega \setminus \{n\}$, we have

$$x_n \vee \bigvee_{j<n} y_j = f(\omega \setminus \{j : j \leq n\}) \vee \bigvee_{j<n} f(\{j\}) \leq f(\omega \setminus \{n\}).$$

Thus $y_n \nleq x_n \vee \bigvee_{j<n} y_j$ as required.

Conversely, suppose that there are two sequences satisfying (i)–(iii). Define $f : \wp(\omega) \to T$ by setting $f(X) := \bigvee\{y_n : n \in X\}$ for every $X \subseteq \omega$, with the convention that $f(\varnothing)$ is the least element of T. Clearly, this map is order preserving. To show that it is an order embedding, it suffices to prove that if $X \nsubseteq X'$ then $f(X) \nleq f(X')$. Suppose that $n \in X \setminus X'$. Then $X' \subseteq \omega \setminus \{n\}$. Since f is order preserving, it follows that $f(X') \leq f(\omega \setminus \{n\})$. By definition of f, we have

$$f(\omega \setminus \{n\}) = \bigvee_{j>n} f(\{j\}) \vee \bigvee_{j<n} f(\{j\}) = \bigvee_{j>n} y_j \vee \bigvee_{j<n} y_j \leq x_n \vee \bigvee_{j<n} y_j$$

since $y_j \leq x_n$ for $j > n$. Thus, $f(X') \leq x_n \vee \bigvee_{j<n} y_j$. Since $n \in X$, we have $y_n = f(\{n\}) \leq f(X)$. Since $y_n \nleq x_n \vee \bigvee_{j<n} y_j$, this yields $f(X) \nleq f(X')$, as required. □

We note that the conditions in the preceding lemma were introduced in [17] to study pregaps under lattice homomorphisms.

Lemma 4.6. *Let T be a complete lattice. If $\mathcal{C}_L(T)$ is not complete then there is an embedding from $\wp(\omega)$ into T.*

Proof. Suppose that $\mathcal{C}_L(T)$ is not complete. We begin by the following claim.

Claim 4.7. *$\mathcal{C}_L(T)$ contains a special gap $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are, respectively, up-directed with a least element A_0 and down-directed with a largest element B_0.*

Proof of Claim 4.7. First $\mathcal{C}_L(T)$ contains a gap, not necessarily special, with these properties. Indeed, since $\mathcal{C}_L(T)$ is not complete, it contains a gap, say $(\mathcal{A}, \mathcal{B})$. Since $C_L(T)$ has a least element and a largest element (namely $\{0_T\}$ and $\{1_T\}$, where 0_T and 1_T are the least and largest elements of T), $\mathcal{A}$ and $\mathcal{B}$ are both nonempty. Since $\mathcal{C}_L(T)$ is a lattice, we may suppose that $\mathcal{A}$ is up-directed and $\mathcal{B}$ down-directed (otherwise, replace $\mathcal{A}$ by $L(\mathcal{B})$ and $\mathcal{B}$ by $U(L(\mathcal{B}))$) and also that $\mathcal{A}$ and $\mathcal{B}$ have, respectively, a least element A_0 and a largest element B_0. The pair $(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}})$ is a special gap with the required properties. Indeed, according to Lemma 4.2, $(\mathcal{A}, \mathcal{B})$ is a gap in $C(T)$. Furthermore $I_{\mathcal{B}}$ and $F_{\mathcal{A}}$ are, respectively, up- and down-directed. Hence, by Lemma 3.13, $(\mathcal{A}_{\mathcal{B}}, \mathcal{B}_{\mathcal{A}})$ is a gap in $C(T)$, and thus in $\mathcal{C}_L(T)$, and $\mathcal{A}_{\mathcal{B}}$ and $\mathcal{B}_{\mathcal{A}}$ are, respectively, up-directed with a least element and down-directed with a largest element. □

Claim 4.8. *If Y and X are two subsets of T such that $Y \cap A$ and $X \cap B$ are nonempty for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $\bigvee Y \nleq \bigwedge X$.*

Proof of Claim 4.8. Suppose that the conclusion does not hold. Let d satisfy $\bigvee Y \leq d \leq \bigwedge X$. Since $d \leq \bigwedge X$, $d \in \bigcap_{B \in \mathcal{B}} \downarrow B = I_{\mathcal{B}}$ Similarly, since $\bigvee Y \leq d$, $d \in \bigcap_{A \in \mathcal{A}} \uparrow A = F_{\mathcal{A}}$. Hence $d \in I_{\mathcal{B}} \cap F_{\mathcal{A}}$. This is impossible as $I_{\mathcal{B}} \cap F_{\mathcal{A}} = \varnothing$. □

Now the following claim provides the constructions of the desired elements.

Claim 4.9. *There are $(x_n : n \in \omega) \subseteq B_0$ and and $(y_n : n \in \omega) \subseteq A_0$ satisfying conditions $(i) - (iii)$ of Lemma 4.5.*

Proof of Claim 4.9. Let $n \in \omega$ and suppose x_k and y_k have been defined for all $k < n$. Define x_n and y_n as follows. If $n = 0$, set $Y_0 := A_0$ and $X_0 := B_0$. The hypotheses of Claim 4.8 are satisfied, hence there are $y_0 \in Y_0$ and $x_0 \in X_0$ such that $y_0 \nleq x_0$. If $n > 0$, set

$$Y_n := \downarrow x_{n-1} \cap A_0,\ Z_n := \downarrow x_{n-1} \cap B_0, \text{ and}$$
$$X_n := \ Z_n \vee \bigvee_{k<n} y_k (= \{x \vee \bigvee_{k<n} y_k : x \in Z_n\}).$$

The sets X_n and Y_n satisfy the hypotheses of Claim 4.8. Indeed, let $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since $A \leq B_0$ and $x_{n-1} \in B_0$, there is some $t \in A$ with $t \leq x_{n-1}$. Since $A \subseteq A_0$, $t \in Y_n \cap A$ proving that $Y_n \cap A \neq \varnothing$. Since $B \leq B_0$ and $x_{n-1} \in B_0$ there is some $x \in B$ such that $x \leq x_{n-1}$, and since $B \subseteq B_0$, $x \in B_0$ and, thus, $x \in Z_n$. Also, since $A_0 \leq B$ and $\bigvee_{k<n} y_k \in A_0$, $x \vee \bigvee_{k<n} y_k \in B$, hence $X_n \cap B \neq \varnothing$. From Claim 4.8 there are $y_n \in Y_n$ and $x_n \in X_n$ such that $y_n \nleq x_n \vee \bigvee_{k<n} y_k$. With the fact that necessarily $x_n < x_{n-1}$, all conditions of Lemma 4.5 are satisfied. □

Finally, according to Lemma 4.5, there is an embedding from $\wp(\omega)$ into T and this completes the proof. □

Lemma 4.10. *Let P and Q be lattices. If Q is an order retract of P or a lattice quotient of P then $\mathcal{C}_L(Q)$ is an order retract of $\mathcal{C}_L(P)$.*

Proof. Suppose that Q is an order retract of P. Let $s : Q \to P$ and $r : P \to Q$ be two order preserving maps such that $r \circ s = 1_Q$ and let $\overline{s} : \mathcal{C}(Q) \to \mathcal{C}(P)$ and $\overline{r} : \mathcal{C}(P) \to \mathcal{C}(Q)$ be given by Lemma 3.23. These two maps send $\mathcal{C}_L(Q)$ into $\mathcal{C}_L(P)$ and $\mathcal{C}_L(P)$ onto $\mathcal{C}_L(Q)$. Thus, we have coretraction and retraction maps and $\mathcal{C}_L(Q)$ is an order retract of $\mathcal{C}_L(P)$.

Suppose that Q is a quotient of P. Let $r : P \to Q$ be a surjective lattice homomorphism. Let $\overline{s} : \mathcal{C}_L(Q) \to \mathcal{C}_L(P)$ and $\overline{r} : \mathcal{C}_L(P) \to \mathcal{C}_L(Q)$ be defined by setting $\overline{s}(Y) := r^{-1}(Y)$ and $\overline{r}(X) := Conv_Q(r[X])$ for all $Y \in \mathcal{C}_L(Q)$ and $X \in \mathcal{C}_L(P)$. These maps are order preserving. Moreover, if $Y \in \mathcal{C}_L(Q)$ then, since $r[r^{-1}(Y)] = Y$, $\overline{r} \circ \overline{s}(Y) = Y$. Hence, $\overline{s}$ and $\overline{r}$ are coretraction and retraction maps; in particular $\mathcal{C}_L(Q)$ is an order retract of $\mathcal{C}_L(P)$. □

Since a retract of a complete lattice is complete, Lemma 4.10 yields immediately:

Corollary 4.11. *If Q is a lattice quotient of a retract of P and $\mathcal{C}_L(P)$ is complete, then $\mathcal{C}_L(Q)$ is complete too.*

Another consequence is this:

Corollary 4.12. *If Q is a lattice quotient of a complete lattice P and P is order embeddable in a lattice T then $\mathcal{C}_L(Q)$ is a retract of $\mathcal{C}_L(T)$.*

Proof. Since every complete lattice which is embeddable in a lattice is an order retract of that lattice, P is an order retract of T. From Lemma 4.10, $\mathcal{C}_L(P)$ is a retract of $\mathcal{C}_L(T)$ and $\mathcal{C}_L(Q)$ is a retract of $\mathcal{C}_L(P)$. Thus $\mathcal{C}_L(Q)$ is a retract of $\mathcal{C}_L(T)$ as claimed. □

For any set E, let $\wp(E)$ denote the set of all subsets of E ordered by containment and let $\wp(E)/Fin$ be the quotient of $\wp(E)$ by the ideal Fin of finite subsets of E. Define $p : \wp(E) \to \wp(E)/Fin$ to be the canonical projection. For $X, Y \in \wp(E)$, we set $X \leq_{Fin} Y$ if $X \setminus Y \in Fin$. This defines a quasi-order on $\wp(E)$, its image under p is the order on $\wp(E)/Fin$.

With all these tools in hand, we are ready to provide the proof of Main Theorem 1.

4.1. **Proof of Main Theorem 1.** $(i) \Rightarrow (ii)$: Let Q be a quotient of a retract of T. According to Corollary 4.12, $\mathcal{C}_L(Q)$ is a retract of $\mathcal{C}_L(T)$. Since $\mathcal{C}_L(T)$ is complete, $\mathcal{C}_L(Q)$ is complete too. According to Corollary 4.4, Q complete.

$(ii) \Rightarrow (iii)$: Since T is a quotient and a retract of itself, it is complete. Let $P := \wp(\omega)$ and $Q := \wp(\omega)/Fin$. Clearly, the lattice Q is not complete. Thus P cannot be a retract of T.

$(iii) \Rightarrow (i)$: Apply Corollary 4.4 and Lemma 4.6. □

4.2. **Selection property, fixed point property and completeness of the lattice of convex sublattices.**

Proposition 4.13. *CLFPP is preserved under retraction and lattice quotient.*

Proof. Let P be a lattice. Let Q be an order retract of P or a lattice quotient of P. According to Lemma 4.10, $\mathcal{C}_L(Q)$ is an order retract of $\mathcal{C}_L(P)$. More precisely, if Q is an order retract, let $s : Q \to P$ and $r : P \to Q$ be two order preserving maps such that $r \circ s = 1_Q$ and let $\overline{s} : \mathcal{C}(Q) \to \mathcal{C}(P)$ and $\overline{r} : \mathcal{C}(P) \to \mathcal{C}(Q)$ be given by Lemma 3.23. These two maps provide a retraction of $\mathcal{C}_L(P)$ onto $\mathcal{C}_L(Q)$.

If Q is a lattice quotient of P let $r : P \to Q$ be a surjective lattice homomorphism. Let $\overline{s} : \mathcal{C}_L(Q) \to \mathcal{C}_L(P)$ and $\overline{r} : \mathcal{C}_L(P) \to \mathcal{C}_L(Q)$ be defined by setting

$$\overline{s}(Y) := r^{-1}(Y) \text{ and } \overline{r}(X) := Conv_Q(r[X])$$

for all $Y \in \mathcal{C}_L(Q)$ and $X \in \mathcal{C}_L(P)$. According to Lemma 4.10, $\overline{s}$ and $\overline{r}$ are coretraction and retraction maps. Thus, in both cases, if $g : Q \to \mathcal{C}_L(Q)$ is an order preserving map, we may proceed as in the proof of Corollary 3.25, defining $h : P \to \mathcal{C}_L(P)$ by $h := \overline{s} \circ g \circ r$. Then, if x is a fixed point of h, $y := r(x)$ is a fixed point of g. □

Corollary 4.14. *The lattice $\wp(\omega)$ does not have CLFPP.*

Proof. The lattice $\wp(\omega)/Fin$ is a lattice quotient of $\wp(\omega)$. Since it is not complete, it does not have FPP [4]. In particular, it does not have CLFPP. According to Lemma 4.13, $\wp(\omega)$ cannot have CLFPP. □

Definition 4.15. *Recall that a lattice T has the* selection property for convex sublattices *(CLSP for short) if there is an order preserving map $\varphi : \mathcal{C}_L(T) \to T$ such that $\varphi(S) \in S$ for every $S \in \mathcal{C}_L(T)$.*

Proposition 4.16. *Let us consider the following properties of a lattice T:*

(i) T is complete and has CLSP;

(ii) T has CLFPP;

(iii) $\mathcal{C}_L(T)$ is a complete lattice.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof. $(i) \Rightarrow (ii)$. Let $h : T \to \mathcal{C}_L(T)$ be an order preserving map. Let $\varphi : \mathcal{C}_L(T) \to T$ be an order preserving selection map guaranteed by (i). The map $\varphi \circ h : T \to T$ is order preserving. Since T is a complete lattice, $\varphi \circ h$ has a fixed point [24], say, $x = \varphi \circ h(x)$. Since φ is a selection, $x \in h(x)$. Hence, (ii) holds.

$(ii) \Rightarrow (iii)$. Suppose that (ii) holds. According to Lemma 4.13 and Corollary 4.14, $\wp(\omega)$ is not a retract of T. Since $\wp(\omega)$ is a complete lattice, it cannot be embedded in T. Hence, according to implication $(iii) \Rightarrow (i)$ of Main Theorem 1, $\mathcal{C}_L(T)$ is a complete lattice. □

Remark 4.17. *Here is a more direct route to the implication $(ii) \Rightarrow (iii)$. As in the proof of Lemma 4.6, observe that in the non complete lattice $\mathcal{C}_L(T)$ there is a totally ordered gap $(\mathcal{A}, \mathcal{B})$. According to Lemma 4.2, $I_{\mathcal{B}} \cap F_{\mathcal{A}} = \varnothing$. This gap is then a gap in $\mathcal{C}(T)$. Lemma 3.15 yields a fixed point free map of T to $\mathcal{C}_L(T)$.*

Trivially, we have:

Proposition 4.18. *The dual of a lattice with CLSP has CLSP.*

Proposition 4.19. *Given lattices L_0 and L_1, let $L := L_0 \times L_1$ be their direct product and $\pi_i : L \to L_i$ be the i-th projection for $i < 2$. Then the map $\pi : \mathcal{C}_L(L) \to \mathcal{C}_L(L_0) \times \mathcal{C}_L(L_1)$ defined by $\pi(S) := (\pi_0[S], \pi_1[S])$ for all $S \in \mathcal{C}_L(L)$ is an isomorphism from $\mathcal{C}_L(L)$ onto $\mathcal{C}_L(L_0) \times \mathcal{C}_L(L_1)$.*

Proof. Let $\pi' : \mathcal{C}_L(L_0) \times \mathcal{C}_L(L_1) \to \mathcal{C}_L(L)$ be defined by $\pi'(S_0, S_1) := S_0 \times S_1$. Observe that π' is the inverse of π. □

Corollary 4.20. *The class of lattices T such that $\mathcal{C}_L(T)$ is complete is closed under finite product.*

Proposition 4.21. *CLSP is preserved under finite products.*

Proof. Let L_0 and L_1 be lattices with CLSP. Let $L := L_0 \times L_1$, let π_i be the i-th projection, and let $\varphi_i : \mathcal{C}_L(L_i) \to L_i$ be an ordered preserving selection map for $i < 2$. Let $\varphi : \mathcal{C}_L(L) \to L$ be defined by $\varphi(S) := (\varphi_0(\pi_0[S]), \varphi_1(\pi_1[S]))$ for $S \in \mathcal{C}_L(L)$. Since $S = \pi_0[S] \times \pi_1[S]$ for all $S \in \mathcal{C}_L(L)$, φ is an order preserving selection map. □

Proposition 4.22. *Every quotient of a lattice with CLSP is a retract of that lattice.*

Proof. Let Q be a quotient of a lattice P with CLSP with a surjective homomorphism $q : P \to Q$. For each $y \in Q$, the set $q^{-1}(y)$ belongs to $\mathcal{C}_L(P)$. Moreover, the map $q^{-1} : Q \to \mathcal{C}_L(P)$ is order preserving. Let $s : \mathcal{C}_L(P) \to P$ be an order preserving selection. Let $f := s \circ q^{-1}$. Clearly $q \circ f = 1_Q$. Hence, q is a retraction and f a coretraction. Thus Q is a retract of P. □

Proposition 4.23. *CLSP is preserved under retraction.*

Proof. Let P be a lattice with CLSP and let Q be an order retract of P. Let $s : Q \to P$ and $r : P \to Q$ be order preserving maps such that $r \circ s = 1_Q$. Let $\overline{s} : \mathcal{C}_L(Q) \to \mathcal{C}_L(P)$ and $\overline{r} : \mathcal{C}_L(P) \to \mathcal{C}_L(Q)$ be defined as in the proof of Lemma 4.10. Let $\varphi : \mathcal{C}_L(P) \to P$ be an order preserving selection. Let $\psi : \mathcal{C}_L(Q) \to Q$ be defined by $\psi := \overline{r} \circ \varphi \circ \overline{s}$. Clearly, this map is order preserving. We claim that this is a selection map. Indeed, let $Y \in \mathcal{C}_L(Q)$. Since $\varphi(\overline{s}(Y)) \in \overline{s}(Y)$ it follows that $\psi(Y) = \overline{r}(\varphi(\overline{s}(Y))) \in \overline{r} \circ \overline{s}(Y)$. Since, according to Lemma 4.10, $\overline{r} \circ \overline{s}(Y) = Y$, it follows that $\psi(Y) \in Y$. □

Since a complete lattice which is embeddable in a poset is a retract of that poset, Proposition 4.23 immediately yields:

Corollary 4.24. *Every complete lattice which is embeddable in a lattice with CLSP has CLSP.*

Combining Propositions 4.22 and 4.23, we get:

Corollary 4.25. *CLSP is preserved under lattice quotients.*

Several examples of lattices with *CLSP* can be obtained from the following result of [8]. For the reader's convenience, we recall the proof.

Proposition 4.26. *Every chain has CLSP.*

Proof. Let $C := (E, \leq)$ be a chain and let $\leq_{wo}$ be a well ordering on E. Define $\varphi : \mathcal{C}_L(C) \to C$ by setting $\varphi(S)$ to be the least element of $S \in \mathcal{C}_L(C)$ with respect to the well-ordering $\leq_{wo}$. This map is order preserving. Indeed, let $S', S'' \in \mathcal{C}_L(C)$ such that $S' \le S''$. Let $x' := \varphi(S')$ and $x'' := \varphi(S'')$. If $x', x'' \in S' \cap S''$ then $x' \le_{wo} x''$ and $x'' \le_{wo} x'$, thus $x' = x''$. If $x' \in S' \setminus (S' \cap S'')$ then, since $S' \le S''$, $S' \cap S''$ is a final segment of S''. Thus $x' < x''$. If $x'' \in S'' \setminus (S' \cap S'')$, a similar argument yields $x' < x''$. From this, φ is an order preserving selection. □

Combining Propositions 4.21, 4.23, and 4.26, we get:

Proposition 4.27. *Every retract of a product of finitely many chains has $CLSP$.*

Corollary 4.28. *Every complete finite dimensional lattice has $CLSP$.*

Proof. A poset P of dimension n embeds in a direct product L of n chains (and not fewer). If P is complete then it is a retract of L, thus from Proposition 4.28 it has $CLSP$. □

Countable lattices have the $CLSP$ as well. One proof follows directly from Proposition 4.23 and the following two facts.

Proposition 4.29. *(Proposition 7* [18]*). A sublattice of the lattice of all finite unions of intervals of some chain, ordered by containment, has the $CLSP$.*

Proposition 4.30. *(Corollary 3* [18]*) Every countable lattice is a retract of a convex sublattice of a Boolean algebra generated by a countable chain.*

We also provide a direct argument based on a variant of the lattice "weaving argument" (see [5]).

Theorem 4.31. *Every countable lattice has $CLSP$.*

Proof. Let $(x_n)_{n<\omega}$ be an enumeration of the elements of the countable lattice T. Let $(\mathcal{D}_n)_{n<\omega}$ and $(\mathcal{C}_n)_{n<\omega}$ be the sequences of subsets of $\mathcal{C}_L(T)$ defined inductively by the following conditions: for all $k < \omega$

$$\begin{aligned} \mathcal{D}_0 &:= \varnothing; \\ \mathcal{C}_k &:= \{C \in \mathcal{C}_L(T) : x_k \in C\} \setminus \mathcal{D}_k; \\ \mathcal{D}_{k+1} &:= \mathcal{D}_k \cup \mathcal{C}_k. \end{aligned}$$

Then $\mathcal{C}_0 = \{C \in \mathcal{C}_L(T) : x_0 \in C\}$, $\mathcal{D}_n := \bigcup_{i<n} \mathcal{C}_i$ for $n < \omega$, and $T = \bigcup_{n<\omega} \mathcal{D}_n$.

We wish to define a map $\varphi : \mathcal{C}_L(T) \to T$ such that for each $n < \omega$:

(i) $\varphi(C) \in C$ for all $C \in \mathcal{D}_n$;
(ii) $\varphi_{\restriction D_n}$ takes only finitely many values and
(iii) $\varphi_{\restriction D_n}$ is order preserving.

Set $\varphi(C) := x_0$ for $C \in \mathcal{C}_0$. Hence, (i) - (iii) hold for $n \le 1$. Let $k \ge 1$ and suppose that φ is defined on $\mathcal{D}_k$ such that (i) - (iii) hold for $n = k$. Extend φ on $\mathcal{D}_{k+1}$ as follows. For $C \in \mathcal{C}_k$, set

$$\begin{aligned} \varphi_k^+(C) &= \bigwedge\{\varphi(D) : D \in \mathcal{D}_k \text{ and } C < D\}, \\ \varphi_k^-(C) &= \bigvee\{\varphi(D) : D \in \mathcal{D}_k \text{ and } D < C\}, \text{ and} \\ \varphi(C) &= (x_k \wedge \varphi_k^+(C)) \vee \varphi_k^-(C). \end{aligned}$$

Clearly, (i) and (ii) hold for $n = k+1$, so it remains to verify (iii). Let $C', C'' \in \mathcal{D}_{k+1}$ with $C' \leq C''$. We consider three cases:

Case 1. $C', C'' \in \mathcal{C}_k$. Since (iii) holds for $n = k$, $\varphi^+_k(C') \leq \varphi^+_k(C'')$ and $\varphi^-_k(C') \leq \varphi^-_k(C'')$. This yields $\varphi(C') \leq \varphi(C'')$.

Case 2. $C' \in \mathcal{C}_k$ and $C'' \in \mathcal{D}_k$. Since (iii) holds for $n = k$, $\varphi^+_k(C') \leq \varphi(C'')$ and $\varphi^-_k(C') \leq \varphi(C'')$. Since $\varphi(C') \leq \varphi^+_k(C') \vee \varphi^-_k(C')$,we get $\varphi(C') \leq \varphi(C'')$.

Case 3. $C' \in \mathcal{D}_k$ and $C'' \in \mathcal{C}_k$. Since (iii) holds for $n = k$, $\varphi(C') \leq \varphi^-_k(C'')$. Since by definition $\varphi^-_k(C'') \leq \varphi(C'')$ we get $\varphi(C') \leq \varphi(C'')$.

The map φ is an order preserving selection map. This proves the theorem. □

4.3. **Well-quasi-ordered posets, posets with no infinite antichain and a proof of Main Theorem 2.** As said in the introduction, the cornerstone of Main Theorem 2 is implication $(iv) \Rightarrow (i)$. Here is a restatement.

Lemma 4.32. *If P has no infinite antichain, then $\mathcal{I}(P)$ has $CLSP$.*

The proof of this result follows from a Hausdorff type result due to Abraham *et al.* [1]. In order to state this result, first we recall that if $(P_\alpha)_{\alpha\in A}$ is a family of ordered sets indexed by a poset A, the *lexicographical sum of the P_α's indexed by A* is the poset, that we denote by $\sum_{\alpha\in A} P_\alpha$, obtained by replacing each element $\alpha \in A$ by P_α and by ordering the elements accordingly. Finally, an *augmentation* of a poset P is a poset Q on the same underlying set such that $x \leq y$ in P implies $x \leq y$ in Q.

Let $\mathcal{BP}$ be the class of posets P such that P is either a wqo poset, the dual of a wqo poset, or a linear ordering. Let $\mathcal{P}$ be the smallest class of posets such that

(a) $\mathcal{P}$ contains $\mathcal{BP}$;
(b) $\mathcal{P}$ is closed under lexicographic sums with index set in $\mathcal{BP}$;
(c) $\mathcal{P}$ is closed under augmentation.

The Hausdorff type result is the following:

Theorem 4.33. [1] *$\mathcal{P}$ is the class of posets with no infinite antichain.*

With this result in hand, we prove Lemma 4.32 as follows. We start with $P \in \mathcal{P}$ and we prove that $\mathcal{I}(P)$ has $CLSP$ by induction, distinguishing the following cases:

Case 1: $P \in \mathcal{BP}$.
Case 2: P is a lexicographic sum of posets P_α indexed by a poset $A \in \mathcal{BP}$ such that $P_\alpha \in \mathcal{P}$ and $\mathcal{I}(P_\alpha)$ has $CLSP$ for each $\alpha \in A$.
Case 3: P is an augmentation of Q such that $Q \in \mathcal{P}$ and $\mathcal{I}(Q)$ has $CLSP$.

Case 1. If P is a chain, then $\mathcal{I}(P)$ is a chain, which has $CLSP$ according to Proposition 4.26. If P is well-quasi-ordered then $\mathcal{I}(P)$ is well-founded [11], the fact that it has $CLSP$ is a consequence of the following lemma.

Lemma 4.34. *Every well founded lattice has $CLSP$.*

Proof. Let T be a well-founded lattice. Each nonempty sublattice T' has a least element $min(T')$. The map $s : \mathcal{C}_L(T) \to T$ defined by $s(T') := min(T')$ is order preserving, hence T has $CLSP$. □

If P is dually well-quasi-ordered, note that $\mathcal{I}(P)$ is order isomorphic to $\mathcal{I}(P^*)^*$. From above, $\mathcal{I}(P^*)$ has $CLSP$. Since this property is preserved by duality, it holds for $\mathcal{I}(P^*)^*$, hence for $\mathcal{I}(P)$.
Case 2. The following lemma deals with the case of a lexicographic sum indexed by a chain or a wqo. The case of a lexicographical sum indexed by a reverse wqo, use duality as in Case 1.

Lemma 4.35. *Let A be a chain or a wqo poset, let $(P_\alpha), \alpha \in A$, be a family of posets and let $T_\alpha := \mathcal{I}(P_\alpha)$ have $CLSP$ for each $\alpha \in A$. Then $\mathcal{I}(\sum_{\alpha\in A} P_\alpha)$ has the $CLSP$.*

Proof. According to Case 1 above, $T := \mathcal{I}(A)$ has $CLSP$. Let $s : \mathcal{C}_L(T) \to T$ and $s_\alpha : \mathcal{C}_L(T_\alpha) \to T_\alpha$, $\alpha \in A$, be selection maps. Let $P := \sum_{\alpha\in A} P_\alpha$ and $p : P \to A$ be the projection map.

Let T' be a nonempty convex sublattice of $\mathcal{I}(P)$ and let $\theta(T') := \{p[I] : I \in T'\}$. Since $\theta(T')$ is convex, down-directed and up-directed, it is a nonempty convex sublattice of T and $A' := s(\theta(T'))$ is well defined. Then $A' \in \theta(T')$ so there is some $I' \in T'$ such that $p[I'] = A'$.

Let $\max A'$ be the set of maximal elements of A' (perhaps the empty set). For each $\alpha \in \max A'$, let $T'_\alpha := \{I \cap P_\alpha : I \in T'\}$. Routine checking verifies that T'_α is a convex sublattice of T_α. Let $s_\alpha(T'_\alpha) = P'_\alpha$ and define φ on $\mathcal{C}_L(\mathcal{I}(P))$ as follows:

$$\varphi(T') = \bigcup\{P_\alpha : \alpha \in A' \setminus \max A'\} \cup \bigcup\{P'_\alpha : \alpha \in \max A'\}.$$

Claim 4.36. *The map $\varphi : \mathcal{C}_L(\mathcal{I}(P)) \to \mathcal{I}(P)$ is an order preserving selection map.*

Proof of the Claim. First let us see that $\varphi(T') \in T'$. It is obvious that $\varphi(T') \in \mathcal{I}(P)$. If $\max A' = \varnothing$ then the conclusion is immediate since $\bigcup\{P_\alpha : \alpha \in A'\}$ is the only element $I \in T'$ with $p[I] = A'$.

Let us assume that $\max A' \neq \varnothing$. Then $\bigcup\{P_\alpha : \alpha \in A' \setminus \max A'\} \subseteq I$ for each $I \in T'$ with $p[I] = A'$. For each $\alpha \in \max A'$, select $I_\alpha \in T'$ such that $I_\alpha \cap P_\alpha = P'_\alpha$. Since A has no infinite antichain, $\max A'$ is finite and, hence, these two initial segments belong to T':

$$\bigcap(I_\alpha : \alpha \in \max A') \cap I' \quad \text{and} \quad \bigcup(I_\alpha : \alpha \in \max A') \cup I'.$$

It is easy to see that $\varphi(T')$ is between these two sets in $\mathcal{I}(P)$ and so belongs to the convex sublattice T'.

Now, let $T', T'' \in \mathcal{C}_L(\mathcal{I}(P))$ with $T' \leq T''$. We check that $\varphi(T') \subseteq \varphi(T'')$. Let $x \in \varphi(T')$. Then $x \in P_\alpha$ for some $\alpha \in A$. If α is not maximal in A', defined as above, then since we have $A' \subseteq A''$, α is not maximal in A'', thus $x \in \varphi(T'')$. If α is maximal in A' and not maximal in A'' then the same conclusion holds. The only remaining case is α is maximal in A' and A''. In this case,

$$x \in P'_\alpha = s_\alpha(T'_\alpha) \subseteq s_\alpha(T''_\alpha) = P''_\alpha \subseteq \varphi(T'').$$

This depends on the fact that $T'_\alpha \leq T''_\alpha$, a consequence of $T' \leq T''$. □

With that, the proof of the lemma is complete. □

Case 3. If P is an augmentation of a poset Q then $\mathcal{I}(P)$ is embeddable in $\mathcal{I}(Q)$; if $\mathcal{I}(Q)$ has the $CLSP$ then Lemma 4.24 asserts that $\mathcal{I}(P)$ has the $CLSP$.

5. Conclusion and further developments

The following table summarizes various preservation properties under operations relevant to this investigation, and mentions one remaining open question.

	Property of a lattice T		
Preservation under	CLSP	CLFPP	$\mathcal{C}_L(T)$ is complete
Retracts	yes: Lemma 4.23	yes: Lemma 4.13	yes: Corollary 4.11
Quotient	yes: Corollary 4.25	yes: Lemma 4.13	yes: Corollary 4.11
Finite product	yes: Lemma 4.21	Open	yes: Corollary 4.20

To be specific we pose the following:

Problem 5.1. *Is CLFPP preserved under finite products?*

In Lemma 4.22, we showed that every quotient of a lattice with $CLSP$ is a retract of that lattice. We do not know if the corresponding statement is true for CLFPP and $\mathcal{C}_L(T)$ being complete. We therefore ask:

Problem 5.2.
(1) *Is every quotient of a lattice with $CLFPP$ a retract of that lattice?*
(2) *Is every quotient of a lattice T such that $\mathcal{C}_L(T)$ is complete a retract of that lattice?*

In this paper we have considered multivalued maps defined on a poset P whose values belong to a particular subset $\mathcal{D}$ of $\wp(P)$, this set being either $\mathcal{C}(P)$, or $\mathcal{C}_L(T)$ provided that P is a lattice T. In these two cases, we have tried to relate the fixed point property for those maps to the order structure of $\mathcal{D}$. There are other sets $\mathcal{D}$ to consider, notably the set of *bounded* sets, sets of the form $S(\mathcal{A}, \mathcal{B})$ where $(\mathcal{A}, \mathcal{B})$ is a separable pregap of P. One could rather consider other structures than posets. Metric spaces seems to be appropriate. If we look at a metric space (E, d) as an object similar to a poset, the Hausdorff distance on $\wp(E)$ is the analog of the preorder we defined on the power set of a poset, and the *non-expansive* maps f from E to $\wp(E)$ (satisfying $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in E$) the analog of the order preserving maps. In the theory of metric spaces, the spaces analogous to complete lattices seems to be the *hyperconvex metric spaces* introduced by N.Aronszajn and Panitchpakdi in 1956, e.g. bounded hyperconvex metric spaces have FPP (R.Sine, P.M.Soardi, 1979). Also, analogs of convex sublattices seem to be the up directed unions of intersections of balls (for hyperconvex spaces see [14] and for the analogies between posets and metric spaces see [12]).

We conclude with a specific problem in this direction.

Problem 5.3. *For which metric spaces does every multivalued map into the set of up directed unions of intersections of balls have a fixed point?*

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