# Some quotients of chain products are symmetric chain orders

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**Abstract**

Canfield and Mason have conjectured that for all subgroups $G$ of the automorphism group of the Boolean lattice $B_n$ (which can be regarded as the symmetric group $S_n$) the quotient order $B_n/G$ is a symmetric chain order. We provide a straightforward proof of a generalization of a result of K. K. Jordan: namely, $B_n/G$ is an SCO whenever $G$ is generated by powers of disjoint cycles. In addition, the Boolean lattice $B_n$ can be replaced by any product of finite chains. The symmetric chain decompositions of Greene and Kleitman provide the basis for partitions of these quotients.

**Keywords:** symmetric chain decomposition, Boolean lattice, quotients

## 1 Introduction

There are several familiar notions of symmetry for the family of finite ranked partially ordered sets. This family can be defined in more general ways (see [9]), but for our purposes, all of our finite partially ordered sets $P$ have a minimum element $0_P$ and for all $x \in P$, all saturated chains $C \subseteq P$ with minimum element $0_P$ and maximum $x$ have the same length $r_P(x) \coloneqq |C| - 1$. Such $P$ are called *ranked* posets, $r = r_P$ is the *rank function* and $r(P)$, the maximum over all $r(x), x \in P$, is the rank of $P$. Note that a ranked ordered set satisfies the Jordan-Dedekind chain condition: for all $x \leqslant y$ in $P$, all saturated chains in the interval $[x, y]$ have the same length.

In a ranked order $P$ the chain $x_1 < x_2 < \cdots < x_k$ is a *symmetric chain* if it is saturated and if $r(x_1) + r(x_k) = r(P)$. A *symmetric chain decomposition* or *SCD* of $P$ is a partition of $P$ into symmetric chains. If $P$ has an SCD, call $P$ a *symmetric chain order*, or an *SCO*. Here, we are concerned with ordered sets based on the *Boolean lattice*, denoted $B_n$, which is the power set of $[n] = \{1, 2, \dots, n\}$ ordered by containment. Clearly $B_n$ is a ranked poset, with $\emptyset$ being the minimum element, and $r(A) = |A|$ for all $A \subseteq [n]$. In fact, it is an SCO [2].

We are interested in ordered sets defined by actions of the automorphism group of $B_n$. It is well-known that this group is faithfully induced by the symmetric group $S_n$ of all permutations on the underlying set $[n]$, so we will refer to $S_n$ as the automorphism group of $B_n$. Given any subgroup $G$ of $S_n$, the *quotient* $B_n/G$ has as its elements the orbits in $B_n$ under $G$

$$[A] = \{B \mid B = \sigma(A), \text{for some } \sigma \in G\},$$

$A \in B_n$, ordered by

$$[A] \leqslant [B] \iff X \subseteq Y \text{ for some } X \in [A] \text{ and } Y \in [B].$$

In studying Venn diagrams, Griggs, Killian and Savage [12] explicitly constructed an SCD of the quotient $B_n/G$ for $n$ prime and given that $G$ is generated by a single $n$-cycle. They asked if this *necklace* poset is an SCO for arbitrary $n$. Canfield and Mason [3] made a much more general conjecture: for all subgroups $G$ of $S_n$, $B_n/G$ is a symmetric chain order.

Jordan [14] gave a positive answer to the question of Griggs, Killian and Savage, basing the SCD of the quotient on the explicit construction of an SCD in $B_n$ by Greene and Kleitman [10]. The construction in [14] requires an intermediate equivalence relation and some careful analysis. Here we provide a more direct proof of a generalization of Jordan's theorem by "pruning" the Greene-Kleitman SCD. More generally, we show that $B_n/G$ is an SCO provided that $G$ is generated by powers of disjoint cycles (see Theorem 1). We also provide a different proof that $B_n/G$ is an SCO when $G$ is a 2-element subgroup generated by a reflection, based on an SCD of $B_{\lfloor n/2 \rfloor}$.

The ordered sets $B_n/G$ do share several forms of symmetry or regularity with the Boolean lattice. An SCO $P$ is necessarily rank-symmetric, rank-unimodal, and strongly Sperner (see, for instance, [14] for definitions). A result of Stanley [17] shows that $B_n/G$ has these three properties for all subgroups $G$ of $S_n$. However, these three conditions are not sufficient to yield symmetric chain decompositions.

On the other hand, Griggs [11] showed that a ranked ordered set with the LYM property, rank-symmetry and rank-unimodality is an SCO. It is not the case that all SCOs have the LYM property (see [11] for examples), in fact, not all quotients $B_n/G$ have the LYM property. However, if $n$ is prime and $G$ is generated by an $n$-cycle, then $B_n/G$ does satisfy LYM, giving the Griggs, Killian and Savage result. Pouzet and Rosenberg [16] obtain Stanley's results and "local" families of symmetric chains for more general structures than the quotients $B_n/G$, but their results do not show that $B_n/G$ is an SCO.

## 2 The Main Results

There are two results for quotients of $B_n$ by groups generated by powers of disjoint cycles and for a particular 2-element group. The third theorem concerns quotients of powers of a finite chain, with a corollary for products of chains of differing length.

**Theorem 1.** *Let $G$ be a subgroup of $S_n$ generated by powers of disjoint cycles. Then the partially ordered set $B_n/G$ is a symmetric chain order.*

The proof of Theorem 1 follows from this sequence of results. The new proof of Lemma 2, which is a modest generalization of Jordan's result, is given in Section 3.

**Lemma 2.** *Let $\sigma$ be an $n$-cycle in $S_n$ and let $H$ be a subgroup of the group generated by $\sigma$. Then $B_n/H$ is a symmetric chain order.*

The following fact is well-known and can be proved by an argument much like the original proof in [2] that the divisor lattice of an integer is an SCO. (In [9], this is credited to Alekseev [1].)

**Lemma 3.** *Let $P$ and $Q$ be partially ordered sets. If $P$ and $Q$ are symmetric chain orders then so is $P \times Q$.*

In the following lemma, use this notation. Suppose that $\sigma_j$ $(j = 1, 2, \ldots, t)$ are disjoint cycles in $S_n$ and that $\rho_j = \sigma_j^{r_j}$, for integers $r_1, r_2, \ldots, r_t$. Let $X_j$ be the subset of $[n]$ of elements moved by $\rho_j$ $(j = 1, 2, \ldots, t)$, and let $X_0$ be all elements of $[n]$ fixed by all the $\rho_j$'s. Let $B(X)$ denote the Boolean lattice of all subsets of a set $X$.

**Lemma 4.** *Let $H_j$ be the subgroup of $S_n$ generated by $\rho_j$ $(j = 1, 2, \ldots, t)$ and let $G$ be the subgroup generated by $\{\rho_1, \rho_2, \ldots, \rho_t\}$. Then*

$$B_n/G \;\cong\; B(X_0) \times B(X_1)/H_1 \times \cdots \times B(X_t)/H_t.$$

*Proof.* For any $A \subseteq [n]$, let $[A]$ denote its equivalence class in $B_n/G$, and let $A_j = A \cap X_j$, $j = 0, 1, \ldots, t$. Define a map $\Phi$ on $B_n/G$ by $\Phi([A]) = (A_0, [A_1], \ldots [A_t])$. From the definition of the ordering of the quotient, $[A] \leqslant [B]$ in $B_n/G$ if and only if there is some $\tau \in G$ such that $A \subseteq \tau(B)$. Then $\tau = \rho_1^{i_1} \rho_2^{i_2} \cdots \rho_t^{i_t}$ for nonnegative integers $i_1, i_2, \ldots, i_t$. The claimed isomorphism follows from this fact:

$$A \subseteq \tau(B) \text{ if and only if } A_0 \subseteq B_0 \text{ and } A_j \subseteq \sigma^{i_j}(B_j) \text{ for } j = 1, 2, \ldots, t.$$

□

The following is actually a corollary of Theorem 1. Indeed, a proof based on an approach like that used in the proof of Theorem 1 – a greedy pruning of a Greene-Kleitman SCD – can be shown to provide a basis for the proof offered in Section 4. However, the proof in Section 4 provides some insight into the Greene-Kleitman SCD and may be of use for other choices for the group of permutations, such as the dihedral group, which is an appealing next case for study.

**Theorem 5.** *Let $G$ be a 2-element subgroup with non-unit element a product of disjoint transpositions. Then the partially ordered set $B_n/G$ is a symmetric chain order.*

The last result concerns quotients defined by automorphism groups of products of chains. Given any partially ordered set $P$ and subgroup $G$ of its automorphism group $\mathrm{Aut}(P)$, the quotient $P/G$ has elements the orbits $[x]$ on $P$ defined by $G$ with $[x] \leqslant [y]$ in $P/G$ if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \leqslant y'$ in $P$. For any finite chain $C$, positive integer $m$ and $\alpha \in \mathrm{Aut}(C^m)$, there is some $\phi \in S_m$ such that

$$\alpha(c_1, c_2, \dots, c_m) = (c_{\phi^{-1}(1)}, c_{\phi^{-1}(2)}, \dots, c_{\phi^{-1}(m)}), \text{ for all } (c_1, c_2, \dots, c_m) \in C^m.$$

This follows easily from the action of the automorphisms on the covers of the minimum element of the chain product. (It is also a consequence of a result of Chang, Jónsson and Tarski [4], on the strict refinement property for product decompositions of partially ordered sets.) In particular, automorphism groups of powers of chains behave as those of the Boolean lattice and we can regard $\mathrm{Aut}(C^m)$ as the symmetric group $S_m$ acting on the coordinates of $C^m$.

**Theorem 6.** *Let $C$ be a chain and let $K$ be a subgroup of $S_m$ generated by powers of disjoint cycles. Then $C^m/K$ is an SCO.*

The proof, presented in Section 5, is a consequence of the proof of Lemma 2 and some observations on the Greene-Kleitman SCD. V. Dhand [5] has a new, very interesting result that is more general than the essential part of Theorem 6: if $P$ is any SCO then so is $P^n/\mathbb{Z}_n$. His arguments depend upon algebraic tools. P. Hersh and A. Schilling give a new proof of Jordan's result via an explicit combinatorial construction of an SCD in $B_n/\mathbb{Z}_n$ based on representation of the special linear group [13].

We note that Theorem 6 can be stated more generally for chain products. Let $P = \prod_{i=1}^{n} C_i^{m_i}$ where $C_j \not\cong C_k$ for $j \neq k$. It is easy to see that each automorphism of $P$ factors into an $n$-tuple from $\prod_{i=1}^{n} \mathrm{Aut}(C_i^{m_i})$ and that each $\mathrm{Aut}(C_i^{m_i}) \cong S_{m_i}$. (For a much more general result based [4], see [6].) Thus, if $K$ is a subgroup of $\mathrm{Aut}(P)$ which also factors into a product of subgroups of $S_{m_i}$ of the form covered by Theorem 6 then, by Lemmas 3 and 4, $P/K$ is an SCO. In particular, we have this consequence.

**Corollary 7.** *Let $P$ be a product of chains and let $K$ be a subgroup of* $\mathrm{Aut}(P)$ *that is generated by powers of disjoint cycles. Then $P/K$ is an SCO.*

We use Corollary 7 to deal with some cases where $K$ does not factor so nicely in [8].

# 3 The Proof of Lemma 2

We use the natural order $1 < 2 < \ldots < n$ on $[n]$ and may assume that the $n$-cycle $\sigma$ is $(1\ 2\ \cdots\ n)$. This is valid because any $n$-cycle $\rho$ is a conjugate of $(1\ 2\ \cdots\ n)$ and for any subgroup $K$ of $S_n$ and any $\pi \in S_n$, $B_n/K \cong B_n/\pi^{-1}K\pi$ via $[A] \mapsto [\pi(A)]$.

We first describe the procedure for obaining an SCD of $B_n/H$ based on the Greene-Kleitman SCD of $B_n$ then verify that the procedure yields the claimed SCD.

Let $C_1, C_2, \ldots, C_t$, where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman decomposition, ordered by decreasing length. For all $A \in B_n$, $[A]$ is the equivalence class containing $A$ in $B_n/H$ where $H$ is the subgroup of $S_n$ generated by $\rho = \sigma^s$.

**Claim 1:** There is a family $\mathcal{C} = \{C'_{i_1}, C'_{i_2}, \ldots C'_{i_m}\}$, with $(i_1, i_2, \ldots, i_m)$ a subsequence of $(1, 2, \ldots, t)$, that satisfies these conditions:

**(3.1)** for all $1 \leqslant j \leqslant m$, $C'_{i_j} \subseteq C_{i_j}$ and is a symmetric chain in $B_n$;

**(3.2)** for all $1 \leqslant r < s \leqslant m$ and for all $A \in C'_{i_r}, B \in C'_{i_s}$, $A \notin [B]$; and,

**(3.3)** for all $[X]$ there is some $Y \in [X]$ such that $Y \in C'_{i_j}$ for some $j$.

For $j = 1, 2, \ldots, m$, let $\widehat{C}_j = \{[A] \mid A \in C'_{i_j}\}$. Then the chains $\widehat{C}_1, \widehat{C}_2, \ldots, \widehat{C}_m$ cover $B_n/H$ (by **(3.3)**), the sets are disjoint (by **(3.2)**), and form symmetric chains (by **(3.1)**). Thus, it is enough to verify **Claim 1** in order to prove Lemma 2.

Several properties of the Greene-Kleitman SCD of $B_n$ are needed. For the most part, these are well-known – see, for instance, the descriptions in [9] and [14]. It is useful to regard members of $B_n$ both as subsets of $[n]$ and as binary sequences of length $n$, defined with respect to the natural order. (Indeed, one needs to fix an order to speak of *the* Greene-Kleitman SCD.) The SCD is obtained by a bracketing or pairing procedure that has several equivalent descriptions. Here are two that are useful to us. Let $A \subseteq [n]$.

**(3.4)** If $1 \notin A$ and $2 \in A$, pair 1 and 2; define $p_A(2) = 1$. Suppose that we have considered $1, 2, \ldots, k-1$. If $k \in A$ and there is some $j < k$, $j \notin A$ such that $j$ is unpaired, then let $p_A(k)$ be the maximum such $j$ and say $p_A(k)$ and $k$ are *paired*. Continue for all $k$ in $[n]$.

**(3.5)** For all $x \in A$ such that precisely half of the elements of the interval $[y, x]$ are members of $A$, for some $1 \leqslant y < x$, let $p_A(x)$ be the maximum such $y$.

Let $R(A)$ be the set of all $x$ for which $p_A(x)$ is defined, let $L(A) = \{p_A(x) \mid x \in R(A)\}$, and let $P(A) = L(A) \cup R(A)$. Now set

$$f(A) = A \cup \{z\},\ z = \min([n] - (A \cup L(A))),$$

if $[n] - (A \cup P(A)) \neq \emptyset$; otherwise $f(A)$ is undefined. Then this rule inverts $f$:

$$f^{-1}(B) = B - \{z\},\ z = \max(B - R(B)).$$

Let $\mathcal{C}(A) = \{f^k(A) \mid k \in \mathbb{Z}\}$. As $A$ runs over all of $B_n$, the distinct $\mathcal{C}(A)$'s provide the Greene-Kleitman SCD of $B_n$.

The following hold for all $A \in B_n$.

**(3.6)** For all $x \in R(A)$, $[p_A(x), x] \subseteq P(A)$.

**(3.7)** $\mathcal{C}(A) = \{X \in B_n \mid R(X) = R(A)\}$ and $p_X(a) = p_A(a)$ for all $X \in \mathcal{C}(A)$ and for all $a \in R(A)$.

**(3.8)** $\min(\mathcal{C}(A)) = R(A)$, $\max(\mathcal{C}(A)) = [n] - L(A)$; in fact, $\mathcal{C}(A)$ is the chain

$$R(A) \subset R(A) \cup \{a_1\} \subset R(A) \cup \{a_1, a_2\} \subset \ldots \subset R(A) \cup \{a_1, a_2, \ldots a_t\} = [n] - L(A),$$

where $[n] - (R(A) \cup L(A)) = \{a_1 < a_2 < \ldots < a_t\}$.

The following two lemmas provide properties of this SCD that substantiate **Claim 1**. Given a symmetric chain $C$ in $B_n$ and $X \in C$ with $|X| \leqslant \lfloor n/2 \rfloor$, let $X^*$ be the member of $C$ with $|X^*| = n - |X|$.

**Lemma 8.** *For $i = 1, 2, \ldots, t$ and for all $X \in C_i$ with $|X| \leqslant \lfloor n/2 \rfloor$, $(\sigma(X))^* = \sigma(X^*)$. Thus $(\sigma^j(X))^* = \sigma^j(X^*)$ for all integers $j$, so $(\rho(X))^* = \rho(X^*)$ for all $\rho \in H$.*

A special case of the preceding lemma is in [7]. Since this reference is a technical report and the result does not appear to be available in the literature, we prove this below. (We note that for $n$ odd, the middle edges of the Greene-Kleitman SCD are the edges of the well-known lexicographic matching in the middle two levels graph. The invariance of this matching under rotation is a special case of a result of Kierstead and Trotter [15] on what they call $i$-lexical matchings.)

Here is the second property of the Greene-Kleitman SCD that we require.

**Lemma 9.** *Let $w \in \{1, 2, \ldots, t\}$, and let $A \in C_w$ with $|A| \leqslant \lfloor n/2 \rfloor$. Suppose that there is some $B \in [A]$ such that $B \in C_j$ for some $j < w$. Then there is some $k < w$ and $D \in C_k$ such that $D \in [f^{-1}(A)]$, provided that $f^{-1}(A)$ is defined.*

To prove **Claim 1** from these facts, define $\mathcal{C}$ inductively.

First, let $i_1 = 1$ and $C'_{i_1} = C_1$. Suppose that $C'_{i_1}, C'_{i_2}, \ldots, C'_{i_k}$ are defined. If there exists $i \in \{i_k + 1, \ldots, t\}$ such that for some $X \in C_i$,

$$[X] \cap (\bigcup_{j=1}^{k} C'_{i_j}) = \emptyset \tag{1}$$

let $i_{k+1}$ be the least such $i$ and let

$$C'_{i_{k+1}} = \Big\{ Y \in C_{i_{k+1}} \;\Big|\; [Y] \cap (\bigcup_{j=1}^{k} C'_{i_j}) = \emptyset \Big\}. \tag{2}$$

If there is no such $i$ then $m = k$, that is, $C'_{i_m}$ is the last chain required by **Claim 1** and the procedure is complete.

If $Y \in C'_{i_{k+1}}$, with $|Y| \leqslant \lfloor n/2 \rfloor$ then $Y^* \in C'_{i_{k+1}}$, by the dual of Lemma 8. Also, if $Z \in C_{i_{k+1}}$ and $Y \subseteq Z \subseteq Y^*$ where $Y \in C'_{i_{k+1}}$ then $Z \in C'_{i_{k+1}}$ by Lemma 9 and Lemma 8. Thus, $C'_{i_{k+1}}$ is symmetric in $B_n$ and **(3.1)** holds. Equation (2) verifies **(3.2)**; **(3.3)** follows from (1) and (2).

*Proof of Lemma 8.* The proof is divided into cases depending upon which of $R(X) \subseteq X \subseteq X^*$ contain $n$. It is not possible that $n \in X - R(X)$, because $|X| \leqslant \lfloor n/2 \rfloor$ means that for some $y < n$ precisely half the elements of $[y, n]$ are in $X$, and, hence, $n \in R(X)$ by **(3.5)**. Consequently, there are three cases. In each case, we show that

$$R(\sigma(X^*)) = R(\sigma(X)),$$

apply **(3.7)** to see that $\sigma(X^*)$ and $(\sigma(X))^*$ are both members of $\mathcal{C}(\sigma(X))$, and conclude that $\sigma(X^*) = (\sigma(X))^*$ since these sets both have cardinality $n - |\sigma(X)|$.

**Case 1:** $n \notin X^*$

Since $n \in [n] - L(X) = \max(\mathcal{C}(X))$ and $n \notin X^*$, $X \neq \min(\mathcal{C}(X)) = R(X)$. Thus, there exists $y = \min(X - R(X))$. If $y = 1$ then $p_{\sigma(X)}(2) = 1 = p_{\sigma(X^*)}(2)$. For each $z \in R(X)$, $\sigma(z) = z + 1 \in R(\sigma(X))$ and each $z + 1 \in R(\sigma(X))$ has $z \in R(X)$ apart from $z + 1 = 2$. Thus,

$$\begin{aligned} R(\sigma(X)) &= \sigma(R(X) \cup \{2\} && \text{since } p_{\sigma(X)}(2) = 1, \\ &= \sigma(R(X^*)) \cup \{2\} && \text{by } (\mathbf{3.7}), \\ &= R(\sigma(X^*)) && \text{since } p_{\sigma(X^*)}(2) = 1. \end{aligned}$$

If $y > 1$ we claim that $[1, y-1] \subseteq P(X)$. Note that $y-1 \in X$ as otherwise $y \in R(X)$ with $p_X(y) = y - 1$, contradicting the choice of $y$. By the minimality of $y$, $y - 1 \in R(X)$ and, by **(3.7)**, $[p_X(y-1), y-1] \subseteq P(X)$. Continue in the same manner, with $p_X(y-1) - 1$ in place of $y$, and thereby verify the claim that $[1, y-1] \subseteq P(X)$. The argument is just about the same as when $y = 1$ except we use the fact that $[1, y-1] \subseteq P(X)$ and $1 \notin X$, so, $p_{\sigma(X)}(y+1) = 1$:

$$R(\sigma(X^*)) = \sigma(R(X^*)) \cup \{y+1\} = \sigma(R(X)) \cup \{y+1\} = R(\sigma(X)).$$

**Case 2:** $n \in R(X)$

Every element of $R(\sigma(X))$ is in $\sigma(R(X))$ and every element of $\sigma(R(X))$, except for 1, is in $R(\sigma(X)$. Thus,

$$R(\sigma(X)) = \sigma(R(X)) - \{1\} = \sigma(R(X^*)) - \{1\} = R(\sigma(X^*)).$$

**Case 3:** $n \in X^* - X$

Since $n \in X^* - X$, **(3.8)** shows that $X^* = \max(\mathcal{C}(X)) = [n] - L(X)$ and, thus, $X = \min(\mathcal{C}(X)) = R(X)$.

If $z + 1 \in R(\sigma(X))$ then $z \in X = R(X)$, so $R(\sigma(X)) \subseteq \sigma(R(X))$. Conversely, $1 \notin \sigma(R(X))$, and any $z + 1 \in \sigma(R(X))$ is obviously a member of $R(\sigma(X))$. Thus, $R(\sigma(X)) = \sigma(R(X))$. Similarly, since $n \notin R(X)$, it follows that $R(\sigma(X^*)) = \sigma(R(X^*))$. Hence, $R(\sigma(X^*)) = R(\sigma(X))$.

Since $(\sigma(X))^* = \sigma(X^*)$ for all $X$ with $|X| \leqslant \lfloor n/2 \rfloor$, we can apply induction on $j$ to conclude that $(\sigma^j(X))^* = \sigma^j(X^*)$:

$$(\sigma^j(X))^* = (\sigma(\sigma^{j-1}(X)))^* = \sigma((\sigma^{j-1}(X))^*) = \sigma(\sigma^{j-1}(X^*)) = \sigma^j(X^*). \quad \square$$

*Proof of Lemma 9.* As before, let $C_1, C_2, \dots, C_t$, where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman decomposition, ordered by decreasing length, and let $\sigma = (1\ 2\ \cdots\ n)$. Let $A \in C_w$ with $|A| \leqslant \lfloor \frac{n}{2} \rfloor$. Suppose that there exists a $j < w$ such that $B \in C_j$ and $B \in [A]$. Hence there is an integer $r$ such that $B = \sigma^r(A)$.

Assume that $f^{-1}(A)$ is defined. We show that there is a $k < w$ such that $D \in C_k$ and $D \in [f^{-1}(A)]$. Since $f^{-1}(A)$ is defined, $A - R(A) \neq \emptyset$. Let $y = \max(A - R(A))$. We may assume that $-(y-1) \leqslant r \leqslant n - y$, $r \neq 0$. We consider two cases:

**Case 1:** $y + r \in R(B)$

Then $r > 0$ since otherwise $y$ would also be paired in $A$, contrary to its choice. Each $z \in B$ with $y + r < z$ must be in $R(B)$ since $y < z - r$ so, by the choice of $y$, $z - r \in R(A)$. Now consider the binary sequence $\sigma^r(f^{-1}(A))$ contained in some chain $C_k$. Recall that $f^{-1}(A) = A - \{y\}$ and note that $\sigma^r(f^{-1}(A)) = B - \{y + r\}$. It follows from this that $\sigma^r(f^{-1}(A))$ must have one fewer pairs than $B$, since $y + r$ will be unpaired in $\sigma^r(f^{-1}(A))$ while $y + r$ is paired in $B$, and there are no other differences in the pairings. By **(3.8)**, $|C_k| > |C_j|$, so $k < j < w$, as desired.

**Case 2:** $y + r \in B - R(B)$

If $y + r = \max(B - R(B))$ then we are done, since then we have $f^{-1}(B) = \sigma^r(f^{-1}(A))$. So suppose instead $z = \max(B - R(B))$ where $y + r < z$. If $r > 0$ then $z - r \in (A - R(A))$, contrary to the choice of $y$. Thus $r < 0$. If $z - r \leqslant n$ then $z - r$ would be an unpaired element of $A$, since $y < z - r$ remains unpaired in $A$. This would contradict the choice of $y$. Thus $n < z - r$.

We now prove that for some $p$, $\sigma^p(B)$ is the maximum element of its chain, and its chain is not a singleton. This will contradict the fact that $|A| \leqslant \lceil \frac{n}{2} \rceil$, since $|\sigma^p(B)| = |A|$.

Since $\sigma^{-r}(B) = A$, $-r > 0$, we obtain $A$ from $B$ by applying $\sigma$ $-r$ times. Since $n < z - r$ there is some $p$ such that $\sigma^p(z) = n$. Let $X = \sigma^p(B)$. Then, $\sigma^p(z) \in X - R(X)$. Because $n \in X - R(X)$, $X = [n] - L(X)$. By **(3.8)**, $X$ is the maximum element of its chain. The chain containing $X$ is not a singleton, since $X - R(X) \neq \emptyset$. □

# 4 The proof of Theorem 5

Let $\rho = (i_1 j_i)(i_2 j_2) \cdots (i_k j_k)$, where the transpositions are pairwise disjoint, let $X = \bigcup_{r=1}^{k} \{i_r, j_r\}$, and let $G = \{1, \rho\}$. Then

$$B_n/G \cong B(X)/G \times B([n] - X)$$

via the mapping $[A] \mapsto ([A \cap X], A - X)$ for all $A \subseteq [n]$. By Lemma 3, we may assume that $n$ is even and that $n = 2k$. Using the remark about conjugation at the beginning of Section 3, we may assume that $\rho = (1\ 2k)(2\ 2k-1) \cdots (k\ k+1)$. As noted in the introduction, $G$ can be generated by a power of a $2k$-cycle, so Theorem 1 applies. (In fact, $\rho = \tau^{-1}\sigma\tau$ where $\sigma = (1\ 2 \cdots 2k)$ and $\tau = (k+1\ 2k)(k+2\ 2k-1) \cdots$. ) And, as we shall see, its proof method can be adapted to give the proof we offer here. However, the argument below might help with the most interesting open case, namely, showing that $B_n/D_{2n}$ is an SCO for the dihedral group $D_{2n}$.

Regard each $A \in B_{2k}$ as a concatenated pair of binary strings of length $k$. That is, $A = \mathbf{b}_1\mathbf{b}_2^r$, where $\mathbf{b}_1, \mathbf{b}_2 \in \{0,1\}^k$ and $\mathbf{b}^r$ is the reverse of the binary $k$-sequence $\mathbf{b}$. Then the equivalence classes in $B_n/G$ are the sets $\{\mathbf{b}_1\mathbf{b}_2^r, \mathbf{b}_2\mathbf{b}_1^r\}$; these sets have 2 elements except in the case that $\mathbf{b}_1 = \mathbf{b}_2$.

Let $C_1, C_2, \ldots, C_t$, where $t = \binom{k}{\lfloor k/2 \rfloor}$, be any symmetric chain decomposition of $B_k$, ordered by decreasing length. We define a total ordering $\preccurlyeq$ on $B_k = \{0,1\}^k$ as follows:

$$\mathbf{b}_r \preccurlyeq \mathbf{b}_s \text{ if } \mathbf{b}_r \in C_i, \mathbf{b}_s \in C_j,\ i < j, \text{ or if } \mathbf{b}_r \subseteq \mathbf{b}_s \text{ in } C_i \text{ for some } i. \tag{3}$$

For $1 \leqslant i < j \leqslant t$, let $P_{ij} = C_i \times C_j$, with the coordinate-wise ordering induced by the containment order on $B_k$, for each $i = 1, 2, \ldots, t$, let

$$P_{ii} = \{(\mathbf{b}_r, \mathbf{b}_s) \in C_i \times C_i \mid \mathbf{b}_r \subseteq \mathbf{b}_s\},$$

ordered coordinate-wise, and let

$$P = \bigcup_{1 \leqslant i \leqslant j \leqslant t} P_{ij},$$

again, ordered coordinate-wise. Thus, $P$ is a subset of $B_{2k}$ with the exactly the ordering inherited from the Boolean lattice.

In fact, with $r_P$ and $r_{B_k}$ as the rank functions in $P$ and $B_k$, respectively, then for $1 \leqslant i \leqslant j \leqslant t$ and with $r_{B_k}(\min C_i) = r_i$, and $r_{B_k}(\max C_i) = k - r_i$,

$$r_P(\min P_{ij}) = r_i + r_j,\ r_P(\max P_{ij}) = 2k - (r_i + r_j), \text{ and } l(P_{ij}) = 2k - 2(r_i + r_j).$$

We see that each $P_{ij}$ is a symmetric subset of $B_{2k}$ in which the covering relation is preserved, that is, $(\mathbf{b}_p, \mathbf{b}_q)$ is covered by $(\mathbf{b}_u, \mathbf{b}_v)$ in some $P_{ij}$ if and only if $\mathbf{b}_p\mathbf{b}_q^r$ is covered by $\mathbf{b}_u\mathbf{b}_v^r$ in $B_{2k}$.

Consider the map $\phi$ of $P$ to $B_{2k}/G$ defined by $\phi((\mathbf{b}_1, \mathbf{b}_2)) = \{\mathbf{b}_1\mathbf{b}_2^r, \mathbf{b}_2\mathbf{b}_1^r\}$. Since $\mathbf{b}_1 \preccurlyeq \mathbf{b}_2$ for all $(\mathbf{b}_1, \mathbf{b}_2) \in P$, $\phi$ is injective. It is obviously a surjection. It is also order-preserving: if $(\mathbf{b}_p, \mathbf{b}_q) \leqslant (\mathbf{b}_u, \mathbf{b}_v)$ in $P$ then $\mathbf{b}_p\mathbf{b}_q^r \leqslant \mathbf{b}_u\mathbf{b}_v^r$ in $B_{2k}$.

Since the rank of an equivalence class in $B_{2k}/G$ is the rank of its members in $B_{2k}$, it follows that a symmetric chain in $P$ is a symmetric chain in $B_{2k}/G$. Thus, it is enough to proof the following.

**Claim 2:** $P$ has a symmetric chain decomposition.

Since $P$ is partitioned by $P_{ij}$, $1 \leqslant i \leqslant j \leqslant t$, each of which preserve the covering relation in $P$, it is enough to prove that each $P_{ij}$ has a partition into chains, each of which is symmetric in $P$.

For $1 \leqslant i < j \leqslant t$, $P_{ij} = C_i \times C_j$ is a cover-preserving subset of $P$, with minimum element at level $r_i + r_j$ and maximum element at level $2k - (r_i + r_l)$ in $P$, a partially ordered set of length $2k$. Then the "standard" symmetric partition of a product of two chains (the original partition in [2]) provides symmetric chains in $P$.

For $i = 1, 2, \ldots, t$, $P_{ii} = \{(\mathbf{b}_r, \mathbf{b}_s) \in C_i \times C_i \mid \mathbf{b}_r \subseteq \mathbf{b}_s\}$, where $C_i$ is the chain of binary strings $\mathbf{b}_{r_i} \subset \mathbf{b}_{r_i+1} \subset \cdots \subset \mathbf{b}_{k-r_i}$, where $r_{B_k}(\mathbf{b}_s) = s$ in $B_k$, that is, is an $s$-element set, for $s = r_i, r_i+1, \ldots, k-r_i$. Then $P_{ii}$ is an interval in $P$ with minimum element at level $2r_i$ and maximum element at level $2k-2r_i$ in $P$. Also,

$$(\mathbf{b}_{r_i}, \mathbf{b}_{r_i}) < (\mathbf{b}_{r_i}, \mathbf{b}_{r_i+1}) < \cdots < (\mathbf{b}_{r_i}, \mathbf{b}_{k-r_i}) < (\mathbf{b}_{r_i+1}, \mathbf{b}_{k-r_i}) < \cdots < (\mathbf{b}_{k-r_i}, \mathbf{b}_{k-r_i})$$

is a symmetric chain in $P$ and $P_{ii} - C$ is a cover-preserving subset of $P$, isomorphic to the product of two chains, with minimum element $(\mathbf{b}_{r_i+1}, \mathbf{b}_{r_i+1})$ at level $2r_i+2$ and maximum element $(\mathbf{b}_{k-r_i-1}, \mathbf{b}_{k-r_i-1})$ at level $2k-2r_i-2$. By induction, we have a decomposition of $P_{ii}$ by chains symmetric in $P$. This verifies **Claim 2** and completes the proof of Theorem 5.

# 5 The proof of Theorem 6

With Lemmas 3 and 4, it is enough to prove the result for $K$ generated by a single $m$-cycle. We assume that $C$ is the $k$-element chain $00\ldots0, 10\ldots0, 11\ldots0, 111\ldots1$ in the Boolean lattice $B_{k-1}$. Let $n = (k-1)m$. Then $C^m$ is the sublattice of $B_n$ consisting of all binary sequences of length $n$ of the form

$$\mathbf{b} = \mathbf{b_1 b_2 \ldots b_m}, \text{ where each } \mathbf{b}_i \in C.$$

That is, the elements of $C^m$ are exactly those $n$-sequences which are $m$ $(k-1)$-sequences, each comprised of 1's followed by 0's.

Let $C_1, C_2, \ldots, C_t$, where $t = \binom{n}{\lfloor n/2 \rfloor}$, be the symmetric chains in the Greene-Kleitman SCD of $B_n$, ordered by decreasing length, as in Section 3. We claim that for each $j$, $C_j \subseteq C^m$ or $C_j \cap C^m = \emptyset$.

Suppose that $\mathbf{b} = \mathbf{b_1 b_2 \ldots b_m} \in C_j \cap C^m$. With the notation in **(3.6) - (3.8)**, and applying these to $A = \mathbf{b}$, we can see that $i^{\text{th}}$ entry $b_i$ of $\mathbf{b}$ is determined as follows:

$$b_i = \begin{cases} 0 & i \in L(\mathbf{b}) \cup \{a_r, a_{r+1}, \ldots, a_t\} \\ 1 & i \in R(\mathbf{b}) \cup \{a_1, a_2, \ldots, a_{r-1}\}. \end{cases}$$

for some $r$.

If $\mathbf{b}$ is not the maximum element of $C_j$ then its successor $\mathbf{b}'$ is obtained by changing the 0 in position $a_r$ to a 1. Either $a_r = 1$ or the entry in $\mathbf{b}$ in position $a_r - 1$ is a 1, by **(3.6)**. Thus, $\mathbf{b}'$ consists of $m$ $(k-1)$-sequences of 1's followed by 0's and, so, belongs to $C_j \cap C^m$. If $\mathbf{b}$ is not the minimum element of $C_j$ then its predecessor $\mathbf{b}''$ is obtained by changing the 1 in position $a_{r-1}$ to a 0. Either $a_{r-1} = n$ or the entry in $\mathbf{b}$ in position $a_{r-1}+1$ is a 0, by **(3.6)**. Again $\mathbf{b}''$ consists of $m$ $(k-1)$-sequences of 1's followed by 0's and, so, belongs to $C_j \cap C^m$. Hence, if $C_j \cap C^m \neq \emptyset$ then $C_j \subseteq C^m$.

Let $K = \langle \phi^r \rangle$ where we may assume that $\phi = (12\cdots m)$. We need an SCD for $C^m/K$. We know that $C^m$ is a sublattice of $B_n$, as noted above, and that $\phi^r = \sigma^{(k-1)r}{}_{|C^m}$ where $\sigma = (12\cdots n) \in S_n$. As in the proof of Lemma 2, **Claim 1** gives an SCD $\widehat{C}_j = C'_{i_j}$, $j = 1, 2, \ldots, m$, of $B_n/H$ where $H = \langle \sigma^{(k-1)r} \rangle$. Thus, the subfamily of those $\widehat{C}_j$, $j = 1, 2, \ldots, m$, such that $\widehat{C}_j \subseteq C^m$ is an SCD for $C^m/K$.

## References


[1] V. B. Alekseev. Use of symmetry in finding the width of partially ordered sets (Russian). *Diskret. Analiz.*, 26: 20 - 35, 1974.

[2] N. G. de Bruijn, C. Tengbergen and D. Kruyswijk. On the set of divisors of a number. *Nieuw Arch. Wiskd.*, 23: 191 - 193, 1951.

[3] E. R. Canfield and S. Mason. When is a quotient of the Boolean lattice a symmetric chain order? Preprint, 2006.

[4] C. C. Chang, B. Jónsson and A. Tarski. Refinement properties for relational structures. *Fund. Math.*, 55: 249 - 281, 1964.

[5] V. Dhand. Symmetric chain decomposition of necklace posets. *Electron. J. Combin.*, 19: P26, 2012

[6] D. Duffus. Automorphism and products of ordered sets. *Algebra Universalis*, 19: 366 - 369, 1984.

[7] D. Duffus, P. Hanlon and R. Roth. Matchings and hamiltonian cycles in some families of symmetric graphs. *Emory University Technical Reports*, 1986.

[8] D. Duffus and K. Thayer, Quotients of chain products and symmetric chain decompositions. Preprint, 2012.

[9] K. Engel. *Sperner Theory.* Cambridge University Press, 1997.

[10] C. Greene and D. J. Kleitman. Strong versions of Sperner's theorem. *J. Combinatorial Theory A*, 20: 80 - 88, 1976.

[11] J. R. Griggs. Sufficient conditions for a symmetric chain order. *SIAM J. Appl. Math.* 32: 807 - 809, 1977.

[12] J. R. Griggs, C. E. Killian and C. D. Savage. Venn diagrams and symmetric chain decompositions in the Boolean lattice. *Electron. J. Combin.*, 11: R2, 2004.

[13] P. Hersh and A. Schilling. Symmetric chain decomposition for cyclic quotients of Boolean algebras and relation to cyclic crystals. *Internat. Math. Res. Notices* doi:10.1093/imrn/rnr254

[14] K. K. Jordan. The necklace poset is a symmetric chain order. *J. Combinatorial Theory A* 117: 625 - 641, 2010.

[15] H. Kierstead and W. T. Trotter. Explicit matchings in the middle levels of the Boolean lattice. *Order* 5: 163 - 171, 1988.

[16] M. Pouzet and I. G. Rosenberg. Sperner properties for groups and relations. *Europ. J. Combinatorics* 7: 349 - 370, 1986.

[17] R. P. Stanley. Quotients of Peck posets. *Order*, 1: 29 - 34, 1984.