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Maximal independent sets in bipartite graphs obtained from Boolean lattices

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ABSTRACT

Attempts to enumerate maximal antichains in Boolean lattices give rise to problems involving maximal independent sets in bipartite graphs whose vertex sets are comprised of adjacent levels of the lattice and whose edges correspond to proper containment. In this paper, we find bounds on the numbers of maximal independent sets in these graphs.

1. Introduction

Here we state the main problems, followed by statements of the main results. The notation and terminology are collected in the third subsection.

1.1. Statement of the problems

Some of the earliest investigations of antichains involve Dedekind's Problem [2], dating from 1897, which asks for the cardinality of the free distributive lattice on n generators, or equivalently, the number of antichains $\mathrm{a}(n)$ in the Boolean lattice $\mathbf{2}^n$ of all subsets of an n-element set. Results on Dedekind's problem include explicit computation for small values of n, an asymptotic solution for $\log_2 \mathrm{a}(n)$ obtained by Kleitman [4] and refined by Kleitman and Markowsky [5], and asymptotics for $\mathrm{a}(n)$ due to Korshunov [6] and Sapozhenko [8]. More recently, Kahn [3] has provided alternative proofs for the Kleitman–Markowsky results using entropy arguments.

An antichain is *maximal* if every proper superset contains a comparable pair of elements. Maximal antichains of the Boolean lattice are very familiar objects; indeed, one of the most well-known

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combinatorial results is Sperner's characterization of those maximal antichains of largest cardinality. But what about counting the maximal antichains in $\mathbf{2}^n$? Let $\mathrm{ma}(n)$ be the number of maximal antichains in $\mathbf{2}^n$ and consider this enumeration problem.

Problem 1. Determine the asymptotic value of $\log_2 \mathrm{ma}(n)$.

Our current results on this problem are given below in Theorem 3. The upper and lower bounds for $\log_2 \mathrm{ma}(n)$ are based on quite straightforward observations and differ by a factor of 2.

One approach to Problem 1 is suggested by the methods used to estimate $\mathrm{a}(n)$. Kleitman established that $\log_2 \mathrm{a}(n)$ is asymptotically equal to $\log_2 \binom{n}{\lfloor n/2 \rfloor}$, that is, the log of the number of antichains of $\mathbf{2}^n$ that are contained in a largest level. Korshunov and Sapozhenko obtained the asymptotics for $\mathrm{a}(n)$ by arguing that almost all antichains are contained in the middle three levels of $\mathbf{2}^n$ (cf. [7], Section 1.2). Following this line, let us consider those maximal antichains of $\mathbf{2}^n$ that are confined to consecutive levels in $\mathbf{2}^n$.

For $0 \le k < n$, let $\mathcal{B}_{n,k}$ denote the bipartite graph of k- and $k+1$-element subsets of an n-set, with adjacency defined by proper containment, and let $\mathrm{mis}(n, k)$ denote the number of maximal independent sets in $\mathcal{B}_{n,k}$. It is not difficult to see that every maximal independent set in $\mathcal{B}_{n,k}$ is a maximal antichain of $\mathbf{2}^n$ and every maximal antichain of $\mathbf{2}^n$ that is contained in $\mathcal{B}_{n,k}$ is a maximal independent set in this graph. We are led to:

Problem 2. Determine the asymptotic value of $\log_2 \mathrm{mis}(n, k)$.

Most of this paper concerns bounding $\log_2 \mathrm{mis}(n, k)$. Ideas for relating maximal antichains in $\mathbf{2}^n$ and maximal independent sets in $\mathcal{B}_{n,k}$ are outlined in the concluding section.

1.2. Statement of the results

Here are the results on Problem 2, first for $k = o(n)$.

Theorem 1. *Let $k = k(n)$ be a function satisfying $k/n \to 0$ as $n \to \infty$. Then*

$$\log_2 \mathrm{mis}(n, k) = (1 + o(1)) \binom{n}{k}.$$

For k a constant proportion of n we have these bounds.

Theorem 2. *Let α be fixed, $0 < \alpha < 1$, and let $k = \alpha n$. Then*

$$\binom{n-1}{k} \le \log_2 \mathrm{mis}(n, k) \le 1.3563(1 + o(1)) \binom{n-1}{k},$$

where $o(1) \to 0$ as $n \to \infty$.

While most of the work in this paper is devoted to proving the upper bound, we believe that the lower bound in Theorem 2, more precisely, $(1 + o(1)) \binom{n-1}{k}$, is of the right order for $\log_2 \mathrm{mis}(n, k)$.

Concerning Problem 1, the upper bound in the theorem below follows directly from Kleitman's result for $\mathrm{a}(n)$, and the lower bound in both the following and the preceding theorems follows from a straightforward observation that we make in the next section.

Theorem 3. *For all n,*

$$\binom{n-1}{\lfloor n/2 \rfloor} \le \log_2 \mathrm{ma}(n) \le (1 + o(1)) \binom{n}{\lfloor n/2 \rfloor},$$

where $o(1) \to 0$ as $n \to \infty$.

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1.3. Definitions, notation and the graphs $\mathcal{B}_{n,k}$

We denote a bipartite graph G with vertex sets X and Y and edge set E by $G = (X, Y; E)$. For $X' \subseteq X$, let

$$S(X') = \{y \in Y \,|\, xy \in E \text{ for some } x \in X'\}$$

and call this the *span* of X' in G. Say that $Y' \subseteq Y$ is a *spanned set* or a *span* if $Y' = S(X')$ for some $X' \subseteq X$. Write $S(x)$ in place of $S(\{x\})$.

Let $\Delta(V)$ ($\delta(V)$) be the maximum degree (respectively, the minimum degree) of vertices in $V \subseteq X \cup Y$. Let $\mathcal{I}(G)$ be the family of all maximal independent sets of G.

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. For any set S, let $\binom{S}{k}$ denote the family of all k-element subsets of S. With this notation, for all $0 < k \leq n$, the bipartite graph induced by levels k and $k + 1$ of $\mathbf{2}^n$ is

$$\mathcal{B}_{n,k} = \left(\binom{[n]}{k}, \binom{[n]}{k+1}; \subset \right),$$

where $\subset$ stands for the set of all pairs AB of vertices such that $A \subset B$. In this paper we shall refer to these as *Boolean graphs* to be brief. To simplify notation, we use $\mathcal{X} = \binom{[n]}{k}$ and $\mathcal{Y} = \binom{[n]}{k+1}$. Note that for $\mathcal{B} \subseteq \mathcal{Y}$, the span $S(\mathcal{B})$ is usually called the *shadow* of $\mathcal{B}$.

The edge set of $\mathcal{B}_{n,k}$ is partitioned by the n pairwise disjoint matchings $\mathcal{M}_i$, $i = 1, 2, \ldots, n$, each induced by the vertex set $\mathcal{X}_i \cup \mathcal{Y}_i$ where

$$\mathcal{X}_i = \left\{ A \,|\, A \in \binom{[n]}{k}, \; i \notin A \right\}, \quad \text{and} \quad \mathcal{Y}_i = \{A \cup \{i\} \,|\, A \in \mathcal{X}_i\}. \tag{1}$$

Let $\mathfrak{I}_{n,k}$ denote the family of all maximal independent sets in $\mathcal{B}_{n,k}$. We shall see that a partition of the edge set by matchings, as in (1), provides an upper bound for $|\mathfrak{I}_{n,k}| = \text{mis}(n, k)$.

2. Observations and easy bounds

In this section, we begin with a couple of facts about maximal independent sets and matchings in general bipartite graphs, which are then applied to Boolean graphs. We obtain the easy bounds in Theorems 1 and 2.

2.1. Matchings and independent sets

Here is how matchings and maximal independent sets are related. For a bipartite graph $G = (X, Y; E)$ and $F \subseteq E$, $\cup F$ is the set of vertices of G that belong to edges in F. We say that F is an *induced* matching if the subgraph of G induced by $\cup F$ is a matching. We also say that F is a *maximal* matching of G if it is not properly contained in any other matching of G.

Proposition 1. *Let $G = (X, Y; E)$ be a bipartite graph.*

(a) *For any induced matching M of G, $2^{|M|} \leq |\mathcal{I}(G)|$.*
(b) *For any maximal matching M of G, $|\mathcal{I}(G)| \leq 3^{|M|}$.*

Proof. (a) There are exactly $2^{|M|}$ subsets of $\cup M$ that contain exactly one element from each edge in M. Each such set extends to a maximal independent set of G with the addition of vertices from $X \cup Y - (\cup M)$. Consequently, $2^{|M|} \leq |\mathcal{I}(G)|$.

(b) Each independent set in $\cup M$ contains at most one vertex from each edge in M, so there are at most $3^{|M|}$ independent sets contained in $\cup M$. For all $I \in \mathcal{I}(G)$, $I \cap (\cup M)$ is an independent set in $\cup M$ and for all $z \in X \cup Y - (\cup M)$, $z \in I$ if and only if $z \notin S(I \cap (\cup M))$. Thus, each independent set contained in $\cup M$ can be extended to at most one maximal independent set of G. This shows that $|\mathcal{I}(G)| \leq 3^{|M|}$. $\square$

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2.2. Easy bounds

First, we obtain the lower bounds in Theorems 1 and 2.

Apply Proposition 1(a) to a matching $\mathcal{M}_i$, for fixed i (as defined in (1)) and note that $|\mathcal{M}_i| = \binom{n-1}{k}$. This gives the lower bounds in Theorems 1 and 2.

On the other hand, any maximal independent set in $\mathcal{B}_{n,k}$ is determined by its intersection with $\mathcal{X}$ and by its intersection with $\mathcal{Y}$ and, therefore, $\log_2 \text{mis}(n,k)$ is at most the minimum of $\binom{n}{k}$ and $\binom{n}{k+1}$. Thus, for $k = o(n)$,

$$\log_2 \text{mis}(n,k) \le \binom{n}{k}. \tag{2}$$

For $k = o(n)$, the lower bound obtained in the preceding paragraph and the upper bound in (2) are asymptotically equal. This proves Theorem 1.

The matchings $\mathcal{M}_i$ give a better upper bound for $\text{mis}(n,k) = |\mathfrak{I}_{n,k}|$. Just observe that $\mathcal{M}_i$ is a maximal matching in $\mathcal{B}_{n,k}$ and apply Proposition 1(b):

$$|\mathfrak{I}_{n,k}| \le 3^{\binom{n-1}{k}}.$$

Taking logarithms,

$$\log_2 \text{mis}(n,k) \le \log_2 3 \cdot \binom{n-1}{k} \le 1.5850 \binom{n-1}{k}.$$

The better bound in Theorem 2 requires a bit more work.

3. The upper bound for maximal independent sets

We first prove two lemmas concerning matchings in general bipartite graphs, then obtain the upper bound in Theorem 2.

3.1. Two lemmas

The first result shows how induced matchings in bipartite graphs limit choices of independent subsets. The second provides a greedy algorithm for expansion when degree conditions are assumed.

Lemma 1. *Let $G = (X, Y; E)$, and let $X' = \{x_1, x_2, \ldots, x_m\} \subseteq X$ and $Y' = \{y_1, y_2, \ldots, y_m\} \subseteq Y$ induce a matching in G with edges x_iy_i, $i = 1, 2, \ldots, m$. Let $\widetilde{X} \subseteq X'$ and $\widetilde{Y} \subseteq Y'$. Then the number of choices of pairs of sets (A_0, B_0) such that*

$$A_0 \subseteq X' - \widetilde{X},\ B_0 \subseteq Y' - \widetilde{Y}, \quad \text{and} \quad S(A_0) \cap B_0 = \emptyset$$

is at most

$$2^{|\widetilde{X}|+|\widetilde{Y}|} \cdot 3^{m-(|\widetilde{X}|+|\widetilde{Y}|)}.$$

Proof. Let $I_X = \{i \mid x_i \in \widetilde{X}\}$, and $I_Y = \{i \mid y_i \in \widetilde{Y}\}$. Then:

- if $i \notin I_X \cup I_Y$ then there are three possibilities for x_i and y_i, namely, $x_i \in A_0$ and $y_i \notin B_0$, $x_i \notin A_0$ and $y_i \in B_0$, or $x_i \notin A_0$ and $y_i \notin B_0$;
- if $i \in I_X \Delta I_Y$ then there are two possibilities; and
- if $i \in I_X \cap I_Y$ then there is one possibility.

Thus, if $|I_X \cap I_Y| = t$ then the number of choices of (A_0, B_0) is

$$2^{(|\widetilde{X}|-t)+(|\widetilde{Y}|-t)} \cdot 3^{m-(|\widetilde{X}|+|\widetilde{Y}|)+t} = 2^{|\widetilde{X}|+|\widetilde{Y}|} \cdot 3^{m-(|\widetilde{X}|+|\widetilde{Y}|)} \cdot \left(\frac{3}{4}\right)^t.$$

This quantity is maximized when $t = 0$, so we have the claimed upper bound. □

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Lemma 2. *Let $G = (X, Y; E)$ and let c and d be positive integers such that $\delta(X) \geq c$ and $\Delta(Y) \leq d$. Then for all $p \leq c$ there exists $\widetilde{X} \subseteq X$ such that*

$$|\widetilde{X}| = \left\lceil \frac{|X|(c-p+1)}{c+pd-p+1} \right\rceil, \quad \textit{and} \quad |S(\widetilde{X})| \geq p \cdot |\widetilde{X}|.$$

Proof. We obtain $\widetilde{X} = \{x_1, x_2, \ldots, x_q\}$ in a greedy manner. Choose any $x_1 \in X$ and any p of its neighbors in Y. Suppose that we have chosen $x_1, x_2, \ldots, x_q$ together with pq of their neighbors in Y, say $y_1, y_2, \ldots, y_{pq}$. Let G_q be the subgraph of G induced by the vertex set $(X - \{x_1, x_2, \ldots, x_q\}) \cup (Y - \{y_1, y_2, \ldots, y_{pq}\})$.

If we cannot continue, that is, if there is no vertex $x \in X - \{x_1, x_2, \ldots, x_q\}$ with degree at least p in G_q, then we have

$$|E(G_q)| \ \leq \ (p-1)(|X| - q).$$

On the other hand,

$$|E(G_q)| \ \geq \ c(|X| - q) - pqd$$

and, consequently,

$$c(|X| - q) - pqd \ \leq \ (p-1)(|X| - q)$$

which implies that

$$\frac{|X|(c-p+1)}{c+pd-p+1} \ \leq \ q.$$

We conclude that we can greedily construct a subset $\widetilde{X}$ of X such that

$$|\widetilde{X}| = \left\lceil \frac{|X|(c-p+1)}{c+pd-p+1} \right\rceil \quad \text{and} \quad |S(\widetilde{X})| \geq p|\widetilde{X}|. \quad \square$$

We apply Lemma 2 twice with $p = \sqrt{k}$ to obtain the following.

Corollary 1. *Let $G = (X, Y; E)$ be a bipartite graph, $0 < \alpha < 1$, n a positive integer, and $k = \alpha n$. Then the following hold with $o(1) \to 0$ as $n \to \infty$:*

(a) *if $\delta(X) \geq n - k$ and $\Delta(Y) \leq k$ then there exists $\widetilde{X} \subseteq X$ such that*

$$|\widetilde{X}| = \frac{1-\alpha}{\alpha^{3/2} n^{1/2}} |X|(1 - o(1)), \quad \textit{and} \quad |S(\widetilde{X})| \geq \frac{1-\alpha}{\alpha} |X|(1 - o(1))\,;$$

(b) *if $\delta(X) \geq k + 1$ and $\Delta(Y) \leq n - k - 1$ then there exists $\widetilde{X} \subseteq X$ such that*

$$|\widetilde{X}| = \frac{\alpha^{1/2}}{(1-\alpha) n^{1/2}} |X|(1 - o(1)), \quad \textit{and} \quad |S(\widetilde{X})| \geq \frac{\alpha}{1-\alpha} |X|(1 - o(1)).$$

3.2. The partition of the maximal independent sets in the graphs $\mathcal{B}_{n,k}$

In order to simplify the notation, we prove the upper bound for Theorem 2 for a family of bipartite graphs slightly more general than the Boolean graphs. Let $B_{n,k} = (X, Y; E)$ be a bipartite graph satisfying:

(i) $|X| = \frac{k+1}{n-k}|Y| = N$;
(ii) E is partitioned by $E_1, E_2, \ldots, E_n$ where each E_i is a maximal matching of $B_{n,k}$ and $|E_i| = \frac{n-k}{n} N = M$ for all i; and
(iii) for all $x \in X,\ y \in Y$, $\deg x = n - k$ and $\deg y = k + 1$.

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We shall show that for such $B_{n,k}$,

$$|\mathcal{I}(B_{n,k})| \leq 2^{1.3563(1+o(1))M}. \tag{3}$$

To see that the upper bound in Theorem 2 follows, note that with $N = \binom{n}{k}$ and $\mathcal{M}_i, i = 1, 2, \ldots, n$, as defined in (1), $\mathcal{B}_{n,k}$ satisfies (i)–(iii).

For $i = 1, 2, \ldots, n$, let $X_i = X \cap (\cup E_i)$ and $Y_i = Y \cap (\cup E_i)$. For $A \subseteq X$ and $B \subseteq Y$, let

$$A_i = A \cap X_i,\ A'_i = A - A_i,\ a_i = |A_i|,\ a'_i = |A'_i| \quad \text{and} \tag{4}$$

$$B_i = B \cap Y_i,\ B'_i = B - B_i,\ b_i = |B_i|,\ b'_i = |B'_i|. \tag{5}$$

Proposition 2. *Let $0 \leq k < n$ and let $B_{n,k}$ satisfy (i)–(iii). For all $A \subseteq X$ and $B \subseteq Y$, using the notation in (4), (5),*

(a) $\sum_{i=1}^{n} a_i = (n-k)|A|$,
(b) $\sum_{i=1}^{n} b_i = (k+1)|B|$, *and*
(c) *there is some i, $1 \leq i \leq n$, such that*

$$a_i + \left(1 - \frac{1}{n-k}\right) b_i \ \leq\ \left(\frac{n-k}{k}\right) a'_i + \left(\frac{k+1}{n-k}\right) b'_i. \tag{6}$$

Proof. (a) The ith summand on the left hand side of (a) enumerates the edges of the matching E_i incident with vertices in A. Since the vertices in A are all of degree $n-k$ and the matchings partition E, we have equality.

(b) This is the same argument as above.

(c) Multiply each side of the equation in (a) by $1/k$, and multiply each side of the equation in (b) by $1/(n-k)$. From this, we obtain

$$\sum_{i=1}^{n} \left[\frac{1}{k} a_i + \frac{1}{n-k} b_i\right] = \left(\frac{n-k}{k}\right)|A| + \left(\frac{k+1}{n-k}\right)|B|.$$

By averaging, there exists i such that

$$\frac{n}{k}\, a_i + \frac{n}{n-k}\, b_i \ \leq\ \frac{n-k}{k}(a_i + a'_i) + \frac{k+1}{n-k}(b_i + b'_i),$$

from which the inequality in (c) follows immediately. □

Note that the maximality of the matchings E_i is not required in the proof of Proposition 2.

3.3. *The upper bound in Theorem 2*

We now verify the bound in (3).

Given $I \in \mathcal{I}(B_{n,k})$ let $A = I \cap X$ and $B = I \cap Y$. By Proposition 2(c), we can choose $i = i(\mathcal{I})$ in $[n]$ such that (6) holds.

We shall use the (binary) entropy function,

$$H(\alpha) = \alpha \log_2(1/\alpha) + (1-\alpha)\log_2(1/(1-\alpha)).$$

(See [1] for properties of this function and [3] for its applications in enumeration of independent sets and antichains.) Let γ be the solution of

$$H(\gamma) + \gamma = \gamma + (1-\gamma)\log_2 3, \tag{7}$$

that is, $\gamma \approx 0.3909$.

Let $\mathcal{I}(B_{n,k}) = \mathcal{I}_{\text{small}} \cup \mathcal{I}_{\text{large}}$ where

- $I \in \mathcal{I}_{\text{small}}$ if $a_i + b_i \leq \gamma M$, and
- $I \in \mathcal{I}_{\text{large}}$ if $a_i + b_i > \gamma M$.

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Case 1: The upper bound for $|\mathcal{I}_{\text{small}}|$.

The proof of Proposition 1(b) and the maximality of the matching E_i show that a maximal independent set I is determined by the triple $(i, I \cap X_i, I \cap Y_i)$, that is, by (i, A_i, B_i), following the notation in (4) and (5). The number of such triples is at most

$$n \cdot \sum_{j=0}^{\gamma M} \binom{M}{j} 2^j \le 2^{(H(\gamma)+\gamma)M(1+o(1))} \le 2^{1.3563M(1+o(1))}.$$

Hence, $|\mathcal{I}_{\text{small}}| \le 2^{1.3563M(1+o(1))}$.

Case 2: The upper bound for $|\mathcal{I}_{\text{large}}|$.

Let $I \in \mathcal{I}_{\text{large}}$, let $i = i(I)$ be fixed, and let $|A|$, $|B|$, a_i, b_i, a'_i, and b'_i be as in (4) and (5). By Proposition 2(c), we have

$$\gamma M(1 - o(1)) \le (a_i + b_i)(1 - o(1)) \le \frac{n-k}{k} a'_i + \frac{k+1}{n-k} b'_i$$

where $o(1) \to 0$ as $n - k \to \infty$.

As in Case 1, we shall use the fact that I is determined by the triple (i, A_i, B_i), so, let us find a bound on the number of pairs (A_i, B_i).

Let G be the subgraph of $B_{n,k}$ induced by the vertex set $A'_i \cup Y_i$. We apply Corollary 1(a) with $X = A'_i$ and $Y = Y_i$. We verify that the hypotheses of the corollary are satisfied.

- Each member of A'_i has degree $n - k$ in $B_{n,k}$ and $S(A'_i) \subseteq Y_i$, since the maximality of the matching E_i means that there are no edges between $X - X_i$ and $Y - Y_i$. Thus, $\delta(A'_i) \ge n - k$.
- Each member of Y_i has degree $k + 1$ in $B_{n,k}$, and has a neighbor in X_i, and $A'_i \cap X_i = \emptyset$. Therefore, $\Delta(Y_i) \le k$.

Corollary 1(a) yields $\widetilde{A} \subseteq A'_i$ and $\widetilde{Y} = S(\widetilde{A})$ such that

$$|\widetilde{A}| = \frac{1-\alpha}{\alpha^{3/2} n^{1/2}} |A'_i|(1 - o(1)) \quad \text{and} \quad |\widetilde{Y}| \ge \frac{1-\alpha}{\alpha} |A'_i|(1 - o(1)). \tag{8}$$

A similar application of Corollary 1(b) to the subgraph of $B_{n,k}$ induced by $B'_i \cup X_i$, this time with $X = B'_i$ and $Y = X_i$, gives $\widetilde{B} \subseteq B'_i$ and $\widetilde{X} = S(\widetilde{B})$ such that

$$|\widetilde{B}| = \frac{\alpha^{1/2}}{(1-\alpha) n^{1/2}} |B'_i|(1 - o(1)) \quad \text{and} \quad |\widetilde{X}| \ge \frac{\alpha}{1-\alpha} |B'_i|(1 - o(1)). \tag{9}$$

See Fig. 1 for a schematic diagram of these sets.

Let us bound the number of possible pairs $(\widetilde{X}, \widetilde{Y})$. Such a pair is determined by the choice of $(\widetilde{A}, \widetilde{B})$. We know that $\widetilde{A} \subseteq A'_i$ and satisfies (8), and that

$$|A'_i| \le |X| - M = N - \frac{n-k}{n} N = \alpha N.$$

Also, $\widetilde{B} \subseteq B'_i$ and satisfies (9), and

$$|B'_i| \le |Y| - M = \frac{n-k}{k+1} N - \frac{n-k}{n} N = \frac{(n-k)^2}{nk} N(1 - o(1)) = \frac{(1-\alpha)^2}{\alpha} N(1 - o(1)).$$

Hence, the number of pairs $(\widetilde{X}, \widetilde{Y})$ is at most

$$\left[\sum_{j=0}^{\frac{1-\alpha}{\alpha n^{1/2}} N(1-o(1))} \binom{\alpha N}{j} \right] \left[\sum_{j=0}^{\frac{1-\alpha}{\alpha n^{1/2}} N(1-o(1))} \binom{\frac{(1-\alpha)^2}{\alpha} N(1 + o(1))}{j} \right] = 2^{o(M)}. \tag{10}$$

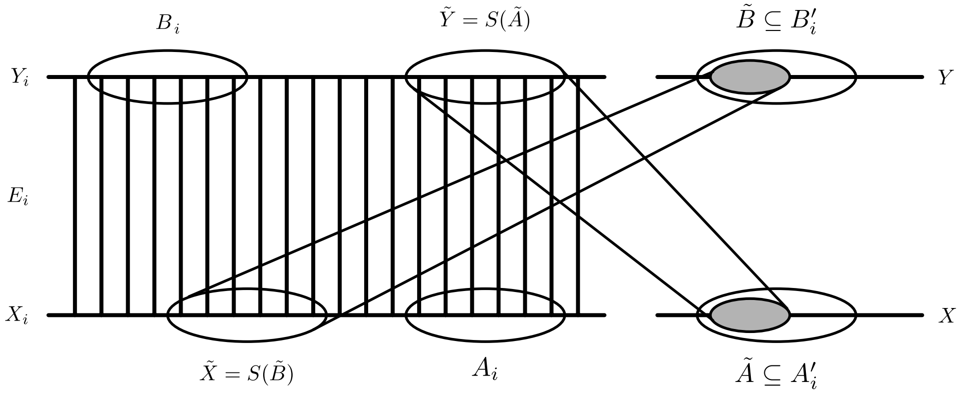

Fig. 1. Restricting pairs (A_i, B_i) of subsets of $I \in \mathcal{I}_{\text{large}}$.

We now apply Lemma 1 with $X' = X_i$, $Y' = Y_i$, and $\widetilde{X}, \widetilde{Y}$ as defined above, to deduce that, given $(\widetilde{X}, \widetilde{Y})$, the number of choices of (A_i, B_i) is at most

$$2^{|\widetilde{X}|+|\widetilde{Y}|} \cdot 3^{|X_i|-(|\widetilde{X}|+|\widetilde{Y}|)}. \tag{11}$$

In view of (8) and (9), and because $I \in \mathcal{I}_{\text{large}}$ and $k = \alpha n$, we infer that

$$\begin{aligned} |\widetilde{X}| + |\widetilde{Y}| &\geq \frac{1-\alpha}{\alpha} |A_i'| + \frac{\alpha}{1-\alpha} |B_i'| \\ &= \frac{n-k}{k} a_i' + \frac{k}{n-k} b_i' \\ &\geq (a_i + b_i)(1 - o(1)) \\ &\geq \gamma M(1 - o(1)). \end{aligned}$$

Consequently, one can bound the quantity in (11) above by

$$3^{(1-\gamma)M} 2^{\gamma M(1-o(1))} = \left(2^\gamma 3^{1-\gamma}\right)^{(1-o(1))M}. \tag{12}$$

Summarizing, we have the number of choices for i, the upper bound on the number of pairs $(\widetilde{X}, \widetilde{Y})$ given in (10), and the upper bound on selections of (A_i, B_i) given in (12). Thus, there are at most

$$n \cdot 2^{o(M)} \cdot (2^\gamma 3^{1-\gamma})^{(1-o(1))M} = (2^\gamma 3^{1-\gamma})^{(1-o(1))M} \tag{13}$$

choices for $I \in \mathcal{I}_{\text{large}}$. Evaluating either side of (7) at the solution $\gamma \approx .3909$ gives the value 1.3563. Therefore, (13) yields

$$|\mathcal{I}_{\text{large}}| \leq 2^{1.3563(1-o(1))M}.$$

Combining this with the bound obtained in Case 1, we have

$$|\mathcal{I}(B_{n,k})| = |\mathcal{I}_{\text{small}}| + |\mathcal{I}_{\text{large}}| \leq 2^{1.3563(1+o(1))M}.$$

This completes the proof of Theorem 2.

4. Concluding remarks

4.1. Closed subsets of a bipartite graph

A likely first thought regarding maximal antichains in the Boolean lattice, or maximal independent sets in the graphs $B_{n,k} = (X, Y; E)$, is just to choose $C \subseteq X$ and consider the independent set

$C \cup (Y - S(C))$. Were all such independent sets maximal, Problems 1 and 2 would have trivial solutions. However, $C \cup (Y - S(C))$ is a maximal independent set if and only if for all $x \in X$,

$$S(x) \subseteq S(C) \text{ implies } x \in C.$$

In particular, enumerating maximal independent sets amounts to counting such *closed* subsets of X or closed subsets of Y.

4.2. Shadows

We note that for a bipartite graph $G = (X, Y; E)$ there is a 1-1 correspondence between $\mathcal{I}(G)$ and the family $\mathcal{S}(X)$ of all spans of subsets of Y. The map $I \mapsto S(I \cap Y)$ is a bijection between the two families. Bounds on $|\mathcal{I}(G)|$ provide bounds on the number of spans, or shadows, of all subsets of Y, or of X. For the graphs $\mathcal{B}_{n,k}$, in particular, shadows are very familiar objects of study so the bounds in Theorems 1 and 2 may be of independent interest.

4.3. From antichains to independent sets

Let us finish by returning to Problem 1 and considering how maximal independent sets in $\mathcal{B}_{n,k}$ might be tied to maximal antichains in $\mathbf{2}^n$. Obviously, every maximal independent set is a maximal antichain, so $\text{mis}(n, k) \leq \text{ma}(n)$ for any k. On the other hand, is it true that almost all maximal antichains are contained in a few of the middle levels of $\mathbf{2}^n$? If yes, then Problems 1 and 2 would be closely related (if almost all were in the middle two levels, Problems 1 and 2 would be the same).

Here is a way to map maximal antichains to maximal independent sets. Given a maximal antichain M in $\mathbf{2}^n$, define the subset $\widehat{M}$ of the vertex set of $\mathcal{B}_{n,k}$ as follows:

$$\begin{aligned} \widehat{M} &= \{A \mid |A| = k+1, \text{ and } A \subseteq A' \text{ for some } A' \in M\} \\ &\cup \{B \mid |B| = k, \text{ and } B \supseteq B' \text{ for some } B' \in M\}. \end{aligned}$$

It is not difficult to argue that $\widehat{M}$ is a maximal independent set. The map $M \mapsto \widehat{M}$ is certainly not one-to-one. Perhaps analysis of preimages of this map could help settle the following.

Conjecture. *There exists* $\alpha < 1$ *such that* $\log_2 \text{ma}(n) \leq \alpha \binom{n}{n/2}$.

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