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Retracts of posets: the chain-gap property and the selection property are independent

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Dedicated to George Grätzer and E. Tamás Schmidt on their 70th birthdays

Abstract. Posets which are retracts of products of chains are characterized by means of two properties: the *chain-gap property* and the *selection property* (Rival and Wille [9]). Examples of posets with the selection property and not the chain-gap property are easy to find. To date, the Boolean lattice $\mathcal{P}(\omega_1)/\operatorname{Fin}$ has been the sole example of a lattice without the selection property [9]. We prove that it also fails to have the chain-gap property. In addition, we provide an example of a lattice which has the chain-gap property but not the selection property. This answers questions raised in [9].

1. Introduction

Given posets P and Q, P is a *retract* of Q if there are order-preserving maps $f\colon P \to Q$ and $g\colon Q \to P$ such that $g \circ f = 1_P$. The maps f and g are called a *coretraction* and a *retraction*, respectively. I. Rival and the first author [2] defined an *order variety* to be a class of posets closed under direct products and retracts. I. Rival and R. Wille [9] characterized members of the order variety generated by the class of chains as posets satisfying two properties: the chain-gap property and the selection property. Briefly, a poset P has the *selection property* if for each separable gap of P one can select an element of P separating the gap such that the overall selection preserves the natural order on these gaps. We say that P has the *chain-gap property* if each gap can be mapped into a gap within a chain in an order-preserving fashion. Rival and Wille gave examples of lattices with the selection property for which the chain-gap property fails. They showed that $\mathcal{P}(\omega_1)/\operatorname{Fin}$, the quotient of the power set of ω_1 by the ideal Fin of finite sets, does

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not have the selection property. They asked if it has the gap-property, and we answer this in the negative.

Theorem 1.1. *If E is infinite, $\mathcal{P}(E)/\operatorname{Fin}$ does not have the chain-gap property.*

They also asked if there is a lattice with the chain-gap property but without the selection property, and we answer this question positively. Our example is a distributive lattice of size $\aleph_1$ which does not embed the ordinal ω_1. It is built from a Sierpinskization of a subchain $\mathbb{S}$ of the real line $\mathbb{R}$ which is $\aleph_1$-dense, that is, $\big|\,]a,b[\,\cap \mathbb{S}\big| \geq \aleph_1$ for every $a < b$ in $\mathbb{S}$, it has no end points and it has size $\aleph_1$. (The existence of such chains is well known and easily proved.) Let $\leq_{\omega_1}$ be an ordering on $\mathbb{S}$ such that the chain $(\mathbb{S}, \leq_{\omega_1})$ has order type ω_1, and let $\leq_{\mathbb{R}}$ be the usual ordering on the reals. The Sierpinskization of $\mathbb{S}$ is the poset $(\mathbb{S}, \leq)$, where $\leq$ is the ordering on $\mathbb{S}$ defined by $x \leq y$ iff $x \leq_{\omega_1} y$ and $x \leq_{\mathbb{R}} y$. Let $L(\mathbb{S}, \leq)$ be the lattice generated within the lattice of subsets of $\mathbb{S}$ by the principal initial segments of $(\mathbb{S}, \leq)$. So $L(\mathbb{S}, \leq)$ consists of all the finite unions of finite intersections of initial segments of the form $\downarrow x$ for $x \in \mathbb{S}$, where $\downarrow x := \{y \in \mathbb{S} : y \leq x\}$. With this construction in mind we show:

Theorem 1.2. *$L(\mathbb{S}, \leq)$ has the chain-gap property, but not the selection property.*

More generally the above notions relate to a central objective in the study of retracts of posets, namely to find conditions (denoted by (C), say) that a map $f \colon P \to Q$ must satisfy in order to be a coretraction. Posets P for which maps satisfying (C) are necessarily coretractions are called *absolute retracts with respect to maps satisfying* (C). For example, each coretraction must be an order-embedding. As is well known, *the absolute retracts with respect to order-embeddings* are the complete lattices. As well, the complete lattices are precisely those posets that are *injective with respect to order-embedding.* That is, every order-preserving map from a poset Q to P extends to an order-preserving map of every poset Q' in which Q order-embeds. Moreover, *there are enough* absolute retracts, in the sense that every poset order-embeds into a complete lattice, that is, into an absolute retract.

Every coretraction must be gap-preserving. A somewhat similar situation to the case of order-embeddings was observed by Duffus and Pouzet [1], and by Nevermann and Rival [6]:

> A poset P is an *absolute retract with respect to gap-preserving maps* if and only if P has the selection property. Moreover, absolute retracts coincide with *injective objects with respect to gap-preserving maps*, and there are enough of them.

The class of absolute retracts is preserved under retraction and products (Rival and Wille [9]), it contains the chains (Duffus, Rival and Simonovits [3]) and, hence, the variety generated by the class of chains. According to Rival and Wille [9]:

A poset P embeds by a gap-preserving map into a product of chains iff P has the chain-gap property.

The chain-gap property implies that P is a lattice. Every countable lattice belongs to the variety generated by the class of chains [8], hence satisfies the chain-gap property. However, there are many lattices for which the chain gap property fails (see [2], [9]).

2. Preliminaries

Let P be a partially ordered set and let $C \subseteq P$. We use this notation and terminology:

- $U(C) := \{x \in P : y \leq x \ \text{ for all } \ y \in C\}$ is the set of upper bounds of C and $L(C)$, the set of lower bounds, is defined dually;
- $\downarrow C = \{x \in P : x \leq y \text{ for some } y \in C\}$ is the *initial segment generated by* C and $\uparrow C$ is defined dually and called the *final segment generated by* C;
- C is *cofinal* in P if $\downarrow C = P$ and is *coinitial* in P if $\uparrow C = P$;
- the *cofinality* of P, $cf(P)$, is the least cardinality of a cofinal subset, the *coinitiality* of P is defined dually and denoted by $ci(P)$.

For a singleton $x \in P$, we use $\downarrow x$ instead of $\downarrow \{x\}$. If reference to P is needed, particularly in case of several orders on the same ground set, we use the notation $\downarrow_P C$ instead of $\downarrow C$.

Let (A, B) be a pair of subsets of P. Here are terms and notation associated to these pairs:

- the *cardinality* of a pair (A, B) is the pair $(|A|, |B|)$;
- call (A, B) *regular* if $|A|$ and $|B|$ are both regular cardinal numbers, or if one is regular and the other is 0;
- say that (A, B) is a *pregap* of P if $A \subseteq L(B)$ or, equivalently, if $B \subseteq U(A)$;
- a pregap (A, B) is *separable* if $U(A) \cap L(B) \neq \emptyset$;
- a pregap (A, B) is a *gap* if $U(A) \cap L(B) = \emptyset$;
- a pair (A', B') is a *subpair* of (A, B) if $A' \subseteq A$ and $B' \subseteq B$, and if both pairs are gaps, call (A', B') a *subgap* of (A, B).

We denote by $B(P)$ the set of separable pregaps of P. Pregaps are quasiordered as follows: $(A, B) \leq (A', B')$ if $A \subseteq \downarrow A'$ and $B' \subseteq \uparrow B$.

A gap (A, B) of P is said to be *minimal* if all subgaps have the same cardinality as (A, B). Call a gap (A, B) *irreducible* if for all subpairs (A', B'), (A', B') is a gap if and only if it has the same cardinality as (A, B). It is straightforward to show that every gap has a subgap which is minimal. On an other hand, irreducible gaps are just minimal gaps all of whose subpairs, of its cardinality, are gaps.

We now come to the main concepts of this paper.

Definition 2.1. (1) The poset P has the *selection property* (the *strong selection property* in the terminology of Nevermann and Wille [7]) if there is an order-preserving map φ from $B(P)$ to P which associates to every pair $(A, B) \in B(P)$ an element of $U(A) \cap L(B)$.

(2) An order-preserving map g from P into a poset Q *preserves* a gap (A, B) of P if $(g[A], g[B])$ is a gap of Q. If g preserves all gaps of P, it is *gap-preserving.* A poset Q *preserves* a gap (A, B) of P if there is an order-preserving map $g\colon P \to Q$ which preserves (A, B). The poset P is said to have the *chain-gap property* if each gap of P is preserved by some chain.

The relationship between the chain-gap property and regular irreducible gaps is given by the following result by Duffus and Pouzet.

Theorem 2.2. [1] *An ordered set P has the chain-gap property if and only if every gap of P contains a regular irreducible gap.*

Although this will not be needed here, they proved a bit more in presence of the selection property.

Proposition 2.3. [1] *Let (A, B) be a minimal gap of P with $\lambda := |A|$ and $\mu := |B|$ both infinite. If P has the selection property then there are two chains C and D of type respectively $cf(\lambda)$ and $cf(\nu)^*$ such that (C, D) is a gap and $(A, B) \leq (C, D)$.*

Moreover, if (A, B) is an irreducible gap then the ordinal sum $C \oplus D$ is a retract of P which preserves (A, B).

We conclude this section with some notation and remarks necessary for the proof of Theorems 1.1 and 1.2.

For E any set, let $\mathcal{P}(E)$ be the Boolean algebra of all subsets of E and let $\mathcal{P}(E)/\operatorname{Fin}$ be the quotient of $\mathcal{P}(E)$ by the ideal Fin of finite subsets of E. Define $p\colon \mathcal{P}(E) \to \mathcal{P}(E)/\operatorname{Fin}$ to be the canonical projection. For $X, Y \in \mathcal{P}(E)$, we set $X \leq_{\mathrm{Fin}} Y$ if $X \setminus Y \in \operatorname{Fin}$. This defines a quasiorder on $\mathcal{P}(E)$; its image under p is the order on $\mathcal{P}(E)/\operatorname{Fin}$.

Remark 2.4. Since $\mathcal{P}(E)/\operatorname{Fin}$ is a lattice, there are no gaps of cardinality (λ, μ) where either λ or μ is finite. Moreover, by a countable diagonalization argument as first observed by Hadamard [5], there are no gaps of cardinality (ω, ω) either.

To avoid trivialities, let us assume that E is infinite in what follows. Gaps of $\mathcal{P}(E)$ under the above quasiorder correspond under p to gaps in the poset $\mathcal{P}(E)/\operatorname{Fin}$, so for notational simplicity all our discussion regarding gaps in $\mathcal{P}(E)$ can be translated in the latter structure if necessary.

We also recall that the usual Hausdorff topology on $\mathcal{P}(E)$ is obtained by identifying each subset of E with its characteristic function and giving the resulting space $\{0, 1\}^E$ the product topology. A basis of open sets consists of subsets of the form

$O(F,G) := \{X \in \mathcal{P}(E) : F \subseteq X \text{ and } G \cap X = \emptyset\}$, where F, G are finite subsets of E. We can therefore talk about closed sets and more generally F_σ sets, that is, countable unions of closed sets. Endowed with this topology, $\mathcal{P}(E)$ is compact and Hausdorff, therefore a Baire space, that is, a space in which any countable union of closed sets with empty interior has empty interior.

3. Proof of Theorem 1.1

Consider $E = T_2$ the binary tree of finite sequences of 0 and 1, and $T_2(n)$ those sequences of length at most n. We denote by $(\,)$ the empty sequence and by $s.(i)$ the sequence obtained by adding $i \in \{0,1\}$ to the sequence s. As mentioned above, for notational simplicity we will consider the quasiorder $\leq_{\mathrm{Fin}}$ on $\mathcal{P}(E)$ as opposed to the poset $\mathcal{P}(E)/\operatorname{Fin}$ itself.

For $B \subseteq \mathcal{P}(E)$ set $B^c = \{E \setminus X : X \in B\}$. We will be particularly interested in the set $\mathcal{B}$ of maximal branches of T_2, a closed subset of $\mathcal{P}(E)$ with no isolated points. Notice that for $(A,B) \in \mathcal{P}(\mathcal{B}) \times \mathcal{P}(\mathcal{B})$, (A, B^c) is a pregap if and only if A and B are disjoint.

Proposition 3.1. *Let A and B be disjoint subsets of $\mathcal{B}$. Then (A, B^c) is separable if and only if A and B are contained in disjoint F_σ sets.*

Proof. Let $X \in U(A) \cap L(B^c)$ separate (A, B^c), and consider $A' := \bigcup_n \{Y \in \mathcal{B} : Y \setminus X \subseteq T_2(n)\}$ and $B' := \bigcup_n \{Y \in \mathcal{B} : Y \cap X \subseteq T_2(n)\}$. Then A' and B' are disjoint, are F_σ sets (since $\mathcal{B}$ is closed) and contain A and B respectively.

Conversely, let A', B' be disjoint F_σ sets containing A, B, respectively. Since $\mathcal{B}$ is closed, we may assume without loss of generality that $A' = \bigcup A'_n$ and $B' = \bigcup B'_n$ are increasing chains of closed sets in $\mathcal{B}$. For any fixed n, we claim that there must be an integer k_n such that any $s \in X \cap Y$ has length at most k_n for any $X \in A'_n$ and $Y \in B'_n$. Indeed otherwise for infinitely many k we could find $s_k \in X_k \cap Y_k$ of length at least k for some $X_k \in A'_n$ and $Y_k \in B'_n$. But then we could find a subsequence of $\{s_m : m \in \mathbb{N}\}$ converging to a maximal branch which by closure would be in $A'_n \cap B'_n$, a contradiction. We can also assume that the sequence produced, $\{k_n : n \in \mathbb{N}\}$, is strictly increasing. Now if $X_n := (\cup A'_n) \setminus T_2(k_n)$, then $X := \bigcup_n X_n \in U(A) \cap L(B^c)$ and therefore (A, B^c) is separable. □

By considering A consisting of a single branch and with $B = \mathcal{B} \setminus A$, one concludes that the above result cannot be strengthened to a covering by disjoint closed sets.

Although the first part of the proof does generalize to any separable pregap in $\mathcal{P}(E)$, it is interesting that the converse is not true as is shown by an example given by Todorcevic [11]. Indeed for $Y \in \mathcal{B}$ let $a_Y = \{s \in E : s.0 \in Y\}$ and $b_Y = \{s \in E : s.1 \in Y\}$. Then $A = \{a_Y : Y \in \mathcal{B}\}$ and $B = \{b_Y : Y \in \mathcal{B}\}$ form two disjoint closed sets in $\mathcal{P}(E)$. Observe moreover that $a_Y \cap b_Y = \emptyset$ for each Y, and that for $Y \neq Y'$, either $a_Y \cap b_{Y'} \neq \emptyset$ or $a_{Y'} \cap b_Y \neq \emptyset$. This last property guarantees that (A, B^c) is a so-called Luzin gap in $\mathcal{P}(E)$.

Since as mentioned above $\mathcal{P}(E)$ is a Baire space, we further have:

Corollary 3.2. *If $A \subseteq \mathcal{B}$ and $B = \mathcal{B} \setminus A$ are both dense then the pair (A, B^c) is a gap in $\mathcal{P}(E)$.*

We finally arrive at the main reason for considering this structure.

Proposition 3.3. *Let A and B be disjoint subsets of $\mathcal{B}$. If (A, B^c) is a gap, then it does not contain a regular irreducible gap.*

Proof. For $s \in E$ and $D \subseteq \mathcal{P}(E)$, we set $D(s) = \{X \in D : s \in X\}$ and $\widehat{D} = \{s \in E : |D(s)| = |D|\}$.

Now observe that for an infinite $D \subseteq \mathcal{B}$, the least element of T_2, namely the empty sequence $()$, belongs to $\widehat{D}$. Moreover if $s \in \widehat{D}$, then either $s.(0)$ or $s.(1)$ belongs to $\widehat{D}$, so we conclude that $\widehat{D}$ contains a branch and so certainly is infinite. Moreover, if $|D|$ is regular and uncountable, then $\widehat{D}$ must contain more than a branch and is therefore itself not a chain.

With this, suppose for contradiction that (A, B^c) contains a regular irreducible gap of size (λ, μ) in $\mathcal{P}(E)$. This means that there is a pair (A', B') such that $A' \subseteq A$, $|A'| = \lambda$, $B' \subseteq B$, $|B'| = \mu$ such that (A', B'^c) is an irreducible gap.

As noted in Remark 2.4, λ and μ must be infinite and one of them uncountable. With no loss of generality, we may suppose that this is λ. According to the above observation, $\widehat{A'}$ is not a chain and $\widehat{B'}$ is infinite, hence there are $s \in \widehat{A'}$, $t \in \widehat{B'}$ which are incomparable with respect to the order on T_2. Let $A'' := A'(s)$ and $B'' := B'(t)$. We have $A'' \subseteq A'$, $|A''| = |A'| = \lambda$, $B'' \subseteq B'$, $|B''| = |B'| = \mu$, and therefore (A'', B''^c) must be a gap by the irreducibility assumption. On the other hand for $Z := \bigcup A''$, we have $X \leq_{\mathrm{Fin}} Z \leq_{\mathrm{Fin}} Y$ for every $X \in A''$ and $Y \in B''^c$, a contradiction. $\square$

With this in hand, the proof of Theorem 1.1 breaks into two cases.

Case 1. E is countably infinite. We deduce Theorem 1.1 as follows. We identify E with T_2, and choose $A \subseteq \mathcal{B}$ and $B = \mathcal{B} \setminus A$ both dense in $\mathcal{B}$. According to Corollary 3.2, (A, B^c) is a gap of $\mathcal{P}(E)$, and according to Proposition 3.3, it does not contain a regular irreducible gap. According to Theorem 2.2, $\mathcal{P}(E)/\mathrm{Fin}$ does not have the chain-gap property.

Case 2. E is uncountable. Let E' be a countably infinite subset of E. The identity map $1_{E'}$ on E' extends to a map φ from $\mathcal{P}(E')/\mathrm{Fin}$ into $\mathcal{P}(E)/\mathrm{Fin}$. This map is gap-preserving.

By Case 1, $\mathcal{P}(E')/\,\mathrm{Fin}$ has a gap (A, B) containing no regular irreducible subgap. But then $(\varphi(A), \varphi(B))$ is a gap in $\mathcal{P}(E)/\,\mathrm{Fin}$ containing no regular irreducible subgap. Thus, it does not have the chain-gap property.

4. Proof of Theorem 1.2

The proof naturally breaks into two main parts.

Part 1: $L(\mathbb{S}, \leq)$ does not have the selection property.
It suffices to prove the following.

Proposition 4.1. (1) ω_1 *does not embed into* $L(\mathbb{S}, \leq)$.
(2) $L(\mathbb{S}, \leq)$ *has a minimal gap* $(A, \emptyset)$ *of cardinality* $(\aleph_1, 0)$.

Indeed to see how the statement of Part 1 follows, let $(a_\alpha)_{\alpha<\omega_1}$ be an enumeration of the elements of A. Set $A_\alpha = \{a_\beta : \beta < \alpha\}$. If the selection property holds, then to every pair $(A_\alpha, \emptyset)$ we can associate an element $x_\alpha \in U(A_\alpha) \cap \emptyset_* = U(A_\alpha)$ such that $(A_\alpha, \emptyset) \leq (A_{\alpha'}, \emptyset)$ implies $x_\alpha \leq x_{\alpha'}$. In particular, for $\alpha \leq \alpha'$ we must have $x_\alpha \leq x_{\alpha'}$. If ω_1 does not embed into $L(\mathbb{S}, \leq)$ then the sequence x_α must eventually be stationary, and in particular have an upper bound. If u is such an upper bound, then $u \in U(A_\alpha)$ for every α, thus $u \in U(A)$. This is impossible since A is unbounded.

Proof of Proposition 4.1. We first prove that (2) holds.

Lemma 4.2. *Fix* $r \in \mathbb{S}$ *arbitrary and let* $A = \{\downarrow x : x \in \mathbb{S}$ *and* $x \leq_{\mathbb{R}} r\}$. *Then* $(A, \emptyset)$ *is a minimal gap in* $L(\mathbb{S}, \leq)$ *of size* $(\aleph_1, 0)$.

Proof. The proof will follow after these two claims.

Claim 4.3. *Any two elements of* $(\mathbb{S}, \leq)$ *have an upper bound.*

Proof of Claim 4.3. Let $x, y \in \mathbb{S}$. The set $X := \{z \in \mathbb{S} : z \leq_{\omega_1} x$ or $z \leq_{\omega_1} y\}$ is countable, but on the other hand the set $Y := \{z \in \mathbb{S} : x, y \leq_{\mathbb{R}} z\}$ is uncountable. Thus, $Y \setminus X$ is non empty, and every $z \in Y \setminus X$ majorizes x and y in $(\mathbb{S}, \leq)$, proving our claim. □

Claim 4.4. *A subset* B *of* $L(\mathbb{S}, \leq)$ *has an upper bound if and only if* $\bigcup B$ *has an upper bound in* $(\mathbb{S}, \leq)$.

Proof of Claim 4.4. If B has an upper bound in $L(\mathbb{S}, \leq)$, then there is some member X of $L(\mathbb{S}, \leq)$ which includes every element of B, hence $\bigcup B \subseteq X$. This set X is a finite union of finite intersections of principal initial segments of $(\mathbb{S}, \leq)$; hence B is a subset of a finite union $\downarrow x_1 \cup \cdots \cup \downarrow x_k$ of principal initial segments of $(\mathbb{S}, \leq)$. By Claim 4.3, there is some x which majorizes $x_1, \ldots, x_k$, and such an x majorizes $\bigcup B$. The converse is trivial and the claim is verified. □

With these claims, the proof of Lemma 4.2 goes as follows. From the fact that $\mathbb{S}$ is $\aleph_1$-dense of size $\aleph_1$, A has size $\aleph_1$. Next, let's see that $(A, \emptyset)$ is a gap.

Now if A is bounded, then Claim 4.4 implies that $\bigcup A \subseteq \downarrow z$ for some $z \in \mathbb{S}$. In particular, the uncountable initial segment of $\mathbb{S}$ below r under $\leq_{\mathbb{R}}$ from $\mathbb{S}$ is a subset of the countable initial segment of $\mathbb{S}$ below z under $\leq_{\omega_1}$, a contradiction. Therefore, A is unbounded in $L(\mathbb{S}, \leq)$ and $(A, \emptyset)$ is a gap.

Finally, showing that $(A, \emptyset)$ is a minimal gap amounts to showing that every countable subset A' of A is bounded. Indeed, $\{x \in \mathbb{S} : \downarrow x \in A'\}$ is countable and thus bounded in $\leq_{\omega_1}$. If y is such a bound, then according to Claim 4.3 there is some $z \in \mathbb{S}$ such that $y \leq z$ and $r \leq z$. Then $\downarrow z$ is a bound of A'. □

This concludes the verification of statement (2) of the Proposition, and we now turn our attention to (1).

Claim 4.5. *Let $a_1, \ldots, a_n \in (\mathbb{S}, \leq)$ and $A := \downarrow a_1 \cap \cdots \cap \downarrow a_n$. Then there are $i, j \leq n$ such that $A = \downarrow a_i \cap \downarrow a_j$.*

Proof of Claim 4.5. Let i such that $a_i \leq_{\omega_1} a_k$ for all k, $1 \leq k \leq n$, and let j such that $a_j \leq_{\mathbb{R}} a_k$ for all k, $1 \leq k \leq n$. Then $A = \downarrow a_i \cap \downarrow a_j$. □

From this we immediately get:

Claim 4.6. *Every finite intersection of principal initial segments of $(\mathbb{S}, \leq)$ is of the form $\downarrow x \cap \downarrow y$ with $x \leq_{\mathbb{R}} y$ and $y \leq_{\omega_1} x$.*

Now toward a proof that ω_1 does not embed in $L(\mathbb{S}, \leq)$, let $(A_\alpha)_{\alpha<\omega_1}$ be an ω_1-sequence of elements of $L(\mathbb{S}, \leq)$. According to Claim 4.6, for each $\alpha < \omega_1$, we may write $A_\alpha = \bigcup\{A_{\alpha,i} : i \in I_\alpha\}$ where I_α is a finite set and $A_{\alpha,i} = \downarrow x_{\alpha,i} \cap \downarrow y_{\alpha,i}$ with $x_{\alpha,i} \leq_{\mathbb{R}} y_{\alpha,i}$ and $y_{\alpha,i} \leq_{\omega_1} x_{\alpha,i}$. Set $X_\alpha = \{x_{\alpha,i} : i \in I_\alpha\}$, $Y_\alpha = \{y_{\alpha,i} : i \in I_\alpha\}$ and $Z_\alpha = X_\alpha \cup Y_\alpha$.

Claim 4.7. *If $(A_\alpha)_{\alpha<\omega_1}$ is strictly increasing then the sets Z_α cannot be pairwise disjoint.*

Proof of Claim 4.7. For $\alpha < \omega_1$ let $x_\alpha = \max_{\mathbb{R}}(X_\alpha)$. Since $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ we have $\downarrow_{\mathbb{R}} A_\alpha \subseteq \downarrow_{\mathbb{R}} A_\beta$. Since further ω_1 does not embed into $(\mathbb{S}, \leq_{\mathbb{R}})$, it does not embed into the chain of initial segments of $(\mathbb{S}, \leq_{\mathbb{R}})$. Hence, the ω_1-sequence $(\downarrow_{\mathbb{R}} A_\alpha)_{\alpha<\omega_1}$ is eventually constant. Let α_0 such that $\downarrow_{\mathbb{R}} A_\alpha = \downarrow_{\mathbb{R}} A_{\alpha_0}$ for $\alpha_0 \leq \alpha < \omega_1$ and define $A = \downarrow_{\mathbb{R}} A_{\alpha_0}$.

Suppose now that the Z_α's are pairwise disjoint. Then in particular all the X_α's are pairwise disjoint, and therefore there is at most one α such that $A = \downarrow_{\mathbb{R}} x_\alpha$. With no loss in generality, we may assume that $x_\alpha \in \mathbb{S} \setminus A$ for all $\alpha > \alpha_0$. Since ω_1^* does not embed in $\mathbb{S}$, there is $x \in \mathbb{S} \setminus A$ for which $X_x := \{\alpha : \alpha_0 < \alpha < \omega_1 \text{ and } x <_{\mathbb{R}} x_\alpha\}$ is uncountable. Now consider $y_\alpha := \max_{\mathbb{R}}\{y_{\alpha,i} : x_{\alpha,i} = x_\alpha \text{ and } i \in I_\alpha\}$ for $\alpha < \omega_1$.

Since the Y_α's are also pairwise disjoint, all y_α's are distinct. Since $\{z \in \mathbb{S} : z \leq_{\omega_1} x\}$ is countable, there is some $\alpha \in X_x$ such that $x \leq_{\omega_1} x_\alpha$ and $x \leq_{\omega_1} y_\alpha$. But for such an α we have $x <_{\mathbb{R}} x_\alpha \leq_{\mathbb{R}} y_\alpha$. This implies that $x \in A_\alpha$. Since $A_\alpha \subseteq \downarrow_{\mathbb{R}} A_\alpha = A$, we get $x \in A$, contradicting the properties of x. □

Claim 4.8. *If there is a strictly increasing ω_1-sequence of elements of $L(\mathbb{S}, \leq)$ then there is an $\aleph_1$-dense subchain $\mathbb{S}'$ of $\mathbb{R}$ and a strictly increasing ω_1-sequence of elements of $L(\mathbb{S}', \leq')$ for which all Z_α's are pairwise disjoint.*

Proof of Claim 4.8. Start with a strictly increasing ω_1-sequence $(A_\alpha)_{\alpha<\omega_1}$ of members of $L(\mathbb{S}, \leq)$. Since the Z_α's are finite, there is an uncountable subset U of ω_1 and a finite subset F of $\mathbb{S}$ such that for all $\alpha, \beta \in U$, F is an initial segment of Z_α with respect to $(\mathbb{S}, \leq_{\omega_1})$ and $Z_\alpha \cap Z_\beta = F$. That is, $(Z_\alpha)_{\alpha \in U}$ forms an uncountable Δ-system.

Let $x \in \mathbb{S}$ satisfy $F \subseteq \downarrow_{\omega_1} x$ and write $X = \downarrow_{\omega_1} x$. Set $\mathbb{S}' = \mathbb{S} \setminus X$, $A'_\alpha = A_\alpha \setminus X$, $A'_{\alpha,i} = A_{\alpha,i} \setminus X$, and $I'_\alpha = \{i \in I_\alpha : A'_{\alpha,i} \neq \emptyset\}$.

Since X is countable, $\mathbb{S}'$ is again an $\aleph_1$-dense chain with no end points and the well-ordering induced has order type ω_1. The intersection order $\leq'$ is the order induced by $\leq$ on $\mathbb{S}'$.

The ω_1-sequence $(A'_\alpha)_{\alpha<\omega_1}$ is increasing and since X is countable, it contains a strictly increasing subsequence $(A'_\alpha)_{\alpha \in U'}$, for some uncountable $U' \subseteq U$. Let $\alpha \in U' \setminus \min(U')$. Then $A'_\alpha \neq \emptyset$, hence $A'_\alpha = \cup\{A'_{\alpha,i} : i \in I'_\alpha\}$. Since X is an initial segment of $(\mathbb{S}, \leq)$ it follows that $A'_{\alpha,i} = \downarrow_{(\mathbb{S}',\leq')} x_{\alpha,i} \cap \downarrow_{(\mathbb{S}',\leq')} y_{\alpha,i}$. Thus, with $X'_\alpha := \{x_{\alpha,i} : i \in I'_\alpha\}$, $Y'_\alpha := \{y_{\alpha,i} : i \in I'_\alpha\}$, and $Z'_\alpha := X'_\alpha \cup Y'_\alpha$, we see that the Z'_α's for $\alpha \in U \setminus \min(U)$ are pairwise disjoint. □

From Claim 4.7 and Claim 4.8, there is no strictly increasing ω_1-sequence of elements of $L(\mathbb{S}, \leq)$. The proof of Proposition 4.1 is complete. □

Part 2: $L(\mathbb{S}, \leq)$ has the chain-gap property.

Let (A, B) be a gap in $L(\mathbb{S}, \leq)$. By Theorem 2.2 it suffices to show that it contains a regular irreducible gap. The fact that $L(\mathbb{S}, \leq)$ is a distributive lattice allows us to break this into two steps via the following well-known consequence of distributivity.

Lemma 4.9 ([8, Lemma 4])**.** *There is a partition of $L(\mathbb{S}, \leq)$ into a prime ideal I and a prime filter F such that $(A, \emptyset)$ is a gap of I and $(\emptyset, B)$ is a gap of F.*

Proof. Since (A, B) is a pre-gap, $U(A)$ is a filter and $L(B)$ is an ideal. Since (A, B) is a gap, $U(A) \cap L(B) = \emptyset$. By Stone's Lemma [4], there exists a prime ideal I such that $L(B) \subseteq I$ and $I \cap U(A) = \emptyset$. Put $F = L(\mathbb{S}, \leq) \setminus I$. □

Now to show that (A, B) contains a regular irreducible gap in $L(\mathbb{S}, \leq)$, it suffices to show that both $(A, \emptyset)$ and $(\emptyset, B)$ do, within I and F, respectively.

Lemma 4.10. *The gap $(\emptyset, B)$ of F contains a regular irreducible gap.*

Proof. It suffices to show that the coinitiality of F is countable.

Let $K = \{x \in \mathbb{S} : \downarrow x \in F\}$. The coinitiality of any subset of $\mathbb{R}$ is countable, so we can select a countable subset D coinitial in K with respect to the order $\leq_{\mathbb{R}}$. Let $U = \{x \in K : x \leq_{\omega_1} y \text{ for some } y \in D\}$. Since D is countable, U is countable. Moreover, U is coinitial in K. To see this, for any $x \in K$, there is $x_1 \in D$ such that $x_1 \leq_{\mathbb{R}} x$. Now, either $x_1 \leq_{\omega_1} x$, in which case $x_1 \leq x$, or $x <_{\omega_1} x_1$ but then by definition of U, $x \in U$. So in both cases, x majorizes an element of U. Let $\delta(U) = \{\downarrow x \cap \downarrow y : x \in U, y \in U\}$. Since U is countable, $\delta(U)$ is countable. Moreover, it is coinitial in F. Indeed, any $a \in F$ is of the form $a = a_1 \cup \cdots \cup a_n$ where $a_i = \downarrow x_i \cap \downarrow y_i$. Since F is a prime filter, some $a_i \in F$ and since F is a filter, x_i, y_i belong to K. Because U is coinitial in K there are $x', y' \in U$ such that $x' \leq x_i$ and $y' \leq y_i$. Thus, $\downarrow x' \cap \downarrow y' \subseteq a_i \subseteq a$ proving that $\delta(U)$ is coinitial in F. □

Lemma 4.11. *The gap $(A, \emptyset)$ of I contains a regular irreducible gap.*

Proof. Elements of I are of the form $a = a_1 \cup a_2 \cdots \cup a_n$ where $a_i = \downarrow x_i \cap \downarrow y_i$. Since I is a prime ideal, for every a_i one of the sets $\downarrow x_i, \downarrow y_i$ belongs to I. Consequently the set of finite unions of members of I of the form $\downarrow x$ is cofinal in I. Note that for all $X \subseteq I$, $\bigcup X = \{x \in \mathbb{S} : x \in u \text{ for some } u \in X\}$ and observe that: $(A, \emptyset)$ is a gap in I iff $\bigcup A$ is not contained in a finitely generated initial segment of $\bigcup I$.

Let $\leq_*$ denote one of the two orderings $\leq_{\omega_1}, \leq_{\mathbb{R}}$ restricted to $\bigcup I$ and let $(\bigcup I)_* := (\bigcup I, \leq_*)$. We consider two cases:

(1): $\bigcup A$ is an unbounded subset of $(\bigcup I)_*$ for some $\leq_*$.
(2): $\bigcup A$ is a bounded subset of $(\bigcup I)_*$ for the two possible orderings $\leq_*$.

Case (1). To ease reading in this case, let $J = (\bigcup I)_*$.
Since $\bigcup A$ is an unbounded subset of J in this case, J has no largest element. Therefore we can choose $(c_\alpha)_{\alpha<\mu}$ ($\mu = \omega_1$ or ω) to be a strictly increasing cofinal sequence of elements of J. With $I_\alpha := \{u \in I : u \subseteq \downarrow_J c_\alpha\}$, $\alpha < \mu$, we have an increasing sequence. We claim $I = \bigcup_{\alpha<\mu} I_\alpha$. Indeed, let $u \in I$. There are $x_1, \dots, x_k \in \bigcup I$ such that $u \subseteq \downarrow x_1 \cup \cdots \cup \downarrow x_k$, and there exists c_α such that $x_1, \dots, x_k \leq_* c_\alpha$. For $i = 1, \dots, k$, $\downarrow x_i \subseteq \downarrow_J x_i \subseteq \downarrow_J c_\alpha$, so we have $u \subseteq \downarrow_J c_\alpha$ and, thus, $u \in I_\alpha$.

Now let us see that for all α, $A \setminus I_\alpha \neq \emptyset$. Indeed, since $\bigcup A$ is unbounded in J, there is some $x \in \bigcup A$ such that $c_{\alpha+1} \leq_* x$. Since $x \in a$ for some $a \in A$, that $a \notin I_\alpha$. Pick $a_\alpha \in A \setminus I_\alpha$ for each $\alpha < \mu$. Let $A' := \{a_\alpha : \alpha < \mu\}$. Since $(A, \emptyset)$ is a minimal gap and μ is regular, $(A', \emptyset)$ is a regular and irreducible gap. This completes Case (1).

Case (2). To simplify notation in this case, let $W = (\mathbb{S}, \leq_{\omega_1})$ and let $R = (\mathbb{S}, \leq_{\mathbb{R}})$.

Since $\bigcup A$ is a bounded subset of $\bigcup I$ with respect to ω_1, $\bigcup A$ is countable. Hence, there is a least element c of W for which $C := (\bigcup A) \cap \downarrow_W c$ is not contained in a finitely generated initial segment of $\bigcup I$. Let $\widetilde{C} = \{\downarrow x : x \in C\}$. The pair $(\widetilde{C}, \emptyset)$ is a gap in I and is a subgap of $(A, \emptyset)$. Therefore, it suffices to show that $(\widetilde{C}, \emptyset)$ contains a regular irreducible gap.

Let $G = \{z \in \bigcup I : C \subseteq \downarrow_W z\}$. Clearly c is a lower bound of G with respect to $\leq_{\omega_1}$. Let $C_1 = C \cap \downarrow_{\mathbb{S}} G$ and let $C_2 = C \setminus C_1$. Since C is not contained in a finitely generated initial segment of $\bigcup I$, for at least one of $i = 1$ or $i = 2$, C_i has the same property.

Subcase 1. $i = 2$. Due to the choice of c, C_2 is cofinal in $\downarrow_W c$. Thus C_2 is cofinal in C with respect to $\leq_{\omega_1}$. Let C' be a cofinal subset of C_2 with respect to $\leq_{\omega_1}$ having order type ω. We claim that no countable subset of C' can be contained in a finitely generated initial segment of $\bigcup I$. Indeed, if there were one, then there would be one, say C'', contained in some set of the form $\downarrow z$, with $z \in \bigcup I$. But, with respect to the order $\leq_{\omega_1}$, C'' is cofinal in C', C' is cofinal in C_2, and C_2 is cofinal in C, so C'' is cofinal in C. Then from $C'' \subseteq \downarrow_W z$ we get $z \in G$. With the fact that $C'' \subseteq \downarrow_R z$ this implies $C'' \subseteq C_1$, a contradiction. Therefore $\widetilde{C}' := \{\downarrow x : x \in C'\}$ is a regular irreducible gap, as desired.

Subcase 2. Subcase 1 does not hold. Hence $i = 1$. Again, due to the choice of c, C_1 is cofinal in $\downarrow_W c$ and thus in C. Select a cofinal sequence in C_1 with type ω, say $x_0 <_{\omega_1} x_1 <_{\omega_1} \cdots <_{\omega_1} x_n <_{\omega_1} \cdots$. Observe that G has no largest element with respect to the order $\leq_{\mathbb{R}}$. To see this, suppose that u is the largest element. Then we have both $C_1 \subseteq \downarrow_W u$ and $C_1 \subseteq \downarrow_R u$, thus $C_1 \subseteq \downarrow u$, contradicting the unboundedness of C_1. Hence, the cofinality of G with respect to $\leq_{\mathbb{R}}$ is countably infinite and we may select $u_0 <_{\mathbb{R}} u_1 <_{\mathbb{R}} \cdots <_{\mathbb{R}} u_n <_{\mathbb{R}} \cdots$ in G forming a cofinal sequence with respect to the order $\leq_{\mathbb{R}}$.

Claim 4.12. *There is a sequence $D := \{y_n : n < \omega\} \subseteq C_1$, strictly increasing with respect to $\leq = \leq_{\mathbb{S}}$, that is cofinal in $(C_1, \leq_{\omega_1})$ and in $(G, \leq_{\mathbb{R}})$.*

Proof of Claim 4.12. First, we define y_0. Since $x_0 <_{\omega_1} c$, $C_1 \cap \downarrow_W x_0$ is contained in a finitely generated initial segment of $\bigcup I$. Hence, $C_1 \cap \uparrow_W x_0$ is not contained in a finitely generated initial segment of $\bigcup I$. In particular,

$$C_1 \cap \uparrow_W x_0 \;\not\subseteq\; \downarrow u_0 \ . \tag{1}$$

Since $u_0 \in G$ we have $C \subseteq \downarrow_W u_0$. From (1) we get

$$C_1 \cap \uparrow_W x_0 \;\not\subseteq\; \downarrow_R u_0 \ . \tag{2}$$

From (2) there is some $y_0 \in C_1$ such that $y_0 \geq_{\omega_1} x_0$ and $y_0 \geq_{\mathbb{R}} u_0$.

Suppose that $y_0 < y_1 \cdots < y_n$ are defined with $x_i \leq_{\omega_1} y_i$ and $u_i \leq_{\mathbb{R}} y_i$. In order to define y_{n+1} select x_{n_1} and u_{n_1} such that:

$$y_n \leq_{\omega_1} x_{n_1}, x_{n+1} <_{\omega_1} x_{n_1} \text{ and } y_n \leq_{\mathbb{R}} u_{n_1}, u_{n+1} <_{\mathbb{R}} u_{n_1} \ .$$

As above, since $x_{n_1} <_{\omega_1} c$, $C_1 \cap \uparrow_W x_{n_1}$ is not contained in a finitely generated initial segment of $\bigcup I$. Thus, $C_1 \cap \uparrow_W x_{n_1} \not\subseteq \downarrow_{\mathbb{R}} u_{n_1}$, so there is an element, say y_{n+1}, such that $x_{n_1} \leq_{\omega_1} y_{n+1}$ and $u_{n_1} \leq_{\mathbb{R}} y_{n+1}$. Clearly, $y_n < y_{n+1}$, $x_{n+1} \leq_{\omega_1} y_{n+1}$ and $u_{n+1} \leq_{\mathbb{R}} y_{n+1}$. From our construction, D is cofinal in $(C_1, \leq_{\omega_1})$ and in $(G, \leq_{\mathbb{R}})$. $\square$

Since D is cofinal in $(C_1, \leq_{\omega_1})$ and in $(G, \leq_{\mathbb{R}})$, D is unbounded in $\bigcup I$. But $\widetilde{D} := \{\downarrow y_n : y_n \in D\}$ is a chain and is unbounded in I. Hence, $(\widetilde{D}, \emptyset)$ is a regular irreducible gap in I.

With this, the proofs of Case (2) and of Lemma 4.11 are complete. $\square$

Observe that in a lattice L it is not always true that a gap $(A, \emptyset)$ contains a regular irreducible gap, even when A is countable (see [9]). This is indeed the case if L is a non-principal maximal ideal in $\mathcal{P}(\mathbb{N})$ and $A = \{\{n\} : n \in \mathbb{N}\}$. The above proof is therefore more technical, as expected.

We conclude with the following extension question.

Problem 4.13. Let κ be such that $\omega < \kappa \leq 2^{\aleph_0}$, $\mathbb{S}$ be a κ-dense subchain of $\mathbb{R}$ of size κ and $L(\mathbb{S}, \leq)$ be the distributive lattice associated with a Sierpinskization of $\mathbb{S}$. Does $L(\mathbb{S}, \leq)$ have the chain-gap property?

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