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An Ordered Set of Size $\aleph_1$ with Monochromatic Maximal Chains ⋆

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Abstract. Every model of ZFC contains a product of two linear orders, each of size $\aleph_1$, with the property that every subset or complement thereof contains a maximal chain.

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1. Introduction

A finite (partially) ordered set without isolated points can be partitioned into an initial segment containing the minimal elements and a final segment containing the maximals – and every maximal chain intersects both parts. Countable ordered sets also admit a partition into an initial and final segment so that both intersect every maximal chain [2]. This result does not generalize to higher cardinals. In fact, for any cardinal κ there is an ordered set P_κ for which every coloring by κ colors results in a monochromatic maximal chain [2]. The examples P_κ, however, did not resolve all of the questions. The following remained open [2].

PROBLEM 1. Does every finite product of chains have a 2-coloring of its elements so that no monochromatic maximal chains are present?

PROBLEM 2. Are there ordered sets of size $\aleph_1$ such that every 2-coloring leaves some monochromatic maximal chain?

[1] provides a product of two linear orders of cardinality $2^{\aleph_0}$ such that every 2-coloring leaves a monochromatic chain. This settles both problems but only with the aid of the continuum hypothesis ($2^{\aleph_0} = \aleph_1$). By sharpening some of the tools we introduced there and making a more thorough analysis of maximal chains, in this paper we avoid use of the continuum hypothesis and definitively settle the problems.

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The paper is organized as follows. Section 2 contains the terminology and observations required to construct the counterexample and provides the statement of the Main Theorem. Section 3 contains two lemmas which demonstrate the existence of maximal chains in “sufficiently dense” subsets of the example. Section 4 is comprised of four lemmas which exploit the notion of an “expandable chain”. Section 5 knits these lemmas with a Ramsey-style argument that completes the proof of the Main Theorem. Finally, open problems and conclusions are collected in Section 6.

2. The Main Theorem

To present the counterexample, we need two types of products of chains (linearly ordered sets), both defined on the Cartesian product of the ground sets. For (partially) ordered sets X and Y, $X \times Y$ denotes the *direct product* with ordering $\langle x, y\rangle \leq \langle x', y'\rangle$ iff $x \leq x'$ in X and $y \leq y'$ in Y. We use $X \cdot Y$ to denote the *lexicographic product* with ordering $\langle x, y\rangle < \langle x', y'\rangle$ iff $x < x'$ in X, or $x = x'$ in X and $y < y'$ in Y. In case that both X and Y are chains, $X \cdot Y$ is a chain as well; it may be useful to think of this chain as obtained by replacing every element of X with a copy of Y. While this use of $X \cdot Y$ is nonstandard for ordinal arithmetic (contrast with the definition in [5] and earlier comments on p. 21 of [5]), it is better for our purposes. All other notation is standard and can be found in [5] (for linear orders), [4] (for partial orders), and [3] (for set theory).

Let $C_0 = (\omega_1^* + \omega_1) \cdot (-1, 1)$ where $(-1, 1)$ denotes the countable dense chain without endpoints. Let us take elements of C_0 to be of the form $\langle \alpha, x\rangle$ with $\alpha \in \omega_1^* + \omega_1$ and x a rational number, $-1 < x < 1$. We identify the maximum element of ω_1^* with the minimum of ω_1 and label this 0. We also use 0 to denote the rational zero; to avoid at least some confusion, we set $\hat{0} = \langle 0, 0\rangle \in C_0$ in the construction below. Observe that C_0 has cofinality ω_1, coinitiality ω_1^*, and all gaps $\langle A, B\rangle$, with A and B nonempty, have cofinality ω and coinitiality ω^*.

Let C be the set of all ω-sequences of elements of C_0 with all but finitely many of the terms equal to $\hat{0}$, ordered by

$$\langle\langle \alpha_n, x_n\rangle : n < \omega\rangle < \langle\langle \beta_n, y_n\rangle : n < \omega\rangle \quad \text{iff } \langle \alpha_n, x_n\rangle < \langle \beta_n, y_n\rangle$$

at the least n for which $\langle \alpha_n, x_n\rangle \neq \langle \beta_n, y_n\rangle$. So, C is the (ω-fold) lexicographic product of C_0 restricted to sequences “of finite support” and has cardinality $\aleph_1$.

For convenience, let $\widehat{C} = (-1, 1) \cdot C$. Note that C has uncountable cofinality and coinitiality while $\widehat{C}$ has countable cofinality and coinitiality. Both chains are clearly of cardinality $\aleph_1$ (even under ZFC, independent of the continuum hypothesis). Gaps are dense in both C and $\widehat{C}$: that is, for every pair $a < b$ there is a gap $\langle A, B\rangle$ such that $a \in A$ and $b \in B$. We use $\mathbf{x}, \mathbf{y}, \ldots$ to denote elements of C and $\widehat{C}$; in particular, $\mathbf{0}$ denotes the sequence in C with all components $\hat{0}$, and the same symbol labels the element of $\widehat{C}$ obtained by adding the rational 0 to the front of this sequence.

THEOREM 2.1. *Any* 2*-coloring of the direct product* $C \times \widehat{C}$ *leaves monochromatic maximal chains.*

3. Constructing Maximal Chains

Given a sequence $\mathbf{x} = \langle x_0, x_1, x_2, \dots \rangle$, let

$$\mathbf{x}|_n = \langle x_0, x_1, x_2, \dots, x_{n-1} \rangle, \text{ and}$$

$$\pi_n(\mathbf{x}) = x_n,$$

for $n = 0, 1, \dots$. So, π_n is the usual nth projection map defined on a Cartesian product, with indexing beginning at 0.

It is convenient to treat n-tuples and ω-sequences as strings. Indeed, the chains C and $\widehat{C}$ can be thought of as strings described as follows. For each $\mathbf{a} \in C$, $\mathbf{a} = \alpha_0 x_0 \alpha_1 x_1 \dots$ where each $\alpha_k \in \omega_1^* + \omega_1$, each $x_k \in (-1, 1)$, and $\alpha_k = 0$, $x_k = 0$ for all $k \geq k_0$. The ordering on C defined above is just the lexicographic ordering on these strings. The chain $\widehat{C}$ is the same except that its strings commence with an element from $(-1, 1)$, that is, for each $\mathbf{y} \in \widehat{C}$, $\mathbf{y} = x\alpha_0 x_0 \alpha_1 x_1 \dots$.

We shall concatenate strings and concatenate segments of elements from C or $\widehat{C}$. For instance, given

$$\begin{aligned} \mathbf{g} &= \langle\langle \alpha_0, p_0 \rangle, \langle \alpha_1, p_1 \rangle, \dots, \langle \gamma, r \rangle, 0 \rangle \quad \text{and} \\ \mathbf{z} &= \langle r_{-1}, \langle \gamma_0, r_0 \rangle, \langle \gamma_1, r_1 \rangle, \dots \rangle, \end{aligned}$$

with $0 \in \omega_1^* + \omega_1$ and $r_{-1} \in (-1, 1)$ the concatenation $\mathbf{gz}$ is an element of C:

$$\mathbf{gz} = \langle\langle \alpha_0, p_0 \rangle, \langle \alpha_1, p_1 \rangle, \dots, \langle \gamma, r \rangle, \langle 0, r_{-1} \rangle, \langle \gamma_0, r_0 \rangle, \langle \gamma_1, r_1 \rangle, \dots \rangle.$$

For subsets A and B of some ordered set, we write $A < B$ if $a < b$ for each $a \in A$ and each $b \in B$. A subset of a product $G \subseteq P \times P'$ is *crooked* if $\pi_0 x \neq \pi_0 y$ and $\pi_1 x \neq \pi_1 y$ for any two elements $x, y \in G$ (we usually refer to crooked chains).

A subset I of an ordered set P is *convex* if $x_1 < x < x_2$ and $x_1, x_2 \in I$ imply that $x \in I$. A product $I \times J$ of two sets I and J convex in ordered sets P and Q, respectively, is called a *block* of $P \times Q$ if both I and J have at least two elements. Given $A \subseteq B \subseteq P \times Q$, call A *block-dense* in B if every block of $P \times Q$ contained in B contains an element of A.

Each of the lemmas that follows deals with the construction of a maximal chain within some subset $A \subseteq P$, the structure of A depending on the lemma. In the concluding argument, A will play the role of a color-class.

LEMMA 3.1. *Let* $\langle \mathbf{u}^0, \mathbf{v}^0 \rangle$ *belong to a subset* A *block-dense in* $C \times C$. *Then* A *contains a crooked chain extending* $\langle \mathbf{u}^0, \mathbf{v}^0 \rangle$ *that is maximal in* $C \times C$.

Proof. As a preliminary observation, we show that any element $\langle a, b\rangle \in C_0 \times C_0$ can be extended to a crooked chain maximal in $C_0 \times C_0$. Since both $\omega_1^* + \omega_1$ and $(-1, 1)$ are transitive chains, so is C_0. Also, order isomorphisms of a product of the form $\phi = \langle \phi_0, \phi_1\rangle$ preserve crookedness. Thus, we may assume that $\langle a, b\rangle = \langle \hat{0}, \hat{0}\rangle$ and just observe that the diagonal of $C_0 \times C_0$ is a crooked chain containing $\langle \hat{0}, \hat{0}\rangle$, and is maximal because C_0 is dense.

Using the preliminary observation and denoting the empty string by Λ, let $D(\Lambda, \Lambda)$ be a crooked maximal chain in $\pi_0[C] \times \pi_0[C]$ (a copy of $C_0 \times C_0$) which contains the element $\langle \pi_0\mathbf{u}^0, \pi_0\mathbf{v}^0\rangle$. For each $\langle d_0, d_1\rangle \in D(\Lambda, \Lambda)$ other than $\langle \pi_0\mathbf{u}^0, \pi_0\mathbf{v}^0\rangle$ we consider the set of elements of $C \times C$ of the form $\langle d_0\mathbf{x}, d_1\mathbf{y}\rangle$ and incomparable with $\langle d_0\mathbf{0}, d_1\mathbf{0}\rangle$. Since this set is the union of two blocks of $C\times C$ and A is block-dense in $C \times C$, we can choose $\mathbf{x}(d_0), \mathbf{y}(d_1)$ so that $\langle d_0\mathbf{x}(d_0), d_1\mathbf{y}(d_1)\rangle$ is an element of A incomparable with $\langle d_0\mathbf{0}, d_1\mathbf{0}\rangle$. For $\langle d_0, d_1\rangle = \langle \pi_0\mathbf{u}^0, \pi_0\mathbf{v}^0\rangle$ we let $\langle d_0\mathbf{x}(d_0), d_1\mathbf{y}(d_1)\rangle = \langle \mathbf{u}^0, \mathbf{v}^0\rangle$. The set

$$M_0 = \{\langle d_0\mathbf{x}(d_0), d_1\mathbf{y}(d_1)\rangle : \langle d_0, d_1\rangle \in D(\Lambda, \Lambda)\}$$

is a crooked chain in $C \times C$, because $D(\Lambda, \Lambda)$ is a crooked chain, and is contained in A.

Now suppose that the chain M_{n-1} is already constructed in $C \times C$. For each $\langle \mathbf{u}, \mathbf{v}\rangle \in M_{n-1}$, use the preliminary observation to extend $\langle \pi_n\mathbf{u}, \pi_n\mathbf{v}\rangle$ to a crooked maximal chain $D(\mathbf{u}|_n, \mathbf{v}|_n)$ in $\pi_n[C] \times \pi_n[C]$. Then for each $\langle d_0, d_1\rangle \in D(\mathbf{u}|_n, \mathbf{v}|_n)$ other than $\langle \pi_n\mathbf{u}, \pi_n\mathbf{v}\rangle$ choose $\mathbf{x}(\mathbf{u}|_n d_0), \mathbf{y}(\mathbf{v}|_n d_1)$ so that $\langle \mathbf{u}|_n d_0\mathbf{x}(\mathbf{u}|_n d_0), \mathbf{v}|_n d_1\mathbf{y}$ $(\mathbf{v}|_n d_1)\rangle$ is an element of A and is incomparable to $\langle \mathbf{u}|_n d_0\mathbf{0}, \mathbf{v}|_n d_1\mathbf{0}\rangle$. Again, this is possible as A is block-dense and the set of incomparables of $\langle \mathbf{u}|_n d_0\mathbf{0}, \mathbf{v}|_n d_1\mathbf{0}\rangle$ is the union of two blocks. For $\langle d_0, d_1\rangle = \langle \pi_n\mathbf{u}, \pi_n\mathbf{v}\rangle$ we make the selection

$$\langle \mathbf{u}|_n d_0\mathbf{x}(\mathbf{u}|_n d_0), \mathbf{v}|_n d_1\mathbf{y}(\mathbf{v}|_n d_1)\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$$

in order that $M_{n-1} \subseteq M_n$. Observe that the set

$$M_n = \bigcup_{\langle \mathbf{u},\mathbf{v}\rangle \in M_{n-1}} \{\langle \mathbf{u}|_n d_0\mathbf{x}(\mathbf{u}|_n d_0), \mathbf{v}|_n d_1\mathbf{y}(\mathbf{v}|_n d_1)\rangle : \langle d_0, d_1\rangle \in D(\mathbf{u}|_n, \mathbf{v}|_n)\}$$

is a crooked chain contained in A.

Set $M = \bigcup M_n$.

Certainly M is crooked and is contained in A. To prove that it is maximal in $C \times C$, let $\langle \mathbf{u}', \mathbf{v}'\rangle \in C \times C$ and assume that $\langle \mathbf{u}', \mathbf{v}'\rangle \notin M$. Let k be least such that $\langle \pi_m(\mathbf{u}'), \pi_m(\mathbf{v}')\rangle = \langle \hat{0}, \hat{0}\rangle$ for all $m \geq k$.

First suppose that there exists $\langle \mathbf{u}, \mathbf{v}\rangle \in M$ such that $\mathbf{u}|_k = \mathbf{u}'|_k$ and $\mathbf{v}|_k = \mathbf{v}'|_k$. By the construction of the sets M_n we know there is a pair in M_k which agrees with $\langle \mathbf{u}, \mathbf{v}\rangle$ in its first k positions, so we may assume that $\langle \mathbf{u}, \mathbf{v}\rangle \in M_k$. Choose $r \geq k$ least possible so that $\langle \pi_r(\mathbf{u}), \pi_r(\mathbf{v})\rangle \neq \langle \pi_r(\mathbf{u}'), \pi_r(\mathbf{v}')\rangle$ or, equivalently, $\langle \pi_r(\mathbf{u}), \pi_r(\mathbf{v})\rangle \neq \langle \hat{0}, \hat{0}\rangle$. We used the crooked chain $D(\mathbf{u}|_r, \mathbf{v}|_r)$ maximal in $\pi_r[C] \times \pi_r[C]$ to define M_r. Either $\langle \hat{0}, \hat{0}\rangle$ is incomparable to some element of

$D(\mathbf{u}|_r, \mathbf{v}|_r)$ (and $\langle \mathbf{u}', \mathbf{v}' \rangle$ is consequently incomparable to the associated element of M_r) or $\langle \hat{0}, \hat{0} \rangle$ belongs to $D(\mathbf{u}|_r, \mathbf{v}|_r)$. In this latter case we have an element of M_r of the form $\langle \mathbf{u}|_r \hat{0}\mathbf{x}(\mathbf{u}|_r \hat{0}), \mathbf{v}|_r \hat{0}\mathbf{y}(\mathbf{v}|_r \hat{0}) \rangle$ incomparable to $\langle \mathbf{u}|_r \hat{0}\mathbf{0}, \mathbf{v}|_r \hat{0}\mathbf{0} \rangle = \langle \mathbf{u}', \mathbf{v}' \rangle$.

We may now suppose that there is a largest n such that $\langle \mathbf{u}|_n, \mathbf{v}|_n \rangle = \langle \mathbf{u}'|_n, \mathbf{v}'|_n \rangle$ for some $\langle \mathbf{u}, \mathbf{v} \rangle \in M$ and that $n < k$. So, $\langle \pi_n \mathbf{u}', \pi_n \mathbf{v}' \rangle$ is incomparable to some element of $D(\mathbf{u}|_n, \mathbf{v}|_n)$, by the maximality of $D(\mathbf{u}|_n, \mathbf{v}|_n)$ in $\pi_n[C] \times \pi_n[C]$.

Thus M is maximal in $C \times C$. □

DEFINITION 3.2. A sequence of intervals $\{I_\alpha : \alpha < \kappa\}$ is called *doubly decreasing* if for each $\beta > \alpha$ there are non-endpoints $x, y \in I_\alpha$ so that $I_\beta \subseteq (x, y) \subseteq I_\alpha$.

As a bit of shorthand in what follows, when we refer to a chain-indexed sequence of subsets of an ordered set X (for instance, an ω-sequence of convex sets $\{I_n : n < \omega\}$), we are indicating that these sets are ordered as subsets of X (so in our example, $I_0 < I_1 < I_2 \ldots$). Also, saying that $\{I_n : n < \omega\}$ is cofinal or coinitial in X means just that $\bigcup I_n$ is a cofinal or cointial subset of X.

LEMMA 3.3. *Let A contain a union of blocks of the form*

$$\Big(\bigcup I'_n \times J'_n\Big) \cup \Big(\bigcup I_n \times J_n\Big),$$

where $I_0 < I'_0$, $\{I_n\}$ and $\{I'_n\}$ are both doubly decreasing in C, $\{J_n\}$ is an ω^*-sequence of convex sets coinitial in $\widehat{C}$, $\{J'_n\}$ is an ω-sequence of convex sets cofinal in $\widehat{C}$, and $J_0 = J'_0$. Then A contains a maximal chain.

Proof (see Figure 1). We build, in a piecewise fashion, the chain illustrated in Figure 1.

Since I_n is doubly decreasing, let $\{x_n : n < \omega\}$ be a strictly descending sequence in C for which $x_n \in I_n$ and $\{x_n\} > I_{n+1}$ ($n < \omega$). Using the density of gaps in C and the sequence just chosen, select a sequence of gaps $\{\langle A_n, B_n \rangle : n < \omega\}$ so that $x_n \in B_n$ and $x_{n+1} \in A_n$ (for each $n < \omega$). Now define an ω^*-sequence of convex sets K_n by:

$$K_0 = [x_0, \rightarrow) \cap B_0,$$
$$K_n = A_{n-1} \cap B_n.$$

Then $\bigcup K_n$ is convex and has a maximum element x_0. Dually, select an ω-sequence of convex sets without endpoints $\{K'_n\}$ (except for K'_0 which has a minimum element x'_0) with the end result that $\bigcup K'_n$ is convex and for each $n < \omega$, $K'_n \subseteq I'_n$, $K'_n \cap I'_{n+1} \neq \emptyset$, and $K'_n < I'_{n+2}$. Now we must adjust in the middle. Select $y_0 < y'_0$ in $J_0 = J'_0$. Then take two convex sets $L \cup L' = [y_0, y'_0]$ that determine a gap in $[y_0, y'_0]$. Selecting $y_n \in J_n$ and $y'_n \in J'_n$ for each $n < \omega$ we form the maximal chain

$$\Big(\bigcup K_n \times \{y_n\}\Big) \cup (\{x_0\} \times L) \cup (\{x'_0\} \times L') \cup \Big(\bigcup K'_n \times \{y'_n\}\Big). \qquad \square$$

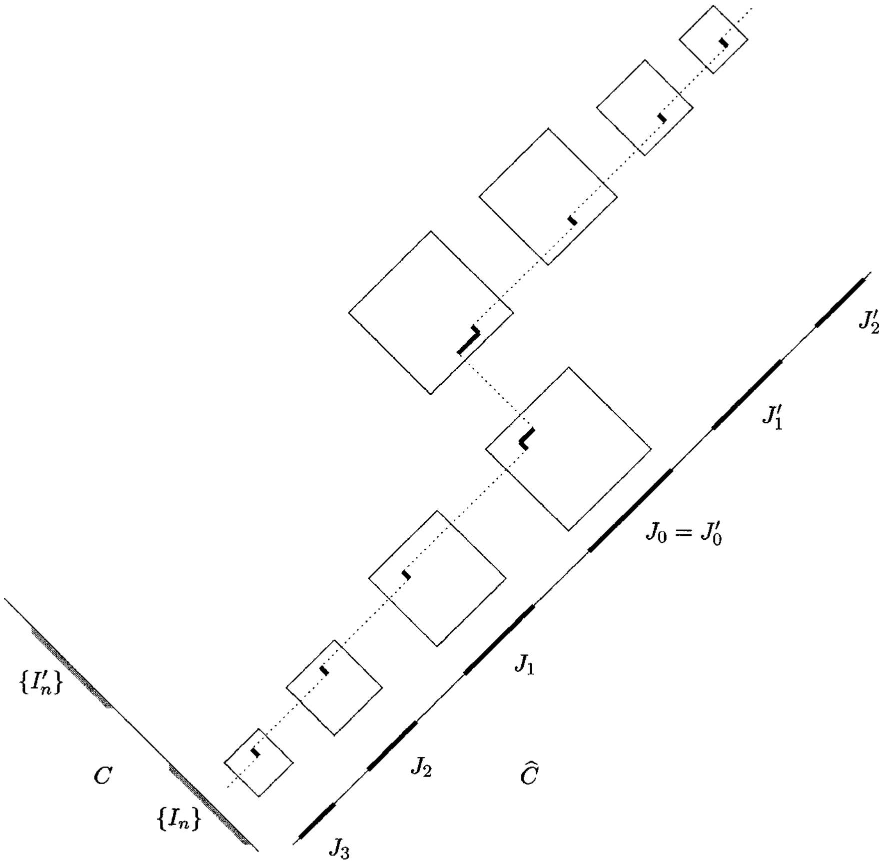

Figure 1. A maximal chain in $(\bigcup I_n' \times J_n') \cup (\bigcup I_n \times J_n)$.

4. Expandable Chains

DEFINITION 4.1. A chain $G \subseteq P$ is called *expandable* if there is a family of convex sets $\{X_g : g \in G\}$ satisfying:

$$g \in X_g,$$
$$g > g' \Rightarrow X_g > X_{g'},$$
$$p \in P \setminus \bigcup_{g \in G} X_g \Rightarrow p \text{ is incomparable to some } g \in G.$$

The import of the expandable chain idea is captured by the following observation: if for each $g \in G$ we have a chain C_g containing g and maximal in X_g, then $\bigcup C_g$ is a maximal chain.

The next lemma uses the previous observation and some special properties of $C \times C$ to allow us to construct a maximal chain within the given set A.

LEMMA 4.2. *Let $G \subseteq A \subseteq C \times \widehat{C}$ be an expandable chain for which X_g is isomorphic to $C \times C$ for each $g \in G$. Then, if A is block-dense in $\bigcup X_g$, A contains a chain extending G that is maximal in $C \times \widehat{C}$.*

Proof. Extend each element g of the expandable chain to a piece of a maximal chain in X_g via Lemma 3.1. □

Now we are ready to construct several types of maximal chains in $C \times \widehat{C}$ using expandable chains and Lemma 4.2. By way of introduction, consider the following lemma.

LEMMA 4.3. *Let A be block-dense in the block $I \times \widehat{C} \subseteq C \times \widehat{C}$. Then A contains a chain maximal in $C \times \widehat{C}$.*

Proof (see Figure 2). First select $\mathbf{g}$ so that $\mathbf{gz} \in I$ for each $\mathbf{z} \in \widehat{C}$. Such a prefix $\mathbf{g}$ can be found by considering any two distinct elements $\langle\langle\alpha_0, p_0\rangle, \langle\alpha_1, p_1\rangle, \ldots\rangle < \langle\langle\beta_0, q_0\rangle, \langle\beta_1, q_1\rangle, \ldots\rangle$ in I that differ first for, say, $\langle\alpha_n, p_n\rangle \neq \langle\beta_n, q_n\rangle$. Let $\langle\gamma, r\rangle$ lie between $\langle\alpha_n, p_n\rangle$ and $\langle\beta_n, q_n\rangle$ and set $\mathbf{g} = \langle\langle\alpha_0, p_0\rangle, \langle\alpha_1, p_1\rangle, \ldots, \langle\gamma, r\rangle, 0\rangle$. The concatenation $\mathbf{gz}$ is an element of C since the last entry of $\mathbf{g}$ belongs to $\omega_1^* + \omega_1$; for example, recall that if $\mathbf{z} = \langle r_{-1}, \langle\gamma_0, r_0\rangle, \langle\gamma_1, r_1\rangle, \ldots\rangle$ then we have

$$\mathbf{gz} = \langle\langle\alpha_0, p_0\rangle, \langle\alpha_1, p_1\rangle, \ldots, \langle\gamma, r\rangle, \langle 0, r_{-1}\rangle, \langle\gamma_0, r_0\rangle, \langle\gamma_1, r_1\rangle, \ldots\rangle.$$

For all $q \in (-1, 1)$ consider the block isomorphic to $C \times C$, $X_{g_q} = \{\langle\mathbf{g}q\mathbf{x}, q\mathbf{y}\rangle : \mathbf{x}, \mathbf{y} \in C\}$. By the block-density of A in $I \times \widehat{C}$, we may define $\mathbf{x}(q)$ and $\mathbf{y}(q)$ so that

$$G = \{g_q = \langle\mathbf{g}q\mathbf{x}(q), q\mathbf{y}(q)\rangle : q \in (-1, 1)\}$$

is contained in A. To see that G is an expandable chain, we must observe every element outside of $\bigcup X_{g_q}$ is incomparable to some element of G, so suppose that $p = \langle\mathbf{p}_0, \mathbf{p}_1\rangle$ is comparable to every element of G. If $\mathbf{p}_0 \neq \mathbf{g}q\mathbf{p}_0'$, p must dominate or be dominated by every element of G, and this is not possible. Therefore, p is of the form $\langle\mathbf{g}q_0\mathbf{p}_0', q_1\mathbf{p}_1'\rangle$. If $q_0 \neq q_1$, then p is incomparable to the element of

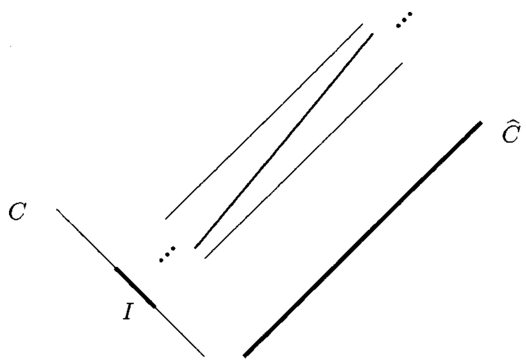

Figure 2. A maximal chain in $I \times \widehat{C}$.

G $\langle \mathbf{g}q\mathbf{x}(q), q\mathbf{y}(q)\rangle$ where q is any element of $(-1, 1)$ between q_0 and q_1. Thus $q_0 = q_1$; but then $p \in X_{g_q}$ (where $q = q_0$). Hence, G is an expandable chain.

Since each of the blocks X_{g_q} is isomorphic to $C \times C$, an application of Lemma 4.2 completes the proof. □

The proofs of the remaining lemmas in this section are quite similar to that of the lemma just presented, and differ only in the particular expandable chain constructed therein. Because of this similarity, we suppress those details that are easily filled in by consideration of the proof of Lemma 4.3.

LEMMA 4.4. *Let A be block-dense in a block $I_0 \times J'$, where J' is a final segment of $\widehat{C}$, and let A contain the union of blocks $\bigcup I_n \times J_n$, where $\{J_n\}$ is an ω^*-sequence of convex sets coinitial in $\widehat{C}$, and the sequence $\{I_n\}$ is doubly decreasing. Then A contains a chain maximal in $C \times \widehat{C}$.*

Proof (see Figure 3). In general, the method is to combine the final segment of a chain from the previous lemma with an initial segment of the chain obtained from Lemma 3.3.

First select $\mathbf{g}$ so that $\mathbf{gz} \in I_0$ and $\{\mathbf{gz}\} > I_1$ for each $\mathbf{z} \in \widehat{C}$. The concatenation $\mathbf{gz}$ is an element of C provided that the last entry of $\mathbf{g}$ belongs to $\omega_1^* + \omega_1$. Let $q_0 \in (-1, 1)$ be large enough that $q_0\mathbf{x} \in J'$ for all $\mathbf{x} \in C$. For all $q > q_0$ consider the set $X_{g_q} = \{\langle \mathbf{g}q\mathbf{x}, q\mathbf{y}\rangle : \mathbf{x}, \mathbf{y} \in C\}$; this is isomorphic to $C \times C$ and a block of $I_0 \times J'$. Then, by the block-density of A in $I_0 \times J'$, we may define $\mathbf{x}(q)$ and $\mathbf{y}(q)$ so that

$$G_0 = \{g_q = \langle \mathbf{g}q\mathbf{x}(q), q\mathbf{y}(q)\rangle : q > q_0\}$$

is contained in A. Set $K' = \{\mathbf{g}q\mathbf{x} : q > q_0, \mathbf{x} \in C\}$. Observe that K' has no minimum element and the set of elements below in C, $\{\mathbf{x} \in C : K' > \{\mathbf{x}\}\}$, has no maximum element.

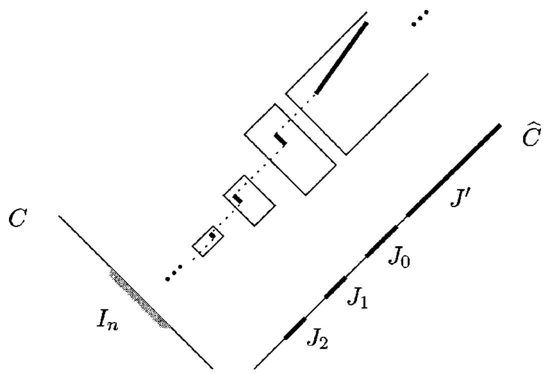

Figure 3. A maximal chain in $(\bigcup I_n \times J_n) \cup (I_0 \times J')$.

Now, just as in Lemma 3.3, select the sequence of gaps $\{\langle A_n, B_n\rangle : n < \omega\}$ based on the doubly decreasing sequence $\{I_n\}$. Set

$$K_0 = B_0 \setminus K',$$
$$K_n = A_{n-1} \cap B_n \quad (0 < n < \omega).$$

It is possible that K_0 is empty in the case that K' and I_1 define a gap, but this does not damage the construction. Next, select $y_n \in J_n$ for each $n < \omega$ and set

$$G_1 = \bigcup K_n \times \{y_n\}.$$

Associate each $g \in G_1$ with its singleton convex set $X_g = \{g\}$. Then $G_0 \cup G_1$ is an expandable chain and we are done by Lemma 4.2. □

As a point of notation in the following lemmas, we use α^* to denote elements of $\omega_1^* + \omega_1$ that are less than 0 (the element obtained by identifying the minimum of ω_1 and the maximum of ω_1^*). For instance, 1^* is the predecessor of 0.

LEMMA 4.5. *Let A be block-dense in a union of blocks of the form $\bigcup I_\alpha \times J$ where J is an initial segment of $\widehat{C}$ and $\{I_\alpha\}$ is an ω_1-sequence of convex sets cofinal in C. Then A contains a maximal chain.*

Proof (see Figure 4). First select $q \in (-1, 1)$ so that $q\mathbf{z} \in J$ for every $\mathbf{z} \in C$. Choose $\mathbf{g}_\alpha$ so that $\mathbf{g}_\alpha\mathbf{z} \in I_\alpha$ for every $\mathbf{z} \in C$. Then define

$$G' = \bigcup_{\alpha<\omega_1} \{\langle \mathbf{g}_\alpha 0 r\mathbf{x}, q\alpha r\mathbf{y}\rangle : r \in (-1, 1)\},$$

choosing $\mathbf{x} = \mathbf{x}(g_\alpha 0r)$ and $\mathbf{y} = \mathbf{y}(q\alpha r)$ so that $G' \subseteq A$.

Let $\psi : (-1, 1) \to (-1, q)$ be an order isomorphism. Then take

$$G = \{\langle \mathbf{g}_0 1^* r\mathbf{x}, \psi(r)\mathbf{y}\rangle : r \in (-1, 1)\},$$

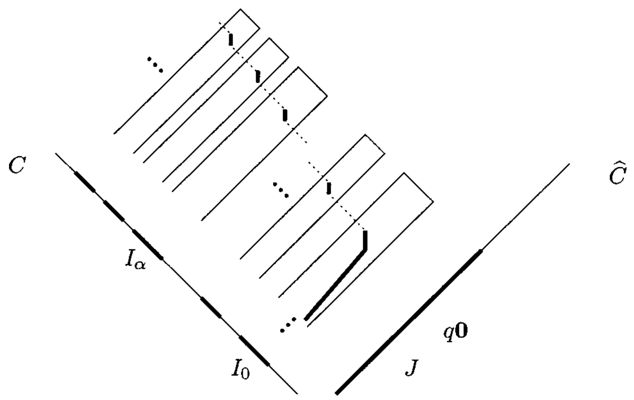

Figure 4. A maximal chain in $\bigcup I_\alpha \times J$.

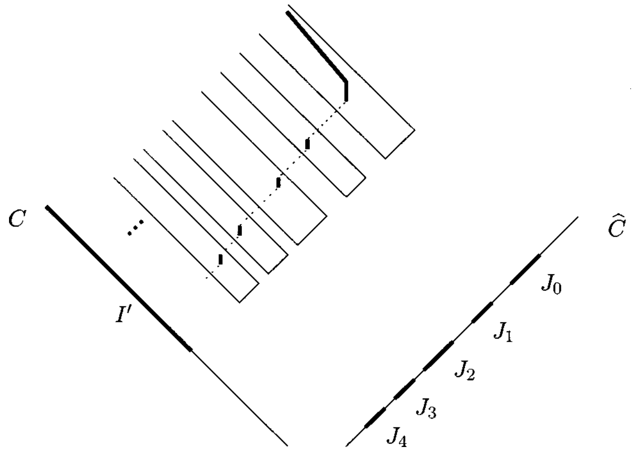

Figure 5. A maximal chain in $I' \times \bigcup J_n$.

choosing $\mathbf{x} = \mathbf{x}(r)$ and $\mathbf{y} = \mathbf{y}(r)$ so that $G \subseteq A$.

The union $G \cup G'$ is an expandable chain provided we take the associated convex sets determined by simply allowing $\mathbf{x}$ and $\mathbf{y}$ in the above chain to range over all possible values of C. An application of Lemma 4.2 then completes the proof. □

LEMMA 4.6. *Let A be block-dense in a union of blocks of the form $I' \times \bigcup J_n$ where I' is a final segment of C and $\{J_n\}$ is an ω^*-sequence of convex sets coinitial in $\widehat{C}$. Then A contains a maximal chain.*

Proof (see Figure 5). For each $n < \omega$ choose $\mathbf{g}_n \in \widehat{C}$ so that $\mathbf{g}_n\mathbf{x} \in J_n$ for every $\mathbf{x} \in C$. Choose $u_0 \in C_0$ so that $u_0\mathbf{z} \in I'$ for every $\mathbf{z} \in C$. Now define

$$G = \bigcup_{n<\omega} \{\langle u_0 n^* r \mathbf{x}(u_0 n^* r), \mathbf{g}_n u_0 n^* r \mathbf{y}(u_0 n^* r)\rangle : r \in (-1, 1)\} \cup$$
$$\{\langle u_0 v \mathbf{x}(u_0 v), \mathbf{g}_0 u_0 v \mathbf{y}(u_0 v)\rangle : v = \langle \alpha, x\rangle \in C_0, \alpha > 0\} \cup$$
$$\{\langle v \mathbf{x}(v), \mathbf{g}_0 v \mathbf{y}(v)\rangle : v \in C_0, v > u_0\},$$

where we define $\langle \mathbf{x}(\mathbf{b}), \mathbf{y}(\mathbf{b})\rangle \in C \times C$ so that $\langle \mathbf{b}\mathbf{x}(\mathbf{b}), \mathbf{g}\mathbf{b}\mathbf{y}(\mathbf{b})\rangle \in A$.

As usual, we observe that G is an expandable chain by taking the convex sets determined by simply allowing $\mathbf{x}$ and $\mathbf{y}$ in the above chain to range over all possible values of C. An application of Lemma 4.2 then completes the proof. □

5. The Ramsey-style Part of the Proof

We are ready to complete the proof of the main theorem.

Suppose on the contrary that $C \times \widehat{C}$ is colored with two colors (say red and blue) in such a way that no maximal chain is monochromatic. Let us first consider conditions that allow us to apply Lemma 3.3. Suppose that there exist a final segment K' of C and an initial segment L of $\widehat{C}$ such that for every non-trivial convex set $I_0 \subseteq K'$, and every non-trivial convex set $J_0 \subseteq L$, the blocks

(a) $I_0 \times J$ (for initial segments $J \subseteq \widehat{C}$),
(b) $I_0 \times J'$ (for final segments $J' \subseteq \widehat{C}$),
(c) $I' \times J_0$ (for final segments $I' \subseteq C$),

each contain a blue block. We argue that these suppositions ensure there is a region as described in Lemma 3.3 colored entirely blue. Following the notation of that lemma, use (a) to obtain a blue block $I_0 \times J_0$ where $J_0 \subseteq L$. Apply (c) to the direct product of the final segment of C generated by I_0 with the convex set J_0. We obtain a blue block $I_0' \times J_0'$ where $I_0 < I_0'$ and, we may assume, $J_0 = J_0'$. Iteratively apply (a) to obtain a sequence of blue blocks $I_n \times J_n$ with I_n doubly decreasing and the ω^*-sequence $\{J_n\}$ coinitial in $\widehat{C}$. Iteratively apply (b) to define a sequence of blue blocks $I_n' \times J_n'$ with I_n' doubly decreasing and the ω-sequence $\{J_n'\}$ cofinal in $\widehat{C}$. Then $(\bigcup I_n' \times J_n') \cup (\bigcup I_n \times J_n)$ is the claimed region colored blue. Hence there is a blue maximal chain. This is a contradiction, so red must be block-dense in some of the blocks of types (a), (b), and (c).

Suppose that red is block dense in a family of blocks $\{I_\alpha \times J_\alpha\}$ where $\{I_\alpha\}$ is an ω_1-sequence of convex sets cofinal in C and each J_α is an initial segment of $\widehat{C}$. By the countable coinitiality of $\widehat{C}$, we may assume that $J = \bigcap J_{\alpha_\beta}$ is nonempty for an ω_1-subsequence $\{I_{\alpha_\beta}\}$ cofinal in C. Thus, $\{I_{\alpha_\beta} \times J\}$ contains a red maximal chain by Lemma 4.5. Hence, there exists a final segment of $K' \subseteq C$ such that for all initial segments $J \subseteq \widehat{C}$ and for all convex $I_0 \subseteq K'$, $I_0 \times J$ contains a blue block.

Suppose that red is block-dense in a family of blocks $\{I_n' \times J_n\}$ where each I_n' is a final segment of C and $\{J_n\}$ is an ω^*-sequence of convex sets coinitial in $\widehat{C}$. Let $I' = \bigcap I_n'$; I' is a nonempty final segment by the uncountable cofinality of C. By Lemma 4.6, the family $\{I' \times J_n\}$ contains a red maximal chain. Hence, there exists an initial segment $L \subseteq \widehat{C}$ such that for all final segments $I' \subseteq C$ and for all convex sets $J_0 \subseteq L$, $I' \times J_0$ contains a blue block.

Now, the only obstacle to constructing a blue region like that needed by Lemma 3.3 is a complete absence of blue blocks (thereby making red block-dense) in blocks of the form $I_0 \times J'$ where $I_0 \subseteq K'$. This must happen for each I_0 selected from some ω_1-sequence of convex sets cofinal in C, as otherwise the absence of blue blocks would be bounded and we would simply invoke the construction of Lemma 3.3 beyond the bound. Consider such an I_0. If every block $I_1 \times J$ contains a red block for $I_1 \subseteq I_0$ and J an initial segment of $\widehat{C}$, then we may construct a region satisfying Lemma 4.4 to obtain a red maximal chain. Thus, for any I_1 selected from some ω_1-sequence of convex sets cofinal in C, we have blocks $I_1 \times J$ that contain no red blocks. But by Lemma 4.5 this is all that we need for a blue maximal chain. The proof is complete – there can be no 2-coloring that 2-colors every maximal chain.

6. Concluding Remarks and Open Problems

Since every countable ordered set can be colored with two colors so that no maximal chain is monochromatic and our example in Theorem 2.1 is of absolute size $\aleph_1$, there is little room for questions about minimal examples for 2-coloring maximal chains, unless one asks about specific order-theoretic conditions. For instance, our construction is minimal with respect to dimension and cardinality, but there may be other measures that are of interest.

We wish to note that the chain C is not the only suitable starting point for the example constructed in this paper. Robert Woodrow pointed our attention to the following chain. Consider ω_1-sequences on $\{-1, 0, 1\}$ with only finitely many nonzero terms, ordered lexicographically ($-1 < 0 < 1$). Call this chain C'. C' has the property that no element is the limit of a countable sequence. Using $C' \times ((\omega_1^* + \omega_1) \cdot C')$, we can prove that all of the lemmas from Lemma 3.3 onward hold, but the maximal chains must be constructed differently (a little more effort is required). We chose to use C and $\widehat{C}$ because we believed that the proofs of the lemmas would be clearer and more explicit through the use of expandable chains.

Perhaps the most dramatic open question for coloring maximal chains in ordered sets of size $\aleph_1$ is the following.

QUESTION. Is there an ordered set of cardinality $\aleph_1$ that contains monochromatic maximal chains under any countable coloring?

As well, many questions remain with regard to coloring maximal antichains, as even for countable ordered sets the situation is not resolved [1].

References

1. Duffus, D. and Goddard, T. (1996) Products of chains with monochromatic maximal chains and antichains, *Order* **13**, 101–117.
2. Duffus, D., Rodl, V., Sauer, N., and Woodrow, R.E. (1992) Coloring ordered sets to avoid monochromatic maximal chains, *Canad. J. Math.* **44**(1), 91–103.
3. Jech, T. (1978) *Set Theory*, Academic Press, San Diego.
4. Milner, E. C. and Pouzet, M. (1982) On the cofinality of partially ordered sets, in I. Rival (ed.), *Ordered Sets*, D. Reidel Publishing Co., Dordrecht, pp. 279–298.
5. Rosenstein, J. G. (1982) *Linear Orderings*, Academic Press, San Diego.