ELSEVIER

Discrete Mathematics 201 (1999) 89–99

DISCRETE MATHEMATICS

An inequality for the sizes of prime filters of finite distributive lattices

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Abstract

Let L be a finite distributive lattice, and let $J(L)$ denote the set of all join-irreducible elements of L. Set $j(L)=|J(L)|$. For each $a\in J(L)$, let $u(a)$ denote the number of elements in the prime filter $\{x\in L\colon x\geqslant a\}$. Our main theorem is

Theorem 1. *For any finite distributive lattice L,*

$$\sum_{a\in J(L)} 4^{u(a)} \geqslant j(L)4^{|L|/2}.$$

The base 4 here can most likely be replaced by a smaller number, but it cannot be replaced by any number strictly between 1 and 1.6159. We also make a few other observations about prime filters and the numbers $u(a)$, $a\in J(L)$, among which is: *every finite distributive non-Boolean lattice L contains a prime filter of size at most* $|L|/3$ *or at least* $2|L|/3$.

The above inequality is certainly *not* true for all finite lattices. However, we give another inequality, equivalent to the above for distributive lattices, which might hold for all finite lattices. If so, this would give an immediate proof of a conjecture known as *Frankl's conjecture.*

AMS classifications: 06D05; 05A05

Keywords: Frankl's conjecture; Union-closed; Distributive lattice; Prime filter; Mean; Inequality

1. Introduction

Let L be a finite distributive lattice, and let $J(L)$ and $M(L)$ denote the sets of all join-irreducible and all meet-irreducible elements of L, respectively. It is well-known that $J(L)$ and $M(L)$ have the same number of elements (in fact they are isomorphic as

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[1] Research done while the author was visiting Emory University.

0012-365X/99/$ – see front matter
PII: S0012-365X(98)00313-6

partially ordered sets), so we may put $j(L)=|J(L)|=|M(L)|$. For each $a \in J(L)$, let $u(a)$ denote the number of elements in the prime filter $\{x \in L\colon x \geqslant a\}$. Our interest in this paper is to study these numbers $u(a)$, in particular to investigate them in relation to the size $|L|$ of L.

Our main theorem is

Theorem 1. *For any finite distributive lattice L,*

$$\sum_{a \in J(L)} 4^{u(a)} \geqslant j(L) 4^{|L|/2}. \tag{1}$$

The base 4 here can most likely be replaced by a smaller number, but we shall see in the last section that it cannot be replaced by any number strictly between 1 and 1.6159.

We also make a few other observations about prime filters and the numbers $u(a)$, $a \in J(L)$, among which is: *every finite distributive non-Boolean lattice L contains a prime filter of size at most* $|L|/3$ *or at least* $2|L|/3$. (See Section 2, Theorem 2.)

Inequality (1) is certainly *not* true for all finite lattices: the modular nondistributive lattice M_3, with 5 elements $\{0, a, b, c, 1\}$ where 0 and 1 are the usual bounds and $\{a, b, c\}$ is an antichain, is a quick counterexample, as it is easy to check that (1) fails, even if the base 4 is replaced by any fixed $c > 1$. However, inequality (2) below is closely related to (1) and might hold for all finite lattices, and in the last section we give two results supporting this possibility. (See the problems and Theorems 3 and 4 in Section 4.)

2. Background

This paper began with our attempt to wrestle with an old problem which has become known as *Frankl's conjecture*, i.e.,

For every (nonempty) finite union-closed family of finite sets, there is an element contained in at least half of them.

For references see [1,6–8,10,11,13–16].

There is an equivalent statement in terms of intersection-closed systems, and two equivalent lattice-theoretic formulations (e.g., [5,11]). We shall focus on this one:

every finite lattice L contains a join-irreducible a such that $u(a) \leqslant |L|/2$.

We have not solved this problem! And in fact, for distributive lattices L, it is known to be true and is easy to prove. Just let a be any *maximal* element of $J(L)$; the set $L - \{x \in L\colon x \geqslant a\}$ has a largest element a' (which is meet-irreducible), and the map $x \to x \wedge a'$ is one-to-one from $\{x \in L\colon x \geqslant a\}$ into $L - \{x \in L\colon x \geqslant a\}$, whence we are done. Similar arguments establish the same fact for the classes of modular, geometric and lower semimodular lattices, among others [11].

Theorem 1 grew out of attempts to verify Frankl's conjecture by an averaging argument, although at first glance it might appear to establish just the opposite! For (1) says that the average size of the quantities $4^{u(a)}$ over all $a \in J(L)$ is at least $4^{|L|/2}$, so some $4^{u(a)} \geqslant 4^{|L|/2}$, which means $u(a) \geqslant |L|/2$, and the inequality is going the wrong way.

Nevertheless, Theorem 1 *does imply* Frankl's conjecture for finite distributive lattices. To see this, begin with the dual of (1):

$$\sum_{a' \in M(L)} 4^{d(a')} \geqslant j(L) 4^{|L|/2},$$

where $d(a') = |\{x \in L\colon x \leqslant a'\}|$ is the size of the prime ideal determined by $a' \in M(L)$. This is equivalent to (1) when quantified over all finite distributive lattices, because the dual of a distributive lattice is another distributive lattice in which the join-irreducibles and meet-irreducibles have switched places. Then divide both sides by $4^{|L|}$, getting

$$\sum_{a' \in M(L)} \frac{1}{4^{|L|-d(a')}} \geqslant \frac{j(L)}{4^{|L|/2}}.$$

Since, for each $a' \in M(L)$, $|L| - d(a')$ is the size of the prime filter generated by the corresponding join-irreducible a, this can be rewritten as

$$\sum_{a \in J(L)} \frac{1}{4^{u(a)}} \geqslant \frac{j(L)}{4^{|L|/2}}. \tag{2}$$

As above, this last inequality implies that the average size of the quantities $1/4^{u(a)}$ over all $a \in J(L)$ is at least $1/4^{|L|/2}$, which this time implies that some $u(a) \leqslant |L|/2$. Therefore a will do as the desired join-irreducible in Frankl's conjecture.

We can see that both (1) and (2) are comparing $|L|/2$ with a certain *mean* of the numbers $\{u(a)\colon a \in J(L)\}$. Given $c > 0$ with $c \neq 1$, and real numbers $x_1, \dots, x_n$, define

$$M_c(x_1, \dots, x_n) = \log_c \left(\frac{1}{n} \sum_{i=1}^{n} c^{x_i} \right).$$

This is a particular kind of *quasiarithmetic mean*, as defined in Example 1 on pp. 218 of [3]. (See also Eq. (3) and Section 4 of [4], or pp. 231–232 of [2] for a more general definition.) Using this notation (1) can be written

$$M_4(\{u(a)\colon a \in J(L)\}) \geqslant |L|/2.$$

Note that the usual arithmetic mean of the $u(a)$'s does not have this property, and in fact might be arbitrarily small in relation to $|L|$. For example, let t be a fixed positive integer, and let L be the distributive lattice $\mathbf{2}^n \oplus C$, where n is sufficiently large, C is a chain of 2^{n-t} elements, $\mathbf{2}^n$ denotes the Boolean lattice with n atoms, and $X \oplus Y$

denotes the *linear sum* of the lattices X and Y. Then the average of the prime filter sizes in L, divided by the size of L, is

$$\frac{1+2+3+\cdots+2^{n-t}+n(2^{n-1}+2^{n-t})}{(n+2^{n-t})(2^n+2^{n-t})}$$

which simplifies to

$$\frac{2^{n-t}+2n(2^{t-1}+1)+1}{2^{n+1}(1+2^{-t})+n2^{t+1}+2n},$$

and this can be made less than any preselected $\varepsilon>0$ as $n\to\infty$, by choosing t large enough. (See Section 4 for some remarks on other means.)

Whereas (1) gives a mean of the $u(a)$'s which is at least $|L|/2$, (2) does the opposite, because (2) can be rewritten as

$$M_{1/4}(\{u(a)\colon\ a\in J(L)\})\leqslant|L|/2.$$

These two inequalities can be considered as strengthenings of the facts that every finite distributive lattice contains some prime filter of size at least half the lattice, and some prime filter of size at most half the lattice. They also agree with the known property of the M_n's that $M_c\leqslant M_d$ whenever $0<c<d$ and $c\neq 1$, $d\neq 1$ (see Remark (8), p. 227 of [3], or Section 4 of [4]). Thus it will follow from Theorem 1 that, for all $c\geqslant 4$ and for all finite distributive lattices L,

$$M_{1/c}(\{u(a)\colon\ a\in J(L)\})\leqslant|L|/2\leqslant M_c(\{u(a)\colon\ a\in J(L)\}),$$

or in other words,

$$\sum_{a\in J(L)}\frac{1}{c^{u(a)}}\geqslant\frac{j(L)}{c^{|L|/2}}\quad\text{and}\quad\sum_{a\in J(L)}c^{u(a)}\geqslant j(L)c^{|L|/2}.$$

It is interesting to note that while M_c as defined above does not exist for $c=1$, the limit of M_c as c approaches 1 does, and is just the usual arithmetic mean. Thus, as we have seen above, $\lim_{c\to 1}M_c$ is not comparable to $|L|/2$ over the class of all finite distributive lattices. We will see in Section 4 that M_c and $M_{1/c}$, where c is any number in the interval $(1, 1.6159)$, are not comparable to $|L|/2$ over this class either.

To describe some other known results about the numbers $u(a)$, in particular as they relate to $|L|$, we introduce one further bit of notation. For L a finite distributive lattice and $a\in J(L)$, let $\upsilon(a)=u(a)/|L|$. So $\upsilon(a)$ is between 0 and 1 for every $a\in J(L)$, and the above example shows that we could have the average of the $\upsilon(a)$'s arbitrarily close to 0. Likewise, by dualizing this example we obtain a distributive lattice where the average is arbitrarily close to 1.

Not much is known about the numbers $\upsilon(a)$. A theorem of Linial and Saks [9], when written in the language of distributive lattices, says that

every finite distributive lattice L contains a prime filter of size between $\upsilon_0|L|$ and $(1-\upsilon_0)|L|$, where $\upsilon_0=(3-\log_2 5)/4\approx 0.17$.

Thus every finite distributive lattice L contains $a\in J(L)$ such that $0.17<\upsilon(a)<0.83$. As reported in [9], an unpublished construction by James Shearer produces a distributive lattice in which no $\upsilon(a)$ lies between 0.197 and 0.803. This is an improvement over the bounds in an earlier paper of the second author [12]. To our knowledge this is where this problem stands today, with the gap between 0.17 and 0.197 still waiting to be closed, the best value of υ_0 in the Linial–Saks result still not known exactly.

The above result tells to what extent every distributive lattice will contain a prime filter of 'about' half the elements. What about the other extreme — when will a distributive lattice contain either a 'small' or a 'large' prime filter? Of course, for every Boolean lattice the prime filters are all exactly half the lattice. But in all other cases the situation is quite different, as the following easy result demonstrates.

Theorem 2. *Every finite distributive non-Boolean lattice L contains either a prime filter of size at most $|L|/3$ or a prime filter of size at least $2|L|/3$.*

Proof. Since L is not Boolean, its poset of join-irreducibles is not an antichain, so we can choose a minimal and b maximal in $J(L)$ so that $a<b$. Then each order ideal I of $J(L)$ that contains b also contains a and, in addition, $I-\{b\}$ is another order ideal of $J(L)$ containing a. Moreover, it is clear that if I_1 and I_2 are distinct order ideals containing b then $I_1-\{b\}$ and $I_2-\{b\}$ are also distinct. Thus the number of order ideals of $J(L)$ which contain a is at least twice the number that contain b. The set of order ideals of $J(L)$ which contain a corresponds to the prime filter in L generated by a, and similarly for b, so the above fact translates in L as: the prime filter $[a,1]$ is at least twice as large as the prime filter $[b,1]$. Now either $[b,1]$ is of size at most $|L|/3$, or $[a,1]$ is of size at least $2|L|/3$. □

In other words, no non-Boolean distributive lattice can have all its $\upsilon(a)$'s strictly between 1/3 and 2/3. This result is best possible in that the distributive lattice **3** (the 3-element chain) contains prime filters of sizes 1 and 2 only. (Or we could take the lattice $\mathbf{2}^n\times\mathbf{3}$ where $\mathbf{2}^n$ is any Boolean lattice.)

3. Proof of Theorem 1

Given a distributive lattice L, let the minimals of $J(L)$ be $a_1,\ldots,a_t$. For each a_i, let n_i be the number of elements of $J(L)$ which are greater than a_i. Then

$$t+\sum_{i=1}^{t} n_i\geqslant j(L).$$

The number of order ideals of $J(L)$ which contain a_i is just $u(a_i)$. Note that for each of the $|L| - u(a_i)$ order ideals I of $J(L)$ which do not contain a_i, $I \cup \{a_i\}$ is an order ideal of $J(L)$ containing a_i, and $I \to I \cup \{a_i\}$ is a one-to-one map into the set of order ideals containing a_i. Moreover, if we choose one such order ideal $I \cup \{a_i\}$, then for every $a \in J(L)$ satisfying $a > a_i$ we can form $I \cup \{x \in J(L)\colon x \leqslant a\}$ to get yet another order ideal of $J(L)$ containing a_i. Since there are n_i such order ideals, we have $u(a_i) \geqslant |L| - u(a_i) + n_i$, or

$$u(a_i) \geqslant \frac{|L|}{2} + \frac{n_i}{2}.$$

By ignoring all terms $4^{u(a)}$ in (1) where $a \in J(L)$ is not minimal, it would be enough to prove

$$\sum_{i=1}^{t} 4^{u(a_i)} \geqslant j(L) 4^{|L|/2},$$

and the above shows that it is enough to prove

$$\sum_{i=1}^{t} 2^{n_i} \geqslant j(L).$$

From $t + \sum_{i=1}^{t} n_i \geqslant j(L)$ it is enough to prove

$$\sum_{i=1}^{t} (2^{n_i} - 1 - n_i) \geqslant 0,$$

which holds for all integral values of n_i since

$$2^x \geqslant 1 + x,$$

is true for all integers x. □

4. On lattices in general

Problem. Is there a constant $c > 1$ such that $M_{1/c}(\{u(a)\colon a \in J(L)\}) \leqslant |L|/2$, that is,

$$\sum_{a \in J(L)} \frac{1}{c^{u(a)}} \geqslant \frac{j(L)}{c^{|L|/2}} \tag{3}$$

holds for every finite lattice L?

If so, then Frankl's conjecture would follow as in the distributive case. In Section 2 we saw that the ordinary arithmetic mean does not yield an inequality from which we could deduce Frankl's conjecture. Indeed, it fails 'arbitrarily badly' even for distributive lattices of the form $C \oplus \mathbf{2}^n$, where C is an appropriately sized chain. The argument given in Section 2 shows that the average size of a prime filter of this lattice, in proportion to the size of the lattice, can be arbitrarily close to 1.

More general means such as weighted arithmetic means, or the rth power mean

$$M^{[r]}(\{u(a)\colon a\in J(L)\})=\left(\frac{1}{j(L)}\sum_{a\in J(L)}[u(a)]^r\right)^{1/r},$$

where $r\neq 0$, do not have the desired property for all finite distributive lattices either. (Indeed, the lattice $\mathbf{1}\oplus\mathbf{2}^n$ shows that $M^{[r]}$ cannot be used to deduce that some $u(a)\leqslant |L|/2$.) These failures led us to consider the inequality in (3).

Note that, unlike (1), (3) holds easily for every $c>1$ for the modular nondistributive lattice $L=M_3$, for the very good reason that all $u(a)=2$ while $|L|=5$, so all $u(a)$'s are less than $|L|/2$! (This is also the real reason why (1) fails when $L=M_3$, as mentioned in Section 1.) And we know from Section 2 that (3) holds for all $c\geqslant 4$ if L is distributive. As further (weak) evidence for an affirmative answer to this problem, we close by showing that two basic constructions preserve this inequality for finite lattices in general.

Theorem 3. *If* (3) *holds (for some* $c>1$*) for finite lattices* A *and* B, *then it holds for the direct product* $A\times B$.

Proof. It is well known that

$$J(A\times B)=\{(a,0_B)\colon a\in J(A)\}\cup\{(0_A,b)\colon b\in J(B)\}.$$

Note that $u((a,0_B))=u(a)\cdot|B|$ and $u((0_A,b))=u(b)\cdot|A|$ for all $a\in A$, $b\in B$. By assumption,

$$\sum_{a\in J(A)}\frac{1}{c^{u(a)}}\geqslant\frac{j(A)}{c^{|A|/2}},\qquad \sum_{b\in J(B)}\frac{1}{c^{u(b)}}\geqslant\frac{j(B)}{c^{|B|/2}}.$$

Thus, by the power mean inequality,

$$\left(\frac{1}{j(A)}\sum_{a\in J(A)}\frac{1}{(c^{u(a)})^{|B|}}\right)^{1/|B|}\geqslant\frac{1}{j(A)}\sum_{a\in J(A)}\frac{1}{c^{u(a)}}\geqslant\frac{1}{c^{|A|/2}},$$

$$\left(\frac{1}{j(B)}\sum_{b\in J(B)}\frac{1}{(c^{u(b)})^{|A|}}\right)^{1/|A|}\geqslant\frac{1}{j(B)}\sum_{b\in J(B)}\frac{1}{c^{u(b)}}\geqslant\frac{1}{c^{|B|/2}}.$$

Hence

$$\sum_{\mathbf{x}\in J(A\times B)}\frac{1}{c^{u(\mathbf{x})}}=\sum_{a\in J(A)}\frac{1}{c^{u(a)\cdot|B|}}+\sum_{b\in J(B)}\frac{1}{c^{u(b)\cdot|A|}}$$
$$\geqslant\frac{j(A)}{(c^{|A|/2})^{|B|}}+\frac{j(B)}{(c^{|B|/2})^{|A|}}=\frac{j(A)+j(B)}{c^{|A|\cdot|B|/2}}=\frac{j(A\times B)}{c^{|A\times B|/2}},$$

and the result follows. □

If A and B are lattices on disjoint vertex sets, let the *short linear sum* $A\oplus' B$ of A and B be the linear sum $A\oplus B$ with the elements 1_A and 0_B identified. In the statement of Theorem 4, $r\approx 1.271187623$ is the largest real root of the polynomial $3x^8-4x^7+1=0$.

Theorem 4. *If* (3) *holds for finite lattices A and B, where $c\geqslant r^2\approx 1.615917973$, then it holds for $A\oplus' B$.*

Proof. For clarity in this proof, we denote $|[x,1_A]|$ by $u_A(x)$ for $x\in J(A)$, and similarly define $u_B(x)$ for $x\in J(B)$, and reserve the notation $u(x)$ for elements of $J(A\oplus' B)$. We are given that

$$\sum_{a\in J(A)}\frac{1}{c^{u_A(a)}}\geqslant\frac{j(A)}{c^{|A|/2}},\qquad \sum_{b\in J(B)}\frac{1}{c^{u_B(b)}}\geqslant\frac{j(B)}{c^{|B|/2}}.$$

Clearly $J(A\oplus' B)=J(A)\cup J(B)$. Also, for $x\in J(A\oplus' B)$,

$$u(x)=\begin{cases}u_A(x)+|B|-1 & \text{if } x\in J(A),\\ u_B(x) & \text{if } x\in J(B).\end{cases}$$

Thus we want to prove that

$$\sum_{a\in J(A)}\frac{1}{c^{u_A(a)+|B|-1}}+\sum_{b\in J(B)}\frac{1}{c^{u_B(b)}}\geqslant\frac{j(A)+j(B)}{c^{(|A|+|B|-1)/2}}.$$

By the assumption it is enough to prove that

$$\frac{j(A)}{c^{|B|-1}c^{|A|/2}}+\frac{j(B)}{c^{|B|/2}}\geqslant\frac{(j(A)+j(B))c^{1/2}}{c^{|A|/2}c^{|B|/2}},$$

which simplifies to

$$j(B)c^{|B|/2}(c^{|A|/2}-\sqrt{c})\geqslant j(A)(c^{|B|/2}\sqrt{c}-c). \tag{4}$$

First we handle the special case $|A|=2$. Then $j(A)=1$, and (4) says

$$j(B)c^{|B|/2}(c-\sqrt{c})\geqslant c^{|B|/2}\sqrt{c}-c,$$

which can be written

$$c^{|B|/2}(j(B)(c-\sqrt{c})-\sqrt{c})+c\geqslant 0. \tag{5}$$

Case (i): $j(B)=1$. Then $|B|=2$ and (5) becomes $c-2\sqrt{c}+1\geqslant 0$, or $(\sqrt{c}-1)^2\geqslant 0$, which is true for all c.

Case (ii): $j(B)=2$. Then $|B|=3$ or 4. If $|B|=3$ then (5) becomes $2c^{3/2}-3c+1\geqslant 0$. But

$$2c^{3/2}-3c+1=(\sqrt{c}-1)^2(2\sqrt{c}+1),$$

so the inequality follows in this case. If $|B|=4$ then (5) becomes

$$c^2(2c-3\sqrt{c})+c \geqslant 0,$$

which by putting $x=\sqrt{c}$ can be written

$$2x^4-3x^3+1 \geqslant 0,$$

which is true for $x \geqslant 5/4$ and thus for $c \geqslant 1.6$.

Case (iii): $j(B)=3$. Then $4 \leqslant |B| \leqslant 8$. Put $z=|B|/2$ so that (5) becomes $c^z(3c-4\sqrt{c})+c \geqslant 0$. Letting

$$g_c(z)=c^z(3c-4\sqrt{c})+c,$$

it is enough to show that $g_c(z) \geqslant 0$ for all $2 \leqslant z \leqslant 4$ and for $c \geqslant r^2=1.61\ldots$. Note that $g_c(z)$ is increasing for $z>2$ as long as $3c>4\sqrt{c}$, that is, $c>16/9$. Thus for $c>16/9$,

$$g_c(z) \geqslant g_c(2)=c^2(3c-4\sqrt{c})+c=c(\sqrt{c}-1)^2(3c+2\sqrt{c}+1)>0$$

as claimed. On the other hand, if $r^2<c<16/9$ then $g_c(z)$ is decreasing, so we need only show that $g_c(4) \geqslant 0$ for c in this range. But by putting $x=\sqrt{c}$,

$$g_c(4)=c^4(3c-4\sqrt{c})+c=x^2(3x^8-4x^7+1),$$

which is greater than 0 for $c>r^2$ by the definition of r.

Case (iv): $j(B) \geqslant 4$. Then from (5) it would be enough to prove

$$4(c-\sqrt{c})-\sqrt{c} \geqslant 0,$$

or $\sqrt{c} \geqslant 5/4$, or $c \geqslant 25/16$, which is true.

Now we can assume that $|A| \geqslant 3$. In fact note that if $|A|=3$, then A must be a three-element chain, which means that

$$A \oplus' B \cong \mathbf{2} \oplus' (\mathbf{2} \oplus' B),$$

and is therefore (in two steps) also handled above. Thus we can in fact assume that $|A| \geqslant 4$.

Now $j(A) \leqslant |A|-1$ (with equality when A is a chain), so it is enough to prove that

$$j(B)c^{|B|/2}(c^{|A|/2}-\sqrt{c}) \geqslant (|A|-1)(c^{|B|/2}\sqrt{c}-c). \tag{6}$$

If $|B|=2$ then $j(B)=1$ and (6) becomes

$$c(c^{|A|/2}-\sqrt{c}) \geqslant (|A|-1)(c\sqrt{c}-c)$$

which simplifies to

$$c^{|A|/2}-|A|\sqrt{c}+|A|-1 \geqslant 0.$$

Letting $g(x)=x^{|A|}-|A|x+|A|-1$, we see that $g(1)=0$ and $g'(x)=|A|x^{|A|-1}-|A| \geqslant 0$ for all $x \geqslant 1$, so $g(x) \geqslant 0$ for all $x \geqslant 1$. Putting $x=\sqrt{c}$ shows (6) in this case.

So we now may assume $|B|\geqslant 3$ and hence $|B|\geqslant 4$ as well, which means $j(B)\geqslant 3$, and so it is enough to prove that

$$3c^{|B|/2}(c^{|A|/2}-\sqrt{c})\geqslant(|A|-1)(c^{|B|/2}\sqrt{c}-c),$$

which simplifies to

$$(|A|+2)c^{|B|/2}\sqrt{c}\leqslant 3c^{|A|/2}c^{|B|/2}+c(|A|-1).$$

Thus it is enough to prove $|A|+2\leqslant 3c^{(|A|-1)/2}$, or that $c^x\geqslant 2x/3+1$ where $x=(|A|-1)/2\geqslant 3/2$. Put $h(x)=c^x-2x/3-1$; then $h(3/2)=c^{3/2}-2>0$ for $c>1.6$, and $h'(x)=c^x\ln c-2/3$ so that $h'(3/2)=c^{3/2}\ln(3/2)-2/3>0$ again for $c>1.6$. Therefore $h(x)>0$ for all $x\geqslant 3/2$ and all $c>r^2$. This finishes the proof. □

Note that equality holds in (3) when $c=r^2$ for the distributive lattice $L=\mathbf{1}\oplus\mathbf{2}^3$, as is suggested by Case (iii) above. Thus, via the same technique as in Section 2, equality will hold in

$$\sum_{a\in J(L)} c^{u(a)}\geqslant j(L)c^{|L|/2} \tag{7}$$

when L is the dual lattice $\mathbf{2}^3\oplus\mathbf{1}$ and for $c=r^2\approx 1.6159$, and (7) with this L will *fail* for any c less than this value. This is the largest value of c yet shown to be necessary for (7) to hold.

Problem. Find better bounds, if not the best value, for c such that (7) holds for all finite distributive lattices. It is now known that the best value c_0 satisfies $1.6159<c_0\leqslant 4$.

In fact the lattice $\mathbf{2}^3\oplus\mathbf{1}$ is the only one known to require a value of c as large as 1.6159, and we do not know the answer to the following:

Problem. For each fixed $c>1$, does (7) hold for all *sufficiently large* finite distributive lattices L?

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