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MAXIMAL CHAINS AND ANTICHAINS IN BOOLEAN LATTICES*

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Abstract. The following equivalent results in the Boolean lattice $\mathbf{2}^n$ are proven.

(a) Every fibre of $\mathbf{2}^n$ contains a maximal chain.

(b) Every cutset of $\mathbf{2}^n$ contains a maximal antichain.

(c) Every red-blue colouring of the vertices of $\mathbf{2}^n$ produces either a red maximal chain or a blue maximal antichain.

(d) Given any n antichains in $\mathbf{2}^n$ there is a disjoint maximal antichain.

Statement (a) is then improved to:

(a′) Every fibre of $\mathbf{2}^n$ contains at least $n!/2^{n-1}$ maximal chains.

One conjecture of Lonc and Rival is supported, and another conjecture disproved, by showing:

(i) Every fibre of $\mathbf{2}^n$ has order $\Omega(1.25^n)$ elements.

(ii) There is a minimal fibre of $\mathbf{2}^n (n \geqq 4)$ of size $2^{n-1} + 2$.

Key words. Boolean lattice, maximal chain, maximal antichain

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1. Introduction. A *cutset* of a finite partially ordered set P is a subset of P that intersects all maximal chains, and a *fibre* of P is a subset intersecting all maximal antichains. The reader may consult [2] for more information on these concepts.

A simple example of a cutset is the set of all minimal elements (or maximal elements) of P. More generally, if all maximal chains of P have the same length, then the *levels* of P are cutsets. (Here the kth level of P is the set of all elements $x \in P$ such that all maximal chains through x contain exactly k elements less than x. Thus the 0th level is just the set of minimal elements of P.)

For fibres, there is also a natural example. The *cone* of an element $x \in P$ is the set of all elements comparable to x (i.e., either $\leqq x$ or $\geqq x$). It is a simple exercise to see that every cone is a fibre (e.g., see [2]).

Both of these constructions reinforce the intuitive idea of a cutset as something stretching "horizontally" through P and a fibre as something stretching "vertically" through P. In particular, every level contains (in fact, is) a *maximal* antichain, and every cone contains a *maximal* chain. This will not be the case for every cutset or fibre of every poset P; for example in the poset of Fig. 1, $\{a, d\}$ is a cutset with no maximal antichain and $\{b, c\}$ a fibre with no maximal chain. The main result of this paper shows that intuition holds for one familiar family of finite posets: finite Boolean lattices.

We denote by $\mathbf{2}^n$ the Boolean lattice with n atoms, that is, the lattice of all subsets of an n-element set.

THEOREM 1. (a) *Every fibre of* $\mathbf{2}^n$ *contains a maximal chain.*

(b) *Every cutset of* $\mathbf{2}^n$ *contains a maximal antichain.*

In fact these two statements are equivalent for any poset P; it is easy to see that if F were a fibre of P containing no maximal chain, then $P - F$ would be a cutset of P containing no maximal antichain, and conversely. We may even note a third equivalent statement of Theorem 1:

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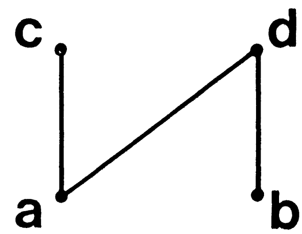

FIG. 1

(c) *If the elements of* $\mathbf{2}^n$ *are coloured red and blue, there is either a red maximal chain or a blue maximal antichain.*

The equivalence again follows for arbitrary posets P since, for example, if the red elements do not contain a maximal chain, then by (a) of Theorem 1 they cannot be a fibre, and therefore must be disjoint from some maximal antichain which is necessarily all blue. Conversely, assuming (c) and given a fibre F of P, colour the elements of F red and everything else blue; then there cannot be a blue maximal antichain, so by (c) F contains a maximal chain.

We give one more equivalent formulation of Theorem 1.

(d) *Given any* n *antichains in* $\mathbf{2}^n$, *there is a maximal antichain disjoint from all of them.*

Note that this is best possible, as $\mathbf{2}^n$ is the union of its $n + 1$ levels. This time we do not get equivalence for arbitrary posets P. Let the shortest maximal chain in P have $l + 1$ elements and the longest have $L + 1$ elements; then we have the following implications:

For any L antichains in P there is a disjoint maximal antichain	$\Rightarrow$	Every fibre of P has a maximal chain	$\Rightarrow$	For any l antichains in P there is a disjoint maximal antichain

For the first implication, if a fibre F of P does not have a maximal chain, then every chain of F has at most L elements. Thus F is the union of at most L antichains, so there is a maximal antichain disjoint from F, a contradiction. For the second implication, the union of l antichains in P cannot contain a maximal chain, so cannot be a fibre, so there is a disjoint maximal antichain.

Of course, when $P = \mathbf{2}^n$ we have $L = l = n$, whence (a) and (d) are equivalent.

In the next section we prove Theorem 1 in form (a), or actually a stronger version which will enable us to deduce the following corollary.

COROLLARY. *Every fibre of* $\mathbf{2}^n$ *contains at least* $n!/2^{n-1}$ *maximal chains.*

In the final section we give some results bearing on two conjectures of Lonc and Rival [2] on the sizes of *minimal* fibres (i.e., fibres none of whose proper subsets is a fibre) of $\mathbf{2}^n$. In particular we prove in the following theorem that all fibres of $\mathbf{2}^n$ are of size exponential in n.

THEOREM 2. *If* $\mathscr{F}$ *is a fibre of* $\mathbf{2}^n$ *then* $|\mathscr{F}| = \Omega(1.25^n)$.

2. Proof of Theorem 1. We take the elements of $\mathbf{2}^n$ to be all subsets of $[n] = \{1, 2, \cdots, n\}$, and shall denote them by capitals. Subsets of $\mathbf{2}^n$ will be denoted by script letters, so a fibre shall be denoted $\mathscr{F}$, for example. If $\mathscr{S} \subseteq \mathbf{2}^n$ we write

$$\mathscr{S}^- = \{X \in \mathbf{2}^n \mid X \subseteq S \text{ for some } S \in \mathscr{S}\}$$

and

$$\mathscr{S}^+ = \{X \in \mathbf{2}^n \mid X \supseteq S \text{ for some } S \in \mathscr{S}\}.$$

Let α be an arbitrary total ordering of the elements of $[n]$. For $X \in \mathbf{2}^n$ we let $X = \{x_1, x_2, \cdots, x_k\}$ where $x_1 <_\alpha x_2 <_\alpha \cdots <_\alpha x_k$ ($<_\alpha$ being determined by α), and we also put $[n] - X = \{y_1, y_2, \cdots, y_{n-k}\}$ where $y_1 <_\alpha y_2 <_\alpha \cdots <_\alpha y_{n-k}$. Define the *$\alpha$-lexical chain through X*, denoted $\mathscr{L}_\alpha(X)$, to be the chain $\mathscr{L}_\alpha^-(X) \cup \mathscr{L}_\alpha^+(X)$ where

$$\mathscr{L}_\alpha^-(X) = \{X, X - \{x_1\}, X - \{x_1, x_2\}, \cdots, X - \{x_1, x_2, \cdots, x_k\} = \varnothing\},$$
$$\mathscr{L}_\alpha^+(X) = \{X, X \cup \{y_1\}, X \cup \{y_1, y_2\}, \cdots, X \cup \{y_1, y_2, \cdots, y_{n-k}\} = [n]\}.$$

That is, $\mathscr{L}_\alpha^-(X)$ is constructed by removing elements from X one at a time, smallest to largest according to α; $\mathscr{L}_\alpha^+(X)$ is constructed by adding elements to X one at a time, smallest to largest according to α; and $\mathscr{L}_\alpha(X)$ is just these two chains put together. Clearly $\mathscr{L}_\alpha(X)$ is a maximal chain of $\mathbf{2}^n$ for each α and X. If $\mathscr{X}$ is a subset of $\mathbf{2}^n$, we write

$$\mathscr{L}_\alpha^-(\mathscr{X}) = \cup\{\mathscr{L}_\alpha^-(X) : X \in \mathscr{X}\},$$

with similar definitions for $\mathscr{L}_\alpha^+(\mathscr{X})$ and $\mathscr{L}_\alpha(\mathscr{X})$. If α is the usual ordering $1 < 2 < \cdots < n$, we drop the α's, writing $\mathscr{L}(X)$ instead of $\mathscr{L}_\alpha(X)$, for instance.

We will prove that

(∗) *every fibre of $\mathbf{2}^n$ contains the lexical chain $\mathscr{L}(X)$ for some $X \in \mathbf{2}^n$*,

and in particular form (a) of Theorem 1 holds. To do this we need the following—somewhat awkward—definition. Call a subset $\mathscr{S}$ of a fibre $\mathscr{F}$ of $\mathbf{2}^n$ *critical* if there do *not* exist subsets $\mathscr{A}$, $\mathscr{B}$ of $\mathbf{2}^n$ satisfying

(1) $\mathscr{A} \cup \mathscr{B}$ is an antichain disjoint from $\mathscr{F}$,
(2) $\mathscr{S} \subseteq \mathscr{A}^- \cup \mathscr{B}^+$, and
(3) $\mathscr{A} \subseteq \mathscr{L}^+(\mathscr{S})$, $\mathscr{B} \subseteq \mathscr{L}^-(\mathscr{S})$.

The idea behind this definition is as follows. Since $\mathscr{F}$ is a fibre, every antichain avoiding $\mathscr{F}$, when extended to a maximal antichain, must meet $\mathscr{F}$. A subset $\mathscr{S}$ of $\mathscr{F}$ for which there is no $\mathscr{A}$ and $\mathscr{B}$ satisfying (1) and (2) *will by itself* prevent any antichain outside $\mathscr{F}$ from being extended to a maximal antichain outside $\mathscr{F}$, in that any such maximal antichain must contain an element of $\mathscr{S}$. It can be assumed, without violating (1) or (2), that $\mathscr{A} \subseteq \mathscr{S}^+$ and $\mathscr{B} \subseteq \mathscr{S}^-$; condition (3), found necessary for our proof to work, requires further that each element of $\mathscr{A}$ be on the upper half of some lexical chain through an element of $\mathscr{S}$, and dually for $\mathscr{B}$.

Note that $\mathscr{F}$ itself is critical; even if $\mathscr{A}, \mathscr{B} \subseteq \mathbf{2}^n$ were to satisfy just (1) and (2) for $\mathscr{S} = \mathscr{F}$, $\mathscr{A} \cup \mathscr{B}$ could be extended to a maximal antichain which would necessarily avoid $\mathscr{F}$, a contradiction. On the other hand the empty set is trivially not critical, as is witnessed by the choice $\mathscr{A} = \mathscr{B} = \varnothing$.

The proof of (∗) is by contradiction. Suppose $\mathscr{F}$ is a fibre of $\mathbf{2}^n$ containing no lexical chain. By the above we may choose a nonempty critical subset $\mathscr{M}$ of $\mathscr{F}$ which is *minimal*, that is, no proper subset of $\mathscr{M}$ is critical. For each $X \in \mathscr{M}$ define the *rank* $r(X)$ of X to be the least positive integer r such that either $X \cup \{1, 2, \cdots, r\}$ or $X - \{1, 2, \cdots, r\}$ is not in $\mathscr{F}$. Note that $r(X)$ exists for each X since the entire lexical chain $\mathscr{L}(X)$ is never in $\mathscr{F}$. Choose $M \in \mathscr{M}$ such that $r(M) \leqq r(X)$ for all $X \in \mathscr{M}$.

Since $\mathscr{M}$ is minimally critical, $\mathscr{M} - \{M\}$ cannot be critical, so there exist subsets $\mathscr{A}$, $\mathscr{B}$ of $\mathbf{2}^n$ satisfying (1), (2), and (3) for $\mathscr{S} = \mathscr{M} - \{M\}$. Since $\mathscr{M}$ is critical, $\mathscr{A}$ and

$\mathscr{B}$ must fail one of (1)–(3) for $\mathscr{S} = \mathscr{M}$. As is easy to see, the only possibility is that $M \notin \mathscr{A}^- \cup \mathscr{B}^+$.

By the definition of $r(M)$ either

(i) $M' = M \cup \{1, 2, \cdots, r(M)\} \notin \mathscr{F}$

or

(ii) $M'' = M - \{1, 2, \cdots, r(M)\} \notin \mathscr{F}$.

We may suppose (i), by a duality argument we now give. For a set $S \in \mathbf{2}^n$, let $\bar{S} = [n] - S$ be the complement. If $\mathscr{S}$ is a subset of $\mathbf{2}^n$ we write

$$\bar{\mathscr{S}} = \{\bar{S} \mid S \in \mathscr{S}\}.$$

(Note carefully that $\bar{\mathscr{S}} \neq \mathbf{2}^n - \mathscr{S}$!) $\bar{\mathscr{S}}$ is the image of $\mathscr{S}$ under the dual automorphism $S \to \bar{S}$ of $\mathbf{2}^n$. Thus it is clear that if $\mathscr{S}$ is a maximal antichain (or maximal chain, or fibre), so is $\bar{\mathscr{S}}$. In particular, if $\mathscr{S}$ is a fibre with no maximal chain, so is $\bar{\mathscr{S}}$. Furthermore, if $X \in \mathbf{2}^n$ then

$$\overline{\mathscr{L}^-(X)} = \mathscr{L}^+(\bar{X}) \text{ and } \overline{\mathscr{L}^+(X)} = \mathscr{L}^-(\bar{X}),$$

and so

$$\overline{\mathscr{L}(X)} = \mathscr{L}(\bar{X}),$$

all as *sets*; as chains they are dual to each other in each case. It now follows that a subset $\mathscr{S}$ of a fibre $\mathscr{F}$ is (minimally) critical if and only if $\bar{\mathscr{S}}$ is a (minimally) critical subset of $\bar{\mathscr{F}}$. Moreover if $\mathscr{F}$ is a fibre containing no lexical chain, then $\bar{\mathscr{F}}$ also contains no lexical chain, and for each $X \in \mathscr{F}$, $r(X) = r(\bar{X})$, where $r(\bar{X})$ denotes the rank of $\bar{X}$ in the fibre $\bar{\mathscr{F}}$. Thus, if (ii) holds for the element M of the fibre $\mathscr{F}$, then (i) will hold for the (equally minimal) element $\bar{M}$ of the fibre $\bar{\mathscr{F}}$. Hence we may assume (i) holds for the given element and fibre.

Now let

$$\mathscr{A}' = (\mathscr{A} - \{M'\}^-) \cup \{M'\}$$

(see Fig. 2 for a schematic).

We claim that the sets $\mathscr{A}'$, $\mathscr{B}$ satisfy (1), (2), and (3) for $\mathscr{S} = \mathscr{M}$. This is impossible since $\mathscr{M}$ is critical, and (*) follows.

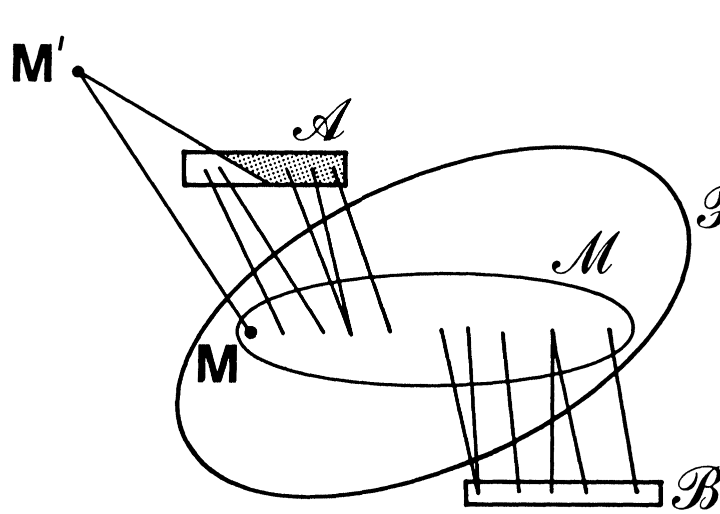

FIG. 2

First, (3) is trivially true since $\mathscr{B}$ has not changed, $\mathscr{S}$ has been enlarged (from $\mathscr{M} - \{M\}$ to $\mathscr{M}$) so $\mathscr{L}^-(\mathscr{S})$ and $\mathscr{L}^+(\mathscr{S})$ could only get bigger, and the only element of $\mathscr{A}'$ not in $\mathscr{A}$ is M' which is an element of $\mathscr{L}^+(M)$ and thus of $\mathscr{L}^+(\mathscr{M})$. (2) is immediate also; $\mathscr{B}^+$ is unchanged, and it is easy to see that $\mathscr{A}'^-$ contains $\mathscr{A}^-$, therefore we need only observe that $M \in \mathscr{A}'^-$ follows from $M \leqq M'$ and $M' \in \mathscr{A}'$.

For (1), $\mathscr{A}' \cup \mathscr{B}$ is of course disjoint from $\mathscr{F}$ since $M' \notin \mathscr{F}$. Thus it remains to check that M' is incomparable to all elements of $\mathscr{A}' - \{M'\} = \mathscr{A} - \{M'\}^-$ and of $\mathscr{B}$. If $A \in \mathscr{A} - \{M'\}^-$, then by definition $A \nleqq M'$, and $M' \nleqq A$ since $M \notin \mathscr{A}^-$. Finally let $B \in \mathscr{B}$. Then by (3) $B \in \mathscr{L}^-(X)$ for some $X \in \mathscr{M} - \{M\}$, and $r(X) \geqq r(M)$ by the choice of M. Thus B cannot contain any of the elements $\{1, 2, \cdots, r(M)\}$; if in X at all, they were the first to be deleted from X in forming the chain $\mathscr{L}^-(X)$. But since $M \notin \mathscr{B}^+$, $B \nleqq M$, and hence $B \nleqq M \cup \{1, 2, \cdots, r(M)\} = M'$. Since $1 \in M' - B$, we also have $M' \nleqq B$, and we are done. □

The reader should note the following examples before attempting to extend this theorem.

(1) If extreme points 0, 1 are added to the poset of Fig. 1, we obtain a poset isomorphic to the product of a two- and a three-element chain, with a fibre $\{0, b, c, 1\}$ which does not contain a maximal chain. Thus Theorem 1 cannot be extended to arbitrary finite products of finite chains.

(2) We now describe a minimal fibre $\mathscr{F}$ of $\mathbf{2}^6$ containing an element X but not containing any maximal chain through X. To simplify notation we will denote elements of $\mathbf{2}^6$ by strings of integers, writing 123 instead of $\{1, 2, 3\}$, for example. Let

$$\mathscr{A} = \{12, 13, 14, 25, 26, 156, 234, 345, 346, 356, 456\}.$$

It can easily be checked that $\mathscr{A}$ is a maximal antichain of $\mathbf{2}^6$ (first check that no two- or three-element subset of $\{1, 2, \cdots, 6\}$ can be added; it then follows that no other subset can be added either). However there is *no* maximal antichain of $\mathbf{2}^6$ contained in

$$\mathscr{B} = (\mathscr{A} \cup \{123, 124, 125, 126\}) - \{12\}.$$

(To avoid our adding 12 to such an antichain, it would have to contain, say, 123; this means that 13 cannot be in the antichain, and now nothing can prevent 135 from being included in the antichain.) This means that $\mathbf{2}^6 - \mathscr{B}$ is a fibre containing 12 but no upper cover of 12, and hence no maximal chain through 12. Let $\mathscr{F}$ be any minimal fibre contained in $\mathbf{2}^6 - \mathscr{B}$; then $X = 12 \in \mathscr{F}$, since otherwise the maximal antichain $\mathscr{A}$ misses $\mathscr{F}$.

(3) We modify a construction of Nowakowski [3] to find a minimal cutset $\mathscr{C}$ of $\mathbf{2}^n$ containing an element X but no maximal antichain containing X. Using a notation analogous to that of the previous example, for $n \geqq 4$ let

$$\mathscr{C} = \{12k : 3 \leqq k \leqq n\} \cup \{1k : 3 \leqq k \leqq n\} \cup \{2k : 3 \leqq k \leqq n\} \cup \{k : 3 \leqq k \leqq n\}.$$

It is easy to see that $\mathscr{C}$ is a minimal cutset of $\mathbf{2}^n$ (consider the smallest element of a given maximal chain of $\mathbf{2}^n$ which is not contained in 12). We claim that, for instance, $\mathscr{C}$ contains no maximal antichain containing 13. For if $\mathscr{A}$ were such an antichain, then for each $k \geqq 4$, to avoid adding $3k$ to $\mathscr{A}$ we must have $k \in \mathscr{A}$, and now nothing can stop 12 from joining $\mathscr{A}$, a contradiction.

In proving above that every fibre of $\mathbf{2}^n$ contains the lexical chain $\mathscr{L}(X)$ for some $X \in \mathbf{2}^n$, by relabeling we have actually shown that *for every ordering* α *of* $\{1, 2, \cdots, n\}$, *every fibre of* $\mathbf{2}^n$ *contains* $\mathscr{L}_\alpha(X)$ *for some* $X \in \mathbf{2}^n$. With this observation and a little counting, form (a) of Theorem 1 can be strengthened considerably.

COROLLARY. *Every fibre of* $\mathbf{2}^n$ *contains at least* $n!/2^{n-1}$ *maximal chains.*

Proof. We first count the number of lexical chains $\mathscr{L}(X)$ in $\mathbf{2}^n$. It is clear that the generating set X of a lexical chain $\mathscr{L}(X)$ must either be the smallest set in $\mathscr{L}(X)$ containing 1 or the largest set in $\mathscr{L}(X)$ missing 1, since 1 is either the first element added to X or the first element deleted, depending on whether $1 \in X$ or not. Thus each lexical chain $\mathscr{L}(X)$ is generated by two of the 2^n subsets of $[n]$, so there are exactly 2^{n-1} lexical chains.

By symmetry, there are exactly 2^{n-1} α-lexical chains for each ordering α. Since there are $n!$ orderings of $[n]$ and $n!$ maximal chains in $\mathbf{2}^n$, by symmetry each maximal chain must be of the form $\mathscr{L}_\alpha(X)$ (for some X) for exactly 2^{n-1} different orderings α. Therefore, since every fibre must contain an α-lexical chain for each α, every fibre must contain at least $n!/2^{n-1}$ different maximal chains. □

To end this section, we note the following stronger version of form (b) of Theorem 1. For an ordering α of $[n]$, call a subset $\mathscr{C}$ of $\mathbf{2}^n$ an *α-generalized cutset* if $\mathscr{C}$ intersects every α-lexical chain $\mathscr{L}_\alpha(X)$, $X \in \mathbf{2}^n$. Obviously every cutset is an α-generalized cutset for each α. Then *for each α, every α-generalized cutset of* $\mathbf{2}^n$ *contains a maximal antichain.* This follows simply because if an α-generalized cutset $\mathscr{C}$ did not contain a maximal antichain, then its complement $\mathbf{2}^n - \mathscr{C}$ would be a fibre and thus by the above observation must contain some α-lexical chain $\mathscr{L}_\alpha(X)$. But then $\mathscr{L}_\alpha(X)$ would be disjoint from $\mathscr{C}$, a contradiction.

Here is an application. Consider $\mathbf{2}^n$ as the disjoint union of posets $\mathscr{P}_1$ and $\mathscr{P}_2$, where

$$\mathscr{P}_1 = \{X \in \mathbf{2}^n : X \subseteq [n-1]\},\ \mathscr{P}_2 = \{X \in \mathbf{2}^n : n \in X\}.$$

Note that both $\mathscr{P}_1$ and $\mathscr{P}_2$ are isomorphic to $\mathbf{2}^{n-1}$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be cutsets of $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively. Then we claim that *$\mathscr{C}_1 \cup \mathscr{C}_2$ contains a maximal antichain of* $\mathbf{2}^n$. For let α be any ordering of $[n]$ ending in the element n (for example, α could be the usual ordering). Then for any $X \in \mathbf{2}^n$, if $n \in X$ the upper cover of $\varnothing$ in $\mathscr{L}_\alpha(X)$ will be $\{n\}$, while if $n \notin X$ the lower cover of $[n]$ in $\mathscr{L}_\alpha(X)$ will be $[n-1]$. Hence for each $X \in \mathbf{2}^n$ either $\mathscr{L}_\alpha(X) - \{\varnothing\} \subseteq \mathscr{P}_2$ or $\mathscr{L}_\alpha(X) - \{[n]\} \subseteq \mathscr{P}_1$, which means that $\mathscr{C}_1 \cup \mathscr{C}_2$ is an α-generalized cutset, and the claim follows.

3. Sizes of minimal fibres of $\mathbf{2}^n$. In [2], Lonc and Rival made the following conjectures:

(i) The minimum size of a fibre of $\mathbf{2}^n$ is

$$2^{(n+1)/2} + 2^{(n-1)/2} - 1, \quad n \text{ odd},$$

$$2^{n/2+1} - 1, \quad n \text{ even}.$$

(ii) The maximum size of a minimal fibre of $\mathbf{2}^n$ is $2^{n-1} + 1$.

Both sizes are attained by cones, (i) by the cone of an element in the middle level(s) of $\mathbf{2}^n$, (ii) by the cone of an atom or co-atom of $\mathbf{2}^n$.

Regarding (i), we can show at least that every fibre of $\mathbf{2}^n$ is of size exponential in n.

THEOREM 2. *If $\mathscr{F}$ is a fibre of $\mathbf{2}^n$ then $|\mathscr{F}| = \Omega(1.25^n)$.*

Proof. Let k be an integer, $1 \leqq k < n/2$, and let S be an element of $\mathbf{2}^n$ of size $2k-1$. Let $\mathscr{T}$ be the family of all k-element subsets of S. We claim that $\mathscr{T} \cup \bar{\mathscr{T}}$ is a maximal antichain in $\mathbf{2}^n$. $\mathscr{T} \cup \bar{\mathscr{T}}$ is certainly an antichain; given $T, U \in \mathscr{T}$, T and U have at least one element of S in common, so $T \nsubseteq \bar{U}$, and $T \cup U \neq [n]$ so $T \nsupseteq \bar{U}$. To show $\mathscr{T} \cup \bar{\mathscr{T}}$ is maximal, let $X \in \mathbf{2}^n$; then X either contains at least k elements of S and thus contains some member of $\mathscr{T}$, or misses at least k elements of S and thus is contained in some member of $\bar{\mathscr{T}}$. Hence X cannot be added to $\mathscr{T} \cup \bar{\mathscr{T}}$ to get a larger antichain.

Consider all such maximal antichains $\mathscr{T} \cup \bar{\mathscr{T}}$ corresponding to elements $S \in \mathbf{2}^n$, $|S| = 2k - 1$. Any fibre $\mathscr{F}$ of $\mathbf{2}^n$ must intersect each of these maximal antichains, of which there are $\binom{n}{2k-1}$, one for each S. Consider a k-element set T in $\mathscr{F}$; T is a k-element subset of exactly $\binom{n-k}{k-1}$ sets S of size $2k - 1$ and so can be the representative in $\mathscr{F}$ of only this many maximal antichains $\mathscr{T} \cup \bar{\mathscr{T}}$. Similarly, an $(n - k)$-element set $\bar{T}$ in $\mathscr{F}$ can only belong to (the $\bar{\mathscr{T}}$-part of) $\binom{n-k}{k-1}$ maximal antichains $\mathscr{T} \cup \bar{\mathscr{T}}$. Thus $\mathscr{F}$ must contain at least

$$\binom{n}{2k-1} \Big/ \binom{n-k}{k-1}$$

elements of size k or $n - k$. This turns out to be optimized for $k = n/5$. The above quotient can then be estimated by Stirling's approximation to yield that $|\mathscr{F}|$ is of order at least

$$\frac{n^n}{(2n/5)^{2n/5}(3n/5)^{3n/5}} \cdot \frac{(n/5)^{n/5}(3n/5)^{3n/5}}{(4n/5)^{4n/5}} = (5/4)^n. \qquad \square$$

Any family $\mathscr{S}$ of $(2k - 1)$-element subsets of $[n]$ with the property that no two members of $\mathscr{S}$ intersect in k or more elements will yield $|\mathscr{S}|$ pairwise disjoint maximal antichains $\mathscr{T} \cup \bar{\mathscr{T}}$ of $\mathbf{2}^n$. Consider the graph whose vertices are all $(2k - 1)$-element subsets of $[n]$ (where k is a fixed proportion of n to be chosen later), two such subsets being adjacent if they have at least k elements in common. Then the degree of any vertex is

$$\sum_{l=k}^{2k-2} \binom{2k-1}{l} \binom{n-2k+1}{2k-l-1}.$$

If $k < n/4$, it can easily be checked that the first term of this sum is the largest and, exponentially, will dominate the sum. Thus there will exist an independent set of order at least

$$\frac{\binom{n}{2k-1}}{\binom{2k-1}{k}\binom{n-2k+1}{k-1}}.$$

Putting $k = \lambda n$ and applying Stirling's approximation, this quotient is exponentially equal to

$$\left(\frac{(1-3\lambda)^{1-3\lambda}}{(16\lambda)^{\lambda}(1-2\lambda)^{2-4\lambda}}\right)^n$$

which is maximized for λ the real root of

$$112x^3 - 88x^2 + 20x - 1,$$

i.e., $\lambda \approx 0.0692304$. Plugging this value of λ into the above expression, we obtain: *there exists a family of* $\Omega(1.0674422^n)$ *pairwise disjoint maximal antichains in* $\mathbf{2}^n$.

Of course $\mathbf{2}^n$ cannot have more than $O(\sqrt{2})^n$ pairwise disjoint maximal antichains since there is a fibre of this size, namely the cone of a middle-level element.

Problem. Find better bounds for the maximum number of pairwise disjoint maximal antichains in $\mathbf{2}^n$.

The base 1.25 in Theorem 2 compares quite well with the base $\sqrt{2}$ in conjecture (i) above. Further evidence in support of conjecture (i) might be presumed from the corollary to Theorem 1, as the cone of a middle-level element of $\mathbf{2}^n$ contains $(n/2)!^2 \approx \pi n(n/2e)^n$ maximal chains, scarcely more than the $n!/2^{n-1} \approx 2\sqrt{2\pi n}\,(n/2e)^n$ maximal chains known to exist in every fibre from the corollary.

On the other hand, we can disprove conjecture (ii) for all $n \geqq 4$, although not resoundingly: the following is an example of a minimal fibre $\mathscr{F}$ of $\mathbf{2}^n$ of size $2^{n-1} + 2$, *one more* than the conjectured maximum size!

We use the notation of example (2) following the proof of Theorem 1, so that 12 means $\{1, 2\}$, etc. Let

$$\mathscr{F} = \{12\}^+ \cup \{\overline{12}\}^- \cup \{2, \bar{1}\}$$

(Fig. 3). Then $|\mathscr{F}| = 2^{n-2} + 2^{n-2} + 2 = 2^{n-1} + 2$. Why is $\mathscr{F}$ a fibre? If $\mathscr{A}$ were a maximal antichain missing $\mathscr{F}$ then we must have $1 \in \mathscr{A}$, since 1 is the only element of the cone of 12 missing from $\mathscr{F}$. Similarly $\bar{2} \in \mathscr{A}$ by considering the cone of $\overline{12}$. But this is nonsense since $1 \subset \bar{2}$. Finally, why is $\mathscr{F}$ minimal? $\mathscr{F} - \{[n]\}$ is obviously not a fibre of $\mathbf{2}^n$ since $\{[n]\}$ is itself a maximal antichain, and $\mathscr{F} - \{2\}$ is not a fibre of $\mathbf{2}^n$ because the maximal antichain $\{2, \bar{2}\}$ misses $\mathscr{F} - \{2\}$. So to finish the proof, by symmetry we need only show that $\mathscr{F} - \{X\}$ is not a fibre whenever $12 \subseteq X \subset [n]$. To do this we claim that

$$\mathscr{A} = \{\bar{2}, X\} \cup \{2y \mid y \notin X\}$$

is a maximal antichain. Since (for $n \geqq 4$) $\mathscr{A}$ intersects $\mathscr{F}$ only in the element X, we would be done. It is easy to check that $\mathscr{A}$ is an antichain. Moreover, if $Y \in \mathbf{2}^n$ with $Y \nsubseteq \bar{2}$ and $Y \nsubseteq X$ then $2 \in Y$ and there is $y \in Y$, $y \notin X$. Thus $2y \subseteq Y$, so Y cannot be added to $\mathscr{A}$.

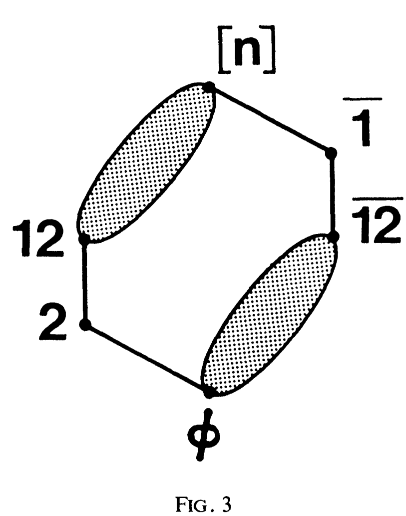

FIG. 3

Recently Füredi, Griggs, and Kleitman [1] have found minimal cutsets of $\mathbf{2}^n$ which contain almost all elements of $\mathbf{2}^n$. We withhold judgment on whether or not the above minimal fibre is largest.

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