Errata for *Spectral Theory of Infinite Area Hyperbolic Surfaces*, 2nd edition

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Chapter 2

p. 17: Lemma 2.12

The end of the proof is potentially confusing, since $[p, p']$ is not compact in the topology of $\mathbb{B}$. The point is that every point on the arc $[T_j z, T_j w]$ lies in $T_j \mathcal{D}_w$. Thus, if these arcs accumulate on $[p, p']$, then some compact neighborhood of an interior point of $[p, p']$ intersects infinitely many of the copies $T_j \mathcal{D}_w$. That is the contradiction with Lemma 2.11.

Also, it should be noted that the restriction the w not be an elliptic fixed point is an artifact of the proof. In fact, $\Lambda(\Gamma)$ is the set of limit points of Γw for any $w \in \mathbb{H}$. See [Beardon, Thm. 5.3.9] for the more general proof.

p. 21: Proof of Theorem 2.16

Although Theorem 2.16 is correct as stated, Lemma 2.17 proves only that the existence of a finite-sided Dirichlet domain implies that the group is finitely generated. To complete the proof, one must show that the existence of a convex finite-sided fundamental domain implies that all convex fundamental domains are finite-sided. See Beardon [20, Thm. 10.1.2] for the proof.

p. 23: second paragraph

The claim that "a small neighborhood of p meets exactly two sides of $\mathcal{D}_w$" requires more justification. One must rule out the possibility that the boundary of $\mathcal{D}_w$ contains infinitely many arcs accumulating at p. For this argument, see Beardon [20, Thm. 9.3.8].

p. 26: Lemma 2.21

In part (3) of the statement, there is a tilde missing: $\overline{O}_p \subset \tilde{N}$.

p. 29: Proof of Theorem 2.23

In the second-to-last paragraph the statement $\mathit{\Gamma}\backslash H_j = \mathit{\Gamma}_j\backslash H_j$ doesn't quite make sense, since $\mathit{\Gamma}$ does not preserve H_j. What is meant here is that if any two points of H_j are related by an element $g \in \mathit{\Gamma}$, then in fact $g \in \mathit{\Gamma}_j$.

p. 44: Proposition 2.39

Should start "Let X be a geometrically finite hyperbolic surface..."

Chapter 4

p. 66: Proof of Proposition 4.2

The derivation of c_s here is sloppy. Although the asymptotic behavior $f'(r) \sim -1/2\pi r$ is indeed universal for a Green's function in two dimensions, the argument given here makes unjustified assumptions about the behavior of $f'(r)$ as $r \to 0$.

For the correct computation of c_s, we note that, by the definition of the resolvent,

$$\varphi(z_0) = \int_{\mathbb{H}} R_{\mathbb{H}}(s; z_0, z)\varphi(z)\, dg(z), \tag{1}$$

for $\varphi \in C_0^\infty(\mathbb{H})$. For convenience, let us use geodesic radial coordinates (r, θ) centered at z_0, and assume that φ is radial in these coordinates, i.e., $\varphi(z) = h(r)$ for some function h. Then (1) translates to

$$h(0) = 2\pi \int_0^\infty f(r) \left[-\frac{1}{\sinh r}(h' \sinh r)' - s(1-s)h \right] \sinh r\, dr,$$

where the primes denote r derivatives. Since $\Delta\varphi$ is smooth, we can write this as a limit

$$h(0) = 2\pi \lim_{\varepsilon\to 0} \int_\varepsilon^\infty f(r) \left[-\frac{1}{\sinh r}(h' \sinh r)' - s(1-s)h \right] \sinh r\, dr.$$

Staying away from $r = 0$ allows us to integrate by parts twice, exploiting the fact that $-\Delta f = s(1-s)f$ in this range, to obtain

$$h(0) = 2\pi \lim_{\varepsilon\to 0} \big[f(\varepsilon)h'(\varepsilon) - f'(\varepsilon)h(\varepsilon) \big] \sinh \varepsilon. \tag{2}$$

The asymptotic behavior of the Legendre function as $u \to 1^+$ is given by [202, §5(12.23)],

$$\mathbf{Q}_{s-1}(u) \sim -\frac{1}{2\Gamma(s)} \log(u-1) + \log\sqrt{2} - \frac{\Gamma'}{\Gamma}(s) - \gamma + O(u-1), \tag{3}$$

for $s \notin -\mathbb{N}_0$, where γ is Euler's constant. Since $\cosh r \sim 1 + r^2/2$ as $r \to 0$, this implies that

$$f(r) \sim -c_s \frac{\log r}{\Gamma(s)}.$$

Since $h'(\varepsilon) = O(\varepsilon)$, by the smoothness of φ, the first term in (2) vanishes.

We can compute the asymptotics of $f'(r)$ from the standard Legendre derivative formula

$$\frac{dQ_{s-1}(u)}{du} = \frac{s}{u^2-1} \big[Q_s(u) - uQ_{s-1}(u) \big]$$

(where $Q_{s-1} = \Gamma(s)\mathbf{Q}_{s-1}$). Using this in conjunction with (3) gives

$$\lim_{\varepsilon\to 0} f'(\varepsilon) \sinh \varepsilon = -\frac{c_s}{\Gamma(s)}.$$

Hence, (2) is satisfied by setting

$$c_s = \frac{\Gamma(s)}{2\pi}.$$

p. 76: Equation (4.33)

The factor of $(2s-1)$ that appears in (4.28) should appear here also:

$$S_{\mathbb{H}}(s; x, x') = (2s-1) \lim_{y,y'\to 0} (yy')^{-s} R_{\mathbb{H}}(s; z, z').$$

This is consistent with (4.6).

p. 82: First sentence

The metric is derived in the (r, θ) coordinates.

Chapter 5

p. 93: Equation (5.29)
The factor of $(2s-1)$ from (5.24) should appear here also:

$$S_{F_\ell}(s;\theta,\theta') = (2s-1)\lim_{r,r'\to\infty}(\rho\rho')^{-s}R_{F_\ell}(s;r,\theta,r',\theta').$$

Chapter 7

p. 138: Equation (7.35)
The factor of $(2s-1)$ from (7.30) should appear here also:

$$S_X(s;\omega,\omega') = (2s-1)\lim_{z\to\omega,z'\to\omega'}(\rho_{\mathrm{f}}\rho'_{\mathrm{f}})^{1-s}(\rho_{\mathrm{f}}\rho'_{\mathrm{f}})^{-s}R_X(s;z,z').$$

p. 141: Proposition 7.15
The formula for $Q^\sharp$ is missing a factor of $(2s-1)$, corresponding to the error in (7.35):

$$Q^\sharp(s;\cdot,\cdot) := (2s-1)\big[(\rho_{\mathrm{f}}\rho'_{\mathrm{f}})^{-s}(\rho_{\mathrm{c}}\rho'_{\mathrm{c}})^{1-s}Q(s;\cdot,\cdot)\big]\Big|_{\partial\overline{X}\times\partial\overline{X}}.$$

Chapter 8

p. 145: First paragraph, second sentence
Should read: "Since the residue of the resolvent..."

Chapter 10

p. 220: Equation (10.15)
The term $S_{F_j}(1-s)\,\partial_s S_{F_j}(s)$ on the first line should have a minus sign.

p. 220: Equation (10.16)
This equation should have an ordinary equals sign (not a definition), a finite part is missing on the left side, and the term $S_{F_j}(1-s)\,\partial_s S_{F_j}(s)$ should have a minus sign:

$$\begin{aligned}\mathop{\mathrm{FP}}_{\varepsilon\to 0} A^{\mathrm{ff}}_{ij}(s,\varepsilon) = &-\mathrm{tr}\Big[S^{\mathrm{ff}}_{jj}(1-s)\,\partial_s S^{\mathrm{ff}}_{jj}(s) - S_{F_j}(1-s)\,\partial_s S_{F_j}(s)\Big]\\ &+\frac{1}{2s-1}\,\mathrm{tr}\Big[S^{\mathrm{ff}}_{jj}(s)S^{\mathrm{ff}}_{jj}(1-s) - I_{F_j}\Big].\end{aligned}$$

Chapter 11

p. 249: Definition 11.1
The fact that the integral of the wave operator against a test function has a well-defined 0-trace should have been justified here. One can deduce this from the functional calculus formula (11.22). It is also not immediately clear that Θ_X defines a distribution on $\mathbb{R}$. For hyperbolic surfaces this fact becomes from the explicit calculation in Theorem 11.3. In the general case, the

discrete component $\Theta_{\rm d}$ is a smooth, exponentially growing function by (11.21). The continuous component $\Theta_{\rm c}$ is a tempered distribution by Lemmas 11.6 and 11.7.

p. 258: Theorem 11.4

The fact that the sum over resonances,

$$u(t) := \sum_{\zeta \in \mathcal{R}_X} e^{(\zeta - \frac{1}{2})|t|},$$

defines a distribution on $\mathbb{R}\backslash\{0\}$ should be justified. In fact, we can show that $tu(t)$ defines a distribution on $\mathbb{R}$.

For convenience, let us write the resonances in terms of $\mu := i(\zeta - \frac{1}{2})$. For $\phi \in C_0^\infty(\mathbb{R})$, define the pairing

$$(tu, \phi) := \sum_\mu \int_{-\infty}^{\infty} e^{-i\mu|t|} t\phi(t)\, dt.$$

We can integrate by parts three times using $e^{\pm i\mu t} = \mp \frac{i}{\mu}\partial_t e^{\pm i\mu t}$. There is no $O(\mu^{-1})$ boundary term at $t = 0$ because of the extra factor of t, and the $O(\mu^{-2})$ cancels between the integrals over positive and negative t. We thus have

$$\int_{-\infty}^{\infty} e^{-i\mu|t|} t\phi(t)\, dt = \frac{4i}{\mu^3}\phi'(0) - \frac{i}{\mu^3}\int_{-\infty}^{\infty} e^{-i\mu|t|}\,\mathrm{sgn}(t)\Big[3\phi''(t) + t\phi^{(3)}(t)\Big]\,dt.$$

Assuming that $\operatorname{supp}\phi \in [-M, M]$ and using the fact that $\operatorname{Re}\zeta < 1$, we can thus estimate

$$|(tu, \phi)| \le Ce^{M/2}\|\phi\|_{C^3} \sum_{\zeta \in \mathcal{R}_X} \frac{1}{|\zeta - \frac{1}{2}|^3}.$$

The sum over ζ is finite by the bound $N_X(r) = O(r^2)$ from Theorem 9.2. This estimate thus shows that tu is well defined as a distribution on $\mathbb{R}$.

p. 259: Lemma 11.6

The distribution $\Theta_{\rm c}$ is not represented by a locally integrable function, as Theorem 11.3 demonstrates, so the left hand side is more properly written as a distributional pairing $(\Theta_{\rm c}, \varphi)$, rather than an integral.

p. 260: Equations (11.22), (11.24), and (11.26)

The resolvent contribution in all of these formulas should be

$$R_X(\tfrac{1}{2} - i\xi) - R_X(\tfrac{1}{2} + i\xi),$$

as in (7.23). Thus (11.22) should read:

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(t)\left[\Pi_{\rm c}\cos\!\Big(t\sqrt{\Delta - \tfrac{1}{4}}\Big)\right] dt \\ = \int_{-\infty}^{\infty} \frac{\xi}{2\pi i}\Big[R_X(\tfrac{1}{2} - i\xi) - R_X(\tfrac{1}{2} + i\xi)\Big]\hat{\varphi}(\xi)\, d\xi.\end{aligned}$$

Similarly, (11.24) should be

$$\int_{-\infty}^{\infty} \varphi(t)\Theta_{\mathrm{c}}(t)\, dt = \frac{1}{4\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi)\Upsilon_X(\tfrac{1}{2}+i\xi)\, d\xi$$
$$+ \lim_{a\to 0}\left(\text{0-tr}\int_{-a}^{a} \frac{\xi}{2\pi i}\Big[R_X(\tfrac{1}{2}-i\xi) - R_X(\tfrac{1}{2}+i\xi)\Big]\hat{\varphi}(\xi)\, d\xi\right).$$

The sign of the Υ_X is correct, because of a factor of $i^2 = -1$. Equation (10.26) should be

$$\text{0-tr}\int_{-a}^{a} \frac{\xi}{2\pi i}\Big[R_X(\tfrac{1}{2}-i\xi) - R_X(\tfrac{1}{2}+i\xi)\Big]\hat{\varphi}(\xi)\, d\xi$$
$$= \operatorname*{FP}_{\varepsilon\to 0} \frac{1}{4\pi}\int_{-a}^{a} A(\tfrac{1}{2}+i\xi,\varepsilon)\hat{\varphi}(\xi)\, d\xi.$$

p. 265: Proof of Theorem 11.4
In the process of preparing the book [74], Dyatlov and Zworski noticed a gap in the proof of the Poisson formula in Guillopé-Zworski [117], which the version printed here is based on. The issue is that the taking the Fourier transform of the expression (11.35) term-by-term is justified only if the sum converges in the topology of $\mathcal{S}'$. This is not necessarily true. (If it were, then the proof of Lemma 11.7 would be much simpler.) A corrected version of the argument for the case of Schrödinger operators in odd dimensions appears in §3.10 of the published version of [74],

> S. Dyatlov and M. Zworski, *Mathematical Theory of Scattering Resonances*, Graduate Studies in Math., **200**, AMS, 2019.

The corrected proof gives a stronger version of the theorem: if (11.19) is multiplied by a factor of t^2, then it holds as a distributional identity on $\mathbb{R}$.

Chapter 14

p. 330: Proof of Proposition 14.6
In the first paragraph, $\partial\mathbb{H}$ should be changed to $\partial\mathbb{B}$.

Appendix

p. 426: Equation (A.20)
Missing absolute value in the denominator:

$$\big(\mathrm{FP}[|x|^{-1}], \varphi\big) := \operatorname*{FP}_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{|x|}\, dx.$$

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