LIST OF ALGEBRAIC GEOMETRY DEFINITIONS AND THEOREMS

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Preface. This note contains a list of definitions used in algebraic geometry, compiled while the author read Ravi Vakil's notes, took courses at Emory University based on Hartshorne's text, and studied for an oral exam in algebraic geometry. The definitions as written here may differ slightly from other common formulations and are listed in *alphabetical* order for ease of access while reading, much like an index. In an effort to make the interdependence of these many definitions more easily decipherable, cross-referencing links are often (but not everywhere) included. All rings are assumed to be commutative with 1.

The following section contains several results, theorems, and constructions which were deemed important by someone who has no authority, and is not complete in any sense. It is also organized alphabetically. Sometimes a proof or a proof sketch is given, sometimes not.

The final section contains a list of practice qualifying exam questions. This grew from a list of questions previously asked, supplied by Prof. David Zureick-Brown of Emory University. It has subsequently been modified and expanded somewhat.

These notes may be incomplete and may contain errors; please consider finding such errors a productive exercise. If you are reading this as you study for your own qualifying exam, study well and good luck!

1. Definitions

abelian category: An **abelian category** is a category $\mathscr{A}$ such that

(i) for all objects A, B, $\mathrm{Hom}(A, B)$ is an abelian group,
(ii) composition distributes: $f \circ (g+h) = (f \circ g) + (f \circ h)$ and $(f+g) \circ h = (f \circ h) + (g \circ h)$,
(iii) biproducts $A \oplus B$ exist (hence so do finite sums),
(iv) for every $f \colon A \to B$, $\ker f$ and $\operatorname{coker} f$ exist,
(v) every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel (the first isomorphism theorem),
(vi) every morphism $f \colon A \to B$ can be factored as $f = g \circ h$ where h is an epimorphism and g is a monomorphism (surjecting onto the image, which injects into the target).

acyclic: Let $\mathscr{A}$ be an abelian category with enough injectives and $F \colon \mathscr{A} \to \mathscr{B}$ a left exact additive functor, i.e. precisely the conditions necessary for the right derived functors R^iF to exist. An object $A \in \mathscr{A}$ is F**-acyclic** (often just "acyclic" when the functor is clear) if the higher right derived functors vanish,

$$R^iF(A) = 0 \text{ for all } i > 0.$$

This also works for left derived functors and F contravariant.

It's worth noting that injective (resp. projective) objects are acyclic, and in fact acyclic resolutions can be used to compute derived functors.

additive functor: A (covariant) functor $F \colon \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}, \mathscr{B}$ are abelian categories, is **additive** if the induced map $\mathrm{Hom}_{\mathscr{A}}(A, A') \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(A'))$ is a homomorphism of abelian groups. That is, $F(f+g) = F(f) + F(g)$ for all $f, g \in \mathrm{Hom}(A, A')$.

adjoint: Suppose $\mathscr{A}$ and $\mathscr{B}$ are categories with functors $F\colon \mathscr{A} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{A}$. F and G are **adjoint** if there is a natural bijection

$$\tau_{AB}\colon \operatorname{Mor}_{\mathscr{B}}(F(A), B) \to \operatorname{Mor}_{\mathscr{A}}(A, F(B))$$

for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. We say F is **left adjoint** and G is **right adjoint**. The adjective "natural" here refers to the fact that if $f\colon A \to A$ is a map in $\mathscr{A}$ then we have

$$\begin{array}{ccc} \operatorname{Mor}_{\mathscr{B}}(F(A), B) & \xrightarrow{\tau_{AB}} & \operatorname{Mor}_{\mathscr{A}}(A, G(B)) \\ {\scriptstyle (Ff)^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \operatorname{Mor}_{\mathscr{B}}(F(A'), B) & \xrightarrow[\tau_{A'B}]{} & \operatorname{Mor}_{\mathscr{A}}(A', G(B)) \end{array}$$

and a similar diagram given a map $g\colon B \to B'$.

affine: A scheme X is **affine** if $X \simeq \operatorname{Spec} A$, as ringed spaces, for some ring A.

A *map* of schemes $\pi\colon X \to Y$ is **affine** if for every open $U \subseteq Y$, $\pi^{-1}(U)$ is an affine open subscheme of X.

affine-local: A property P is said to be **affine-local** if it is sufficient to check it on an affine cover. See local on the ____ for more.

affine space: Classically, **affine** n**-space**, $\mathbb{A}^n_k$, over a field k is the vector space of n-tuples, k^n. As a scheme, we take $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, whose closed points are identified with the classical points when k is algebraically closed.

Over an arbitrary ring, $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \dots, x_n]$. We can extend this to an arbitrary base scheme Y by taking the fiber product, $\mathbb{A}^n_Y = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$.

ample: Let X be a projective (proper is sufficient) A-scheme and $\mathscr{L}$ a line bundle on X. We say $\mathscr{L}$ is **very ample** if there exist $n+1$ sections with no common zero such that the linear system defines a closed embedding $X \hookrightarrow \mathbb{P}^n_A$. Note that very ampleness implies base point freeness.

An equivalent definition (see Exercise 16.6.A) is that $X \simeq \operatorname{Proj} S_\bullet$ where $S_\bullet$ is a finitely generated graded ring over A and $\mathscr{L} \simeq \mathscr{O}_{\operatorname{Proj} S_\bullet}(1)$. If $\pi\colon X \to \operatorname{Spec} A$, we may refer to this as π**-very ample**.

A line bundle $\mathscr{L}$ on X is **ample** if $\mathscr{L}^{\otimes n}$ is very ample for some $n > 0$. This is equivalent to $\mathscr{L}^{\otimes n}$ very ample for *all* $n \gg 0$.

base point: Let X be a k-scheme and $\mathscr{L}$ a line bundle on X. A **base point** $P \in X$ of $\mathscr{L}$ (or of a linear system V) is defined to be a point on which all elements of $\Gamma(X, \mathscr{L})$ vanish. The **base locus** of $\mathscr{L}$ is defined to be the scheme-theoretic intersection of the vanishing loci of each element of $\Gamma(X, \mathscr{L})$.

We say $\mathscr{L}$ is **base point free** if it has no base points, and define the **base point free locus** as the complement of the base locus in X. It is useful to note that base point freeness of $\mathscr{L}$ is equivalent to $\mathscr{L}$ being globally generated.

The utility of this is that an $n+1$ dimensional subset of $\Gamma(X, \mathscr{L})$ defines a morphism $X - \{\text{base locus}\} \to \mathbb{P}^n_k$. This map is given by evaluating (a basis of) the sections at P to obtain and $(n+1)$-tuple not all zero.

canonical bundle/sheaf: Suppose X is a smooth k-variety of dimension n. The **canonical sheaf** $\mathscr{K}_X$ is taken to be the top wedge power of $\Omega_{X/k}$,

$$\mathscr{K}_X = \wedge^n \Omega_{X/k} = \det \Omega_{X/k},$$

which is an invertible sheaf, i.e. a line bundle, on X. (See also exterior algebra, determinant, (co)tangent sheaf.)

When X is projective (proper is sufficient), $\mathscr{K}_X$ is a dualizing sheaf. This is particularly useful in computing the genus, and other cohomological properties. The canonical sheaf/divisor also plays a central role in Riemann–Roch and Riemann–Hurwitz.

Cartier divisor: Let $(X, \mathscr{O}_X)$ be a scheme with sheaf of total quotients $\mathscr{K}$. A **Cartier divisor** on X is a global section of the sheaf $\mathscr{K}^\times/\mathscr{O}_X^\times$. This is equivalent to giving an open cover $X = \cup U_i$ and elements $f_i \in \Gamma(U_i, \mathscr{K}^\times)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j), \mathscr{O}_X^\times)$.

Another characterization of Cartier divisors is to take an effective divisor to be a closed subscheme $D \hookrightarrow X$ such that the ideal sheaf of D is an invertible sheaf on X. The sum of such divisors is seen to be the product of ideal sheaves.

A Cartier divisor is **principal** if it is in the image of the map $\Gamma(X, \mathscr{K}^\times) \to \Gamma(X, \mathscr{K}^\times/\mathscr{O}_X^\times)$. We can define an equivalence relation on Cartier divisors by $D \simeq D'$ if and only if $D - D'$ is a principal divisor. Then we define the **Cartier divisor class group** $\operatorname{CaCl} X$ to be the Cartier divisors mod this equivalence relation.

category of open sets: Let X be a topological space. The **category of open sets on** X is a category whose objects are the open sets of X and whose morphisms are inclusions $V \subseteq U$.

Čech cohomology: Let $\mathscr{F}$ be a sheaf of abelian groups on a quasicompact, separated scheme X with $X = \cup U_i$ a finite open cover. The **Cech complex** of $\mathscr{F}$ with respect to the cover is

$$0 \to \prod_i \Gamma(U_i, \mathscr{F}) \to \prod_{i<j} \Gamma(U_i \cap U_j, \mathscr{F}) \to \cdots .$$

The maps in the complex above are given by plus or minus the restriction map, with the sign determined by the position of the additional index. If I is a (finite) subset and $i \in I$ is at position k, then the map is given by

$$(-1)^k \operatorname{res}_{U_{I-\{i\}}, U_I} .$$

This is necessary to ensure that it is a complex.

The ***i*-th Cech cohomology** of $\mathscr{F}$ with respect to the chosen cover is the cohomology of the complex, denoted $H^i(X, \mathscr{F})$. It is a nontrivial fact that this does not depend on the chosen cover (see Theorem 18.2.2).

closed: A continuous map $\pi\colon X \to Y$ is **closed** if $\pi(K)$ is closed for all closed subsets $K \subseteq X$. π is said to be **universally closed** if for all $Z \to Y$, the induced map $X \times_Y Z \to Z$ is closed.

In a topological space X, we say a subset C is **locally closed** if it the intersection of an open and closed subset. This is equivalent to C being closed in an open subset A with the subspace topology, or C being open in a closed subset B with the subspace topology.

closed embedding: An affine morphism of schemes $\pi\colon X \to Y$ is a **closed embedding** if for every open $\operatorname{Spec} B \subseteq Y$ with preimage $\pi^{-1}(\operatorname{Spec} B) \simeq \operatorname{Spec} A$, the induced map $B \to A$ is a surjection. Hence $A \simeq B/I$ for an ideal I.

coherent: A finitely generated A-module M is **coherent** if for any map (not necessarily surjective) $A^n \to M$ the kernel is finitely generated. Note that coherent implies finitely presented which further implies finitely generated. When A is Noetherian, these three notions coincide.

A sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is **coherent** if it is quasicoherent and locally we have $\mathscr{F}|_{\operatorname{Spec} A} \simeq \widetilde{M}$ for a coherent A-module M. The coherent sheaves on X form a sub abelian category $\operatorname{Coh}_X \subset \operatorname{QCoh}_X$.

cohomology: Let $A^\bullet$ be a complex in an abelian category $\mathscr{A}$. The i^{th} **cohomology** of $A^\bullet$ is $\ker \delta^i / \operatorname{im} \delta^{i-1}$, denoted $h^i(A^\bullet)$.

Given a map of complexes $f\colon A^\bullet \to B^\bullet$, one can check that there is an induced map on cohomology, $h^i(A^\bullet) \to h^i(B^\bullet)$. Moreover, if $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence, then the snake lemma induces a map $\delta^i\colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$, which gives a long exact sequence

$$\cdots \to h^{i-1}(C^\bullet) \xrightarrow{\delta^{i-1}} h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet) \to \cdots .$$

cokernel: The **presheaf cokernel** of a map $\phi\colon \mathscr{F} \to G$ of (pre)sheaves on X is the presheaf given by $(\operatorname{coker}\phi)(U) = \operatorname{coker}\phi(U)$. It satisfies the universal property of cokernels in the category of presheaves on X.

When $\mathscr{F}$ and $\mathscr{G}$ are *sheaves*, the presheaf cokernel is not in general a sheaf. We define the **cokernel** of ϕ as the sheafification of the presheaf cokernel.

complex: In an abelian category $\mathscr{A}$, a **complex** $A^\bullet$ is a sequence of objects A^i for $i \in \mathbb{Z}$ along with maps $\delta^i\colon A^i \to A^{i+1}$ such that $\delta^{i+1} \circ \delta^i = 0$ for all i. This is equivalent to saying that the $\operatorname{im}\delta^i \subseteq \ker \delta^{i+1}$.

A **morphism of complexes** $f\colon A^\bullet \to B^\bullet$ is a collection of maps $f^i\colon A^i \to B^i$ which commutes with the δ^i maps. We write this as $\delta^i \circ f^i = f^{i+1} \circ \delta^i$, without distinguishing between the boundary maps on $A^\bullet$ and $B^\bullet$.

condition (∗): Let X be a scheme. We say X **satisfies condition (∗)** if X is Noetherian, integral, separated, and regular in codimension one.

constructible: The family of **constructible sets** within a Noetherian topological space is the smallest family containing (i) all the open sets, (ii) finite intersections, and (iii) complements. This turns out to be equivalent to finite disjoint unions of locally closed subsets.

degenerate: An embedding $X \to \mathbb{P}^n$ — and equivalently a line bundle $\mathscr{L}$ or linear system $|V|$ on X — is **degenerate** if the image of X is contained in a hyperplane.

degree: There are several definitions of degree, in various contexts, listed below. They all agree with each other when they should.

(*Degree of a point*) If X is locally finite type scheme and $p \in X$ is a closed point, the degree of the function field $\kappa(p) \simeq \mathscr{O}_{X,p}/\mathfrak{m}_p$ as an extension of k is called the **degree of** p.

(*Degree of a map*) Let $\pi\colon X \to Y$ be a map of schemes and $p \in Y$ a point. The **degree of** π **at** p is defined to be the $\kappa(p)$-rank of the fiber of $\pi_*\mathscr{O}_X$ at p. That is, it is the dimension of $(\pi_*\mathscr{O}_X) \otimes \kappa(p)$ as a $\kappa(p)$-vector space. After untangling this definition, it is seen to be equal to the dimension (again as a $\kappa(p)$-vector space) of the space of functions on the fiber above p.

When X is a curve with no embedded points (e.g. reduced), Y is a regular curve and π is finite, one finds that $\pi_*\mathscr{O}_X$ is locally free of finite rank. We take this rank to be the **degree of** π, with no reference to a point needed.

(*Degree of a divisor*) If $D = \sum n_i p_i$ is a Weil divisor on a regular projective curve C/k, the **degree of** D is

$$\deg D = \sum n_i \deg p_i,$$

where the degree of a point is given above.

(*Degree of a line bundle*) If $\mathscr{L}$ is an invertible sheaf on a projective curve C/k, the **degree of** $\mathscr{L}$ is

$$\deg_C \mathscr{L} = \chi(C, \mathscr{L}) - \chi(C, \mathscr{O}_C),$$

where χ denotes the Euler characteristic. Importantly, when $D = \operatorname{div} s$ for a section of $\mathscr{L}$, i.e. $\mathscr{L} \simeq \mathscr{O}(D)$, we have the agreement that $\deg D = \deg_C \mathscr{L}$. This also plays nicely with pullback in that if $\pi\colon C' \to C$ is a finite map, then

$$\deg_{C'} \pi^* \mathscr{L} = (\deg \pi)(\deg_C \mathscr{L}).$$

(*Degree of a coherent sheaf*) If $\mathscr{F}$ is a coherent sheaf on an integral projective curve C, then the **degree of** $\mathscr{F}$ is

$$\deg \mathscr{F} = \chi(C, \mathscr{F}) - (\operatorname{rank} \mathscr{F})\chi(C, \mathscr{O}_C).$$

This is easily seen to agree with the degree of a line bundle when rank $\mathscr{F} = 1$.

(*Degree via hyperplanes*) If $X \hookrightarrow \mathbb{P}^n$ is a projective k-variety, we define the degree by intersecting X with $\dim X$ hyperplanes in general position and counting points (which will not have multiplicity due to the general position).

(*Degree via Hilbert polynomials*) If $i\colon X \hookrightarrow \mathbb{P}^n$ is a projective k-scheme, we define the **degree of** X in terms of the Hilbert polynomial $p_X(m)$. Namely, the degree is $(\dim X)!$ times the coefficient of the $m^{\dim X}$ term.

Note that this is not an intrinsic invariant, and depends on the embedding. For example, $\mathbb{P}^1$ can have degree 1 when embedded into $\mathbb{P}^2$ as a hyperplane. The rational normal curve in $\mathbb{P}^n$, however, is seen to have degree n.

δ-functor: Let $\mathscr{A}, \mathscr{B}$ be abelian categories. A (covariant) **δ-functor** from $\mathscr{A}$ to $\mathscr{B}$ is a collection of functors $T = (T^i)_{i \geq 0}$ along with a boundary morphism $\delta^i\colon T^i(A'') \to T^{i+1}(A')$ for every short exact sequence $0 \to A' \to A \to A'' \to 0$ such that

(i) For each short exact sequence, there is a long exact sequence

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^0} T^1(A') \to T^1(A) \to \cdots$$

(ii) for each morphism f of short exact sequences $A^\bullet \to B^\bullet$ the map commutes with the δ^i, in that $f \circ \delta^i = \delta^i \circ f\colon T^i(A'') \to T^{i+1}(B')$.

derived functors: Let $\mathscr{A}$ be an abelian category with enough injectives, and $F\colon \mathscr{A} \to \mathscr{B}$ be an additive, covariant, left exact functor to another abelian category $\mathscr{B}$. The **right derived functors** $R^iF\colon \mathscr{A} \to \mathscr{B}$ are defined as

$$R^iF(A) = h^i(F(I^\bullet))$$

where $A \to I^\bullet$ is any injective resolution of A.

If F is instead a right exact functor, and $\mathscr{A}$ has enough projectives, one can define the **left derived functors** L^iF similarly by *projective* resolutions,

$$L^iF(A) = h^i(F(P^\bullet))$$

where $P^\bullet \to A$ is any projective resolution.

If F is contravariant, these definitions can be modified to make sense, or one can view F as a covariant functor from $\mathscr{A}^{op}$ to $\mathscr{B}$ and define the derived functors there.

determinant: If $\mathscr{F}$ is a locally free sheaf of rank r, then the top wedge power $\wedge^r \mathscr{F}$ is known as the **determinant**, denoted $\det \mathscr{F}$, which is a locally free sheaf of rank 1. This is a calculation that can be done locally, on the r-th exterior power $\wedge^r A^r$. Possibly the most famous determinant sheaf is the canonical bundle/sheaf, which is $\det \Omega_{X/k}$.

A useful property of determinant sheaves is that if $0 \to \mathscr{F}_1 \to \cdots \to \mathscr{F}_n \to 0$ is an exact sequence of locally free sheaves on X, then

$$\det \mathscr{F}_1 \otimes (\det \mathscr{F}_2)^\vee \otimes \det \mathscr{F}_3 \otimes \cdots \otimes (\det \mathscr{F}_n)^{\vee^{n+1}} \simeq \mathscr{O}_X.$$

diagonal: Let $\pi\colon X \to Y$ be a map. The **diagonal morphism** δ_π is the unique map $X \to X \times_Y X$ induced by mapping $X \to X$ identically to both factors. Note that this makes sense in any category that fiber products do, namely schemes, topological spaces, and sets. In the category of schemes it is a locally closed embedding.

differential: Let $B \to A$ be a map of rings. The **module of (relative or Kähler) differentials** is an A-module $\Omega_{A/B}$ generated by symbols da for $a \in A$ with several relations. It can be viewed as coming from a B-linear "differential" map $d\colon A \to \Omega_{A/B}$ satisfying
- $d(a + a') = da + da'$,
- $d(aa') = ada' + a'da$,
- $db = 0$ for all $b \in \operatorname{im} B$.

direct image: Let $\pi\colon X \to Y$ be a map of schemes and $\mathscr{F}$ an $\mathscr{O}_X$-module. The direct image $\pi_*\mathscr{F}$ naturally has the structure of an $\pi_*\mathscr{O}_X$-module. Since $\pi^\sharp\colon \mathscr{O}_Y \to \pi_*\mathscr{O}_X$, this naturally endows $\pi_*\mathscr{F}$ with the structure of an $\mathscr{O}_Y$-module. We call $\pi_*\mathscr{F}$ with the $\mathscr{O}_Y$-module structure the **direct image** of $\mathscr{F}$ by π.

divisor class group: Let X satisfy $(*)$. We define an equivalence relation on $\operatorname{Div} X$ by $D \sim D'$ if and only if $D - D' = (f)$ for some $f \in K^\times$. The **divisor class group** of X, denoted $\operatorname{Cl}(X)$, is $\operatorname{Div} X/\sim$.

dominant: A rational map (or morphism) $\pi\colon X \dashrightarrow Y$ is **dominant** if there is some open $U \subseteq X$ for which $\pi\colon U \to Y$ has dense image.

dualizing sheaf: Let X be a projective (or proper) scheme over a field k of dimension n. A **dualizing sheaf** for X is a coherent sheaf ω on X, together with a trace map $\operatorname{Tr}\colon H^n(X,\omega) \to k$ such that for any coherent sheaf $\mathscr{F}$ on X we have the pairing

$$\operatorname{Hom}(\mathscr{F},\omega) \times H^n(X,\mathscr{F}) \to H^n(X,\omega) \xrightarrow{\operatorname{Tr}} k$$

induces an isomorphism

$$\operatorname{Hom}(\mathscr{F},\omega) \xrightarrow{\sim} H^n(X,\mathscr{F}))^\vee.$$

The existence of a dualizing sheaf is part of the proof of Serre duality.

effaceable: An additive functor $:\mathscr{A} \to \mathscr{B}$ is **effaceable** if for each object A, there exists a monomorphism $u\colon A \to M$ such that $F(u) = 0$. We say F is **coeffaceable** if there exists an epimorphism $u\colon P \to A$ such that $F(u) = 0$.

étale: A morphism of schemes $\pi\colon X \to Y$ is **étale** if it is smooth of relative dimension 0. This is similar to the notion of a "local isomorphism" (nearly a covering space), or a combination of smooth and unramified.

Euler characteristic: The **Euler characteristic** is generally an alternating sum of dimensions of cohomology vector spaces,

$$\chi = \sum_{i=0}^{\infty} (-1)^i h^i.$$

For example, suppose X is a projective k-scheme, $\mathscr{F}$ is a coherent sheaf on X, and $H^i(X,\mathscr{F})$ denotes the i-th sheaf cohomology (or Čech cohomology) group, which is a k-vector space. Then

$$\chi(X,\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X,\mathscr{F}).$$

This is well defined because for all $i \gg 0$ the i-th cohomology vanishes, and it is invariant upon field extension, i.e. if K/k is a field extension, we get the same number if we compute $\chi(X_K,\pi^*\mathscr{F})$, where $X_K = X \times_k \operatorname{Spec} K$ and $\pi\colon X_K \to X$ is the natural map.

exact: In an abelian category $\mathscr{A}$, a sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** at B if $\ker g = \operatorname{im} f$. We say a sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a **short exact sequence** if f is injective, g is surjective, and $\ker g = \operatorname{im} f$.

An additive functor $F\colon \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}, \mathscr{B}$ are abelian categories, is **left exact** if for all short exact sequences $0 \to A \to B \to C \to 0$, we have

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. Similarly, we call F **right exact** if for all short exact sequences,

$$F(A) \to F(B) \to F(C) \to 0$$

is exact. We call a functor simply **exact** if it is both left and right exact, in which case for all short exact sequences,

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact.

ext: There are two ways to define the **Ext functors/modules**, which are seen to agree, using a spectral sequence argument. First, given an A-module M, we define $\operatorname{Ext}^i_A(M, N)$ to be the right derived functors of $\operatorname{Hom}_A(M, \cdot)$. This gives the usual natural long exact sequence of Ext modules.

Alternatively, for an A-module N, we could take the right derived functors of the (contravariant) left exact functor $\operatorname{Hom}_A(\cdot, N)$ to be $\operatorname{Ext}^i_A(\cdot, N)$. This also gives a natural long exact sequence of Ext modules.

extension by zero: Let $i\colon U \hookrightarrow X$ be an inclusion of an open set. We define a functor $i_!$ called the **extension of i by zero** from $\mathscr{O}|_U$-modules to $\mathscr{O}_X$-modules by the sheafification of the presheaf

$$V \mapsto \begin{cases} \mathscr{F}(V) & \text{if } V \subseteq U \\ 0 & \text{if } V \not\subseteq U \end{cases}.$$

$i_!$ is an exact functor and a left adjoint to the inverse image i^{-1}. This is useful, as it implies i^{-1} is exact in this setting, since it is both a right adjoint (of $i_!$) and a left adjoint (of i_*).

exterior algebra: Let M be an A-module. The **n-th exterior (or wedge) power** $\wedge^n M$ is the quotient of $T^n M$ by the ideal generated by elements of the form $(m_1 \otimes \cdots \otimes m_n)$ where $m_i = m_j$ for some $i \neq j$. Note that this implies $m_1 \otimes m_2 = -(m_2 \otimes m_1)$, and more generally $(m_1 \otimes \cdots \otimes m_n) = (-1)^{\operatorname{sgn} \sigma}(m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)})$.

We can form the graded ring $\wedge^\bullet M$, which satisfies a universal property: if $\phi\colon M \to B$ is a map of A-modules to an A-algebra B such that $(\phi(m))^2 = 0$ for all $m \in M$, then ϕ extends to a unique map $\wedge^\bullet M \to B$.

If $\mathscr{F}$ is a quasicoherent sheaf on X, we can define $\wedge^n \mathscr{F}$ by looking locally: if $\mathscr{F}|_{\operatorname{Spec} A} \simeq M$ we define $\wedge^n \mathscr{F}|_{\operatorname{Spec} A} = \wedge^n M$. One must check that this construction glues.

If $M \simeq A^{\oplus m}$ is a free module of rank m, we have $\wedge^n(A^m) \simeq A^{\binom{m}{n}}$. This can be seen by recognizing that a basis for $\wedge^n(A^m)$ consists of basis elements $e_1 \otimes \cdots \otimes e_n$ where $i < j$, since if any $i = j$ we get zero and we can always reorder by multiplying by ± 1. If $\mathscr{F}$ is a locally free sheaf of rank m, this implies $\wedge^n \mathscr{F}$ has rank $\binom{m}{m}$. In particular, this shows that the top wedge power $\wedge^m \mathscr{F}$ has rank 1. This is called the determinant sheaf.

factorial: A scheme X is **factorial** if all stalks $\mathscr{O}_{X,p}$ are unique factorization domains. Note that this implies X is normal scheme, since UFDs are integrally closed domains. Since localizations of UFDs are UFDs, we also have $\operatorname{Spec} A$ is factorial whenever A is a UFD.

faithful: A (covariant) functor $F\colon \mathcal{A} \to \mathcal{B}$ is **faithful** if for all pairs of objects $A, A' \in \mathcal{A}$ the induced map

$$\operatorname{Mor}_{\mathcal{A}}(A, A') \to \operatorname{Mor}_{\mathcal{B}}(F(A), F(A'))$$

is injective.

faithfully flat: An A-module N is **faithfully flat** if any complex of A-modules

$$M' \to M \to M''$$

is exact if and only if

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N$$

is also exact. Note that flatness implies only one direction of this equivalence. As with flatness, an A-algebra B is **faithfully flat** if B is faithfully flat as an A-module.

A morphism of schemes $\pi\colon X \to Y$ is **faithfully flat** if it is flat and surjective. A map of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat if and only if A is faithfully flat as a B-module, as should be the case.

fiber: Let $\pi\colon X \to Z$ be a map of schemes, and let p be a point of Z with residue field k. Let i be the inclusion $\operatorname{Spec} k \to Z$, with the natural isomorphism of residue fields at p. The **fiber of** π **above** p is the fiber product $\pi^{-1}(p) = \operatorname{Spec} k \times_Z X$.

If Z is irreducible, the fiber above the generic point is called the **generic fiber** of π.

fiber product: Let X, Y be schemes over Z. The **fiber(ed) product**, $X \times_Z Y$, is the scheme satisfying the universal property of fiber products. That is, if W is a scheme with maps to X, Y that agree after composition to Z, there is a unique map $W \to X \times_Z Y$ that is compatible with the others.

finite: If B is a ring and A is a B-algebra, we say A is **finite** if it is finitely generated as a B-module. Note that this is stronger than being finitely generated as a B-algebra.

We say a morphism of schemes $\pi\colon X \to Y$ is **finite** if for every affine open $\operatorname{Spec} B$ of Y, $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$, where A is a finite B algebra. Note that by definition, finite maps are affine.

finite type: A morphism of schemes $\pi\colon X \to Y$ is **locally of finite type** if for every affine open $\operatorname{Spec} B \subseteq Y$, and every affine open of the preimage $\operatorname{Spec} A \subseteq \pi^{-1}(\operatorname{Spec} B)$, the induced ring map $B \to A$ makes A a finitely generated B-algebra. This is equivalent to $\pi^{-1}(\operatorname{Spec} B)$ having a cover by affine opens $\operatorname{Spec} A_i \subseteq X$ for which each A_i is a finitely generated B-algebra.

We say π is **of finite type** if it is locally of finite type and quasicompact. This is equivalent to being able to cover $\pi^{-1}(\operatorname{Spec} B)$ by *finitely many* affine opens $\operatorname{Spec} A_i$.

finitely presented: A ring A is a **finitely presented** B-algebra if A is isomorphic to

$$B[x_1, ..., x_n]/(r_1(x_1, ..., x_n), ..., r_m(x_1, ..., x_n)),$$

that is it is a finitely generated B algebra with finitely many relations. Note that if B is Noetherian, this is the same as finitely generated.

Consequently, a morphism of schemes $\pi\colon X \to Y$ is **locally of finite presentation** if for each affine open $\operatorname{Spec} B \subseteq Y$ we have $\pi^{-1}(\operatorname{Spec} B)$ is covered by $\operatorname{Spec} A_i$ where each A_i is a finitely presented B-algebra. We drop the locally and say π is of **finite presentation** if it is qcqs also. Note that if Y is locally Noetherian, then locally of finite presentation is equivalent to locally of finite type.

A related notion is that of a **finitely presented A-module**. We say M is finitely presented if there is an exact sequence $A^q \to A^p \twoheadrightarrow M \to 0$. That is, M is finitely generated with "finitely generated relations."

flasque: A sheaf $\mathscr{F}$ on X is **flasque** if for every $V \subseteq U$, the restriction map $\text{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

Flasque sheaves are acyclic in sheaf cohomology.

flat: An A-module N is **flat** if the functor $\cdot \otimes_A N$ is exact. A priori it is right exact, so the content of this definition is that $\cdot \otimes_A N$ is left exact; this is equivalent to for every injection $M' \hookrightarrow M$, we have $M' \otimes_A N \hookrightarrow M' \otimes_A N$.

A quasicoherent sheaf $\mathscr{F}$ on a scheme X is **flat at a point** p if the stalk $\mathscr{F}_p$ is a flat $\mathscr{O}_{X,p}$-module. If $\mathscr{F}$ is flat at all $p \in X$, we call $\mathscr{F}$ **flat**. On an affine scheme Spec A, flatness of $\widetilde{M}$ at stalks is equivalent to flatness of M as an A-module (see e.g. Proposition 24.2.3).

A map of schemes $\pi\colon X \to Y$ is **flat at** $p \in X$ if $\mathscr{O}_{X,p}$ is flat as an $\mathscr{O}_{Y,\pi(p)}$-module. Similarly, if π is flat at all $p \in X$, we call it **flat**.

An equivalent formulation is to call $\pi\colon X \to Y$ is **flat** if the pullback functor π^* from quasicoherent sheaves on Y to quasicoherent sheaves on X is exact. This can seen to be equivalent to the above definition because we can check exactness on stalks, and via intermediate results about flatness (Exercise 24.2.C I think).

free: We say an $\mathscr{O}_X$-module $\mathscr{F}$ is **free** if $\mathscr{F} \simeq \oplus_I \mathscr{O}_X$ for some collection I. As usual, the **rank** is the number of copies of $\mathscr{O}_X$ in the direct sum.

If X can be covered by open sets U_i such that $\mathscr{F}|_{U_i}$ is free for all i, then we say $\mathscr{F}$ is **locally free**. Here the rank is only defined on open sets, but if X is connected, then the rank is the same everywhere.

functor of points: Let X be an object in a category $\mathcal{C}$. The **functor of points**, denoted h_X, is a contravariant functor $\mathcal{C} \to$ Sets given by $h_X(T) = \text{Mor}(T, X)$. In the case where $\mathcal{C}$ is the category of schemes, notice that $h_X(T) = X(T)$ are the T-points of X.

full: A (covariant) functor $F\colon \mathcal{A} \to \mathcal{B}$ is **full** if for all pairs of objects $A, A' \in \mathcal{A}$ the induced map

$$\text{Mor}_{\mathcal{A}}(A, A') \to \text{Mor}_{\mathcal{B}}(F(A), F(A'))$$

is surjective.

We say a subcategory $\mathcal{A}'$ of $\mathcal{A}$ is **full** if the inclusion functor is full. One can think of this as the subcategory having possibly fewer objects, but never losing morphisms between objects.

fully faithful: A (covariant) functor $F\colon \mathcal{A} \to \mathcal{B}$ is **fully faithful** if it is both full and faithful. That is, for all pairs of objects $A, A' \in \mathcal{A}$ the induced map

$$\text{Mor}_{\mathcal{A}}(A, A') \to \text{Mor}_{\mathcal{B}}(F(A), F(A'))$$

is a bijection.

general: Something is said to be **general** if it holds in some dense open set. A **general point** of a scheme X has a certain property if there exists a (dense) open set $U \subseteq X$ such that all points $p \in U$ have the property.

Similarly, the **general fiber** of a map $\pi\colon X \to Y$ has a certain property if there is a dense open neighborhood $U \subseteq Y$ for which the fiber of π over $p \in U$ has the property for all $p \in U$.

generic: Let X be an irreducible scheme. The **generic point** of X is a point $\eta \in X$ for which $\overline{\{\eta\}} = X$. We can naturally also talk about generic points for irreducible closed subschemes of a scheme X.

Let $\pi\colon X \to Y$ be a map of schemes. The **generic fiber** of π is the fiber over the generic point $\eta \in Y$.

genus: The **geometric genus** of a projective scheme X is defined to be the dimension of the regular sections of the canonical bundle/sheaf, $h^0(X, \omega_X)$. By Serre duality, this is equivalent to $h^n(X, \mathscr{O}_X)$, where $n = \dim X$.

The **arithmetic genus** $p_a(X)$ is defined in terms of the Euler characteristic of X,

$$p_a(X) = (-1)^n \left(\chi(\mathscr{O}_X) - 1\right),$$

where again $n = \dim X$. In particular we have when $n = 1$, i.e. X is a curve,

$$p_a(X) = 1 - \chi(\mathscr{O}_X).$$

In good circumstances — when X is smooth and projective — the arithmetic and geometric genus coincide. When X is further defined over the field of complex numbers, this number coincides with the topological definition of genus coming from (singular) cohomology.

geometric fiber: A **geometric fiber** of a morphism $\pi\colon X \to Y$ is defined to be the fiber over a geometric point. That is, it is the pullback $\pi^{-1}(p) = X \times_Y \operatorname{Spec} k$ of a geometric point $p\colon \operatorname{Spec} k \to Y$.

geometric point: A **geometric point** of a scheme X is a morphism $\operatorname{Spec} k \to X$ where $k = \overline{k}$ is an algebraically closed field.

geometrically: A morphism $\pi\colon X \to Y$ is **geometrically** ____ if all of every geometric fiber has property ____. Such properties could include connected, irreducible, integral, or reduced.

A k-scheme X is said to be **geometrically** ____ if the structure morphism has geometrically ____ fibers.

germ: If $\mathscr{F}$ is a sheaf on X, a **germ** is an element of a stalk $\mathscr{F}_p$, essentially a local function at a point p. We describe this constructively as the equivalence class of a pair (U, s), where $p \in U$ and $s \in \mathscr{F}(U)$, and the equivalence is given by $(U, s) \sim (V, t)$ if and only if there exists $W \subseteq U \cap V$ such that $s|_W = t|_W$.

globally generated: An $\mathscr{O}_X$-module $\mathscr{F}$ on X is **globally generated** (or **generated by global sections**) if for some index set I we have a surjection

$$\mathscr{O}_X^{\oplus I} \twoheadrightarrow \mathscr{F}.$$

Given such a map, the generators of $\mathscr{F}$ are the images of 1. If I is a finite set, we might say $\mathscr{F}$ is **finitely globally generated**.

If $p \in X$ is a point, we say that $\mathscr{F}$ is **globally generated at** p if there is a morphism $\mathscr{O}_X^{\oplus I} \to \mathscr{F}$ (not necessarily a surjection) such that the induced morphism on stalks is a surjection:

$$\mathscr{O}_{X,p}^{\oplus I} \twoheadrightarrow \mathscr{F}_p.$$

Similarly, if I is finite then we say $\mathscr{F}$ is **finitely globally generated at** p.

A line bundle $\mathscr{L}$ on X is globally generated if and only if it is base point free.

graded ring: A ($\mathbb{Z}$-)**graded ring** is a ring $S_\bullet = \oplus_{n \in \mathbb{Z}} S_n$, where multiplication acts by a map $S_m \times S_n \to S_{m+n}$. Sometimes the grading is assumed to be only for positive integers, i.e. $S_n = 0$ for $n < 0$. The elements of S_n are the **homogeneous elements of degree** n.

$S_\bullet$ is a **graded ring over** S_0, where S_0 is called the **base ring**. Note that $S_\bullet$ is an S_0-algebra, and S_n is an S_0-module for each n. The ideal $I = \oplus_{n\geq 1} S_n$ is called the **irrelevant ideal**.

We say $S_\bullet$ is a **finitely generated graded ring over** S_0 if the irrelevant ideal is finitely generated. We say $S_\bullet$ is **generated in degree 1** if $S_\bullet$ is generated by S_1 as an S_0-algebra.

homogeneous: A **homogeneous element** of graded ring $S_\bullet$ is contained in some S_n.

An ideal $I \subseteq S_\bullet$ is called **homogeneous** if it is generated by homogeneous elements.

Hilbert function: Let $\mathscr{F}$ be a coherent sheaf on a projective k-scheme $X \subset \mathbb{P}^n_k$. The **Hilbert function** of $\mathscr{F}$ is

$$h_{\mathscr{F}}(m) = h^0(X, \mathscr{F}(m)).$$

For $m \gg 0$ this agrees with the Euler characteristic of $\mathscr{F}$; see Hilbert polynomials.

Hilbert polynomial: Let $\mathscr{F}$ be a coherent sheaf on a projective k-scheme $X \subset \mathbb{P}^n_k$. The **Hilbert polynomial of** $\mathscr{F}$ $p_{\mathscr{F}}(m)$, defined by

$$p_{\mathscr{F}}(m) = \chi(X, \mathscr{F}(m))$$

is a polynomial in m. The **Hilbert polynomial of** X is given by the structure sheaf,

$$p_X(m) = p_{\mathscr{O}_X}(m) = \chi(X, \mathscr{O}_X(m)).$$

homotopic: Let $f, g\colon A^\bullet \to B^\bullet$ be morphisms of complexes. We say f and g are **homotopic**, denoted $f \sim g$, if there exist maps $h^i\colon A^i \to B^{i-1}$ such that $f^i - g^i = \delta^{i-1}h^i + h^{i+1}\delta^i$ for all i. Note that homotopic maps induce *the same maps on cohomologies* $h^i(A^\bullet) \to h^i(B^\bullet)$.

ideal sheaf: If Y is a scheme with structure sheaf $\mathscr{O}_Y$, an **ideal sheaf** $\mathscr{I}$ is a sub-$\mathscr{O}_Y$-module. That is, for each open U, $\mathscr{I}(U)$ is an ideal of $\mathscr{O}_Y(U)$.

In particular, given a closed embedding $X \to Y$, we can define $\mathscr{I}_{X/Y}(U)$ to be the kernel of the surjection $\mathscr{O}_Y(U) \to \pi_*\mathscr{O}_X(U)$. This gives an exact sequence

$$0 \to \mathscr{I}_{X/Y} \to \mathscr{O}_Y \to \pi_*\mathscr{O}_X \to 0$$

of $\mathscr{O}_Y$-modules.

image: The **presheaf image** of a map $\phi\colon \mathscr{F} \to G$ of (pre)sheaves on X is the presheaf given by $(\operatorname{im}\phi)(U) = \operatorname{im}\phi(U)$. It satisfies the universal property of images in the category of presheaves on X.

When $\mathscr{F}$ and $\mathscr{G}$ are *sheaves*, the presheaf image is not in general a sheaf. We define the **image** of ϕ as the sheafification of the presheaf image.

For a map of schemes, we have the notion of scheme-theoretic image.

injective: An object I in an abelian category $\mathscr{A}$ is **injective** if $\operatorname{Hom}(\cdot, I)$ is an exact (contravariant) functor from $\mathscr{A}$ to abelian groups.

An *injective resolution* of an object A is a complex $I^\bullet$ with a map $A \to I^0$ such that

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is exact.

If every object in $\mathscr{A}$ is isomorphic to a subobject of an injective object, then we say the category $\mathscr{A}$ has **enough injectives**.

integral: A morphism of schemes $\pi\colon X \to Y$ is **integral** if π is affine and for every open $\operatorname{Spec} B \subseteq Y$ we have $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ and the induced map $B \to A$ is an integral ring morphism.

We say a ring morphism $\phi\colon B \to A$ is **integral** if every element of a is the root of a monic polynomial with coefficients in $\phi(B)$. In the special case that ϕ is an inclusion ($B \subseteq A$) then we call A an *integral extension*.

integrally closed: Let A be an integral domain with fraction field $K(A)$. We say A is **integrally closed** if for all monic polynomials $f(x) \in A[x]$, if $\alpha \in K(A)$ is a root of $f(x)$, then $\alpha \in A$.

inverse image: Let $\pi\colon X \to Y$ be a continuous map of topological spaces and $\mathscr{G}$ a sheaf on Y. The **inverse image functor** from sheaves on Y to sheaves on X is defined to be the left adjoint of pushforward. That is, there is a natural bijection

$$\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$$

functorial in both arguments. More explicitly, we can describe a presheaf

$$(\pi^{-1,\mathrm{pre}}\mathscr{G})(U) = \varinjlim_{V \supset \pi(U)} \mathscr{G}(V)$$

and take $\pi^{-1}\mathscr{G}$ to be the sheafification of this presheaf on X. One then checks that this satisfies the desired adjointness property. Since left adjoints preserve colimits, we find that if $\pi(p) = q$ then

$$\left(\pi^{-1}\mathscr{G}\right)_p = \mathscr{G}_q.$$

Note that the inverse image $\pi^{-1}\mathscr{G}$ of an $\mathscr{O}_Y$-module is an $\pi^{-1}\mathscr{O}_Y$-module, but not necessarily an $\mathscr{O}_X$-module. Just as in the case of modules over rings, this can be fixed by tensoring appropriately with $\mathscr{O}_X$ along the map $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$; see pullbacks.

invertible sheaf: A locally free sheaf of $\mathscr{O}_X$-modules of rank 1 is called an **invertible sheaf**.

irreducible: A topological space X is said to be **irreducible** if it cannot be written as the union of proper nonempty closed subsets. Equivalently, X is irreducible if $X = X_1 \cup X_2$ with X_i closed implies that $X_1 = X$ or $X_2 = X$.

A scheme X is said to be **irreducible** if it is irreducible as a topological space. In the special case of affine schemes $X = \operatorname{Spec} A$, the points $\mathfrak{p} \in \operatorname{Spec} A$ parameterize the irreducible closed subschemes of X.

Jacobian: The **Jacobian matrix** is defined as usual from calculus. That is, given $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ the Jacobian at a point $\overline{x} \in k^n$ is the $n \times r$ matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\overline{x}) & \frac{\partial f_2}{\partial x_1}(\overline{x}) & \cdots & \frac{\partial f_r}{\partial x_1}(\overline{x}) \\ \frac{\partial f_1}{\partial x_2}(\overline{x}) & \frac{\partial f_2}{\partial x_2}(\overline{x}) & \cdots & \frac{\partial f_r}{\partial x_2}(\overline{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(\overline{x}) & \frac{\partial f_2}{\partial x_n}(\overline{x}) & \cdots & \frac{\partial f_r}{\partial x_n}(\overline{x}) \end{pmatrix}$$

where the partial derivatives are defined formally.

The **Jacobian** of a variety X is also used to refer to degree zero divisor classes, i.e. $\operatorname{Pic}^0 X$.

kernel: The **presheaf kernel** of a map $\phi\colon \mathscr{F} \to G$ of (pre)sheaves on X is the presheaf given by $(\ker\phi)(U) = \ker\phi(U)$. It satisfies the universal property of kernels in the category of presheaves on X.

When $\mathscr{F}$ and $\mathscr{G}$ are *sheaves*, the presheaf kernel is itself a sheaf, so we may call it the **kernel**.

linear system: If X is a k-scheme and $\mathscr{L}$ is a line bundle on X, a **linear system** is a vector space V together with a map to $\Gamma(X, \mathscr{L})$. We can define base points and base point freeness just as for line bundles, as well as use $n+1$ dimensional linear systems to give a map $X - \{\text{base locus}\} \to \mathbb{P}^n_k$.

local on the ____: A property of maps of schemes is called **local on the target** if (a) whenever $\pi\colon X \to Y$ has the property, so does the natural restriction $\pi\colon \pi^{-1}(V) \to V$ for any open V in Y, and (b) if $\pi\colon X \to Y$ is a map and there is an open cover $Y = \cup_{i\in I} V_i$ for which the restriction has the property, then π has the property.

Dually, a property of maps is called **local on the source** if when checking if $\pi\colon X \to Y$ has the property, it suffices to check that the restrictions $\pi_i\colon U_i \to Y$ have the property, where $X = \cup_{i\in I} U_i$ is an open cover.

We say a property is **affine-local** on the source or target if it suffices to check on any affine cover of the source/target. This is stronger, because it says that only one affine cover need be checked, rather than all open covers.

locally closed: A subset $S \subseteq X$ is **locally closed** if it is the intersection of an open subset with a closed subset. This is equivalent to S being an open subset of a closed subset, or a closed subset of an open subset.

A morphism of schemes $X \to Y$ is a **locally closed embedding** if it factors into

$$X \to Z \to Y$$

where $X \to Z$ is a closed embedding and $Z \to Y$ is an open embedding. So X is (isomorphic to) a closed subscheme of Z, which is (isomorphic to) an open subscheme of Y.

locally free: A sheaf $\mathscr{F}$ on X is said to be **locally free of rank** n if there exists an open cover $X = \cup U_i$ on which $\mathscr{F}|_{U_i} \simeq \mathscr{O}_{U_i}^{\oplus n}$. Such a cover is called a **trivialization** of $\mathscr{F}$, and sometimes $\mathscr{F}$ is referred to as "locally trivial." Given a trivialization, we have isomorphisms $\phi_{ij}\colon \Gamma(U_i \cap U_j, \mathscr{O}_{U_i}^{\oplus n}) \to \Gamma(U_i \cap U_j, \mathscr{O}_{U_j}^{\oplus n})$ called **transition functions**, satisfying the usual cocycle condition. They are so called because for affine $\operatorname{Spec} A \subset U_i \cap U_j$, ϕ_{ij} can be identified with an invertible $n \times n$ matrix with entries in A.

A locally free sheaf of rank 1 is sometimes called a **line bundle** or an **invertible sheaf**. The reason for the former is that geometrically a locally free sheaf is a line bundle, i.e. a dimension one vector bundle. I'd venture to guess that the latter name is given because the dual $\mathcal{H}om(\mathscr{F}, \mathscr{O}_X)$ is also locally free of rank one, and is inverse to $\mathscr{F}$ in the sense that $\mathscr{F} \otimes \mathcal{H}om(\mathscr{F}, \mathscr{O}_X) \simeq \mathscr{O}_X$. In fact, the invertible sheaves form a group called the Picard group $\operatorname{Pic} X$.

morphism of (pre)sheaves: Let $\mathscr{F}, \mathscr{G}$ be presheaves on X with values in the same category $\mathcal{C}$. A **morphism of presheaves** $\phi\colon \mathscr{F} \to \mathscr{G}$ is a natural transformation, when $\mathscr{F}$ and $\mathscr{G}$ are viewed as contravariant functors. That is, ϕ is the data of maps $\phi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ for all open U such that whenever $V \subseteq U$,

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi(U)} & \mathscr{G}(U) \\ {\scriptstyle \mathrm{res}_{U,V}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{U,V}} \\ \mathscr{F}(V) & \xrightarrow[\phi(V)]{} & \mathscr{G}(V) \end{array}$$

commutes.

We define an *isomorphism* of (pre)sheaves in the categorical sense as a two-sided inverse.

natural isomorphism: Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A **natural isomorphism** $F \to G$ is a natural transformation such that $F(X) \xrightarrow{\sim} G(X)$ is an isomorphism for all objects $X \in \mathcal{C}$.

This is the notion of isomorphism in the category of functors $\mathcal{C} \to \mathcal{D}$, as given a natural isomorphism, we can define a natural isomorphism $G \to F$ such that upon composition, we obtain the identity functor.

natural transformation: Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be (covariant) functors. A **natural transformation** $F \to G$ is the data of maps (in $\mathcal{D}$) $F(X) \to G(X)$ for all $X \in \mathcal{C}$, such that for any $f\colon X \to Y$ the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \longrightarrow & G(Y) \end{array}$$

commutes.

If F, G are *contravariant* functors, we instead ask for the diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & G(X) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(Y) & \longrightarrow & G(Y) \end{array}$$

to commute.

Natural transformations are the correct formulation of a *morphism of functors*, and are taken to be the arrows in the category of covariant/contravariant functors $\mathcal{C} \to \mathcal{D}$.

Noetherian: A scheme X is **locally Noetherian** if it has an affine cover $X = \cup_{i \in I} \operatorname{Spec} A_i$ by Noetherian rings A_i. If X is also quasicompact, so we can make this cover finite, we drop "locally" and call X **Noetherian**.

A ring A is **Noetherian** if it satisfies the ascending chain condition on ideals. That is, any increasing chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals of A eventually stabilizes. This is equivalent to every ideal of A being finitely generated.

A topological space X is **Noetherian** if it satisfies the descending chain condition for closed subsets – any sequence $K_1 \supseteq K_2 \supseteq \cdots$ eventually terminates.

normal scheme: A scheme X is called **normal** if all of its stalks $\mathscr{O}_{X,p}$ are integrally closed domains.

normalization: Let X be an integral k-scheme. A **normalization** of X, often denoted $\widetilde{X}$, is a normal scheme scheme together with a dominant morphism $\widetilde{X} \to X$ satisfying the universal property that for any normal scheme Y with a map to X, it factors uniquely through $\widetilde{X}$.

$$\begin{array}{ccc} Y & \dashrightarrow & \widetilde{X} \\ & \searrow & \downarrow \\ & & X \end{array}$$

In the special case $X = \operatorname{Spec} A$, the normalization is the spectrum of the integral closure of A in its fraction field.

(co)normal sheaf: Let $Z \hookrightarrow X$ be a closed embedding (in fact locally closed is enough) with ideal sheaf $\mathscr{I}$. The **conormal sheaf** $\mathscr{N}^\vee_{Z/X}$ is defined to be $\mathscr{I}/\mathscr{I}^2$, interpreted as a sheaf on Z. Thus the **normal sheaf** $\mathscr{N}_{Z/X} = \left(\mathscr{N}^\vee_{Z/X}\right)^\vee$ is taken to be the dual of the conormal sheaf.

open embedding: Let $(Y, \mathscr{O}_Y)$ be a ringed space (e.g. a scheme). If $(U, \mathscr{O}_U)$ is a ringed space with a map $\pi\colon U \to Y$ (and hence $\mathscr{O}_Y \to \pi_*\mathscr{O}_U$), we call this an **open embedding** if for some open subscheme (subset) $V \subseteq Y$ we have $(U, \mathscr{O}_U) \simeq (V, \mathscr{O}_Y|_V)$. That is, we have a commutative diagram

$$\begin{array}{ccc} (U, \mathscr{O}_U) & \xrightarrow{\sim} & (V, \mathscr{O}_Y|_V) \\ & \searrow^{\pi} & \downarrow \\ & & (Y, \mathscr{O}_Y) \end{array}$$

where the downward map is the natural map.

open subfunctor: Let $h, h' \colon \text{Sch} \to \text{Sets}$ be contravariant functors. We say a natural transformation $h' \to h$ expresses h' as an **open subfunctor of** h if for all representable functors h_X and maps $h_X \to h$, the fibered product $h_X \times_h h'$ is representable, say by a scheme U, such that the map $h_U \to h_X$ corresponds to an open embedding of schemes $U \to X$.

perfect: A (bilinear) pairing is a map $M \otimes N \to L$. By the tensor-hom adjunction, this gives a canonical map $M \to \operatorname{Hom}_A(N, L)$. The pairing is **perfect** if this canonical map is an isomorphism.

Picard group: Let X be a ringed space. The **Picard group** $\operatorname{Pic} X$ is the group of invertible sheaf on X with the operation $\otimes$.

pole: Let X be a scheme with function field K and let Y be a prime divisor. We say $f \in K^\times$ has a pole at Y if $v_Y(f) < 0$, where v_Y is the valuation at Y.

presheaf: Let X be a topological space. A **presheaf** $\mathscr{F}$ on X with values in a category $\mathcal{C}$ is a contravariant functor from the category of open sets on X to $\mathcal{C}$.

More concretely, a presheaf $\mathscr{F}$ is the data of an object $\mathscr{F}(U)$ in $\mathcal{C}$ for every open U and restriction maps $\operatorname{res}_{U,V} \colon \mathscr{F}(U) \to F(V)$ for all pairs of opens $V \subseteq U$. These maps satisfy $\operatorname{res}_{U,U} = \operatorname{id}_{\mathscr{F}(U)}$ and if $W \subseteq V \subseteq U$ the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\operatorname{res}_{U,V}} & \mathscr{F}(V) \\ {\scriptstyle \operatorname{res}_{U,W}}\downarrow & \swarrow_{\operatorname{res}_{V,W}} & \\ \mathscr{F}(W) & & \end{array}$$

commutes. We may also require $\mathscr{F}(\emptyset) = 0$, in the case of abelian groups or rings, or the appropriate final object in $\mathcal{C}$.

principal divisor: Let X satisfy $(*)$ and let K denote the function field of X. For a function $f \in K^\times$, we define the **(Weil) divisor associated to** f to be

$$(f) = \sum v_Y(f) Y,$$

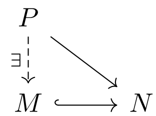
where the sum is taken over prime divisors, and it can be proven that $v_Y(f) = 0$ for all but finitely man Y. The image of K^* in $\operatorname{Div} X$ under the map $f \mapsto (f)$ is called the **principal divisors**.

projective: A scheme is a **projective A-scheme** if it is isomorphic to $\operatorname{Proj} S_\bullet$ for a finitely generated graded ring $S_\bullet$ over A. This is equivalent to having a closed embedding to $\mathbb{P}^n_A$ for some n (see very ample).

A **quasiprojective A-scheme** is a quasicompact open subscheme of a projective A-scheme.

More generally, if Y is a base scheme, we say $\pi \colon X \to Y$ is **projective** if there is an isomorphism of Y-schemes $X \to \mathcal{P}roj\ \mathscr{S}_\bullet$, the relative proj of a quasicoherent sheaf of algebras $\mathscr{S}_\bullet$ finitely generated in degree 1. In this case X is called a **projective Y-scheme**.

An object P in an abelian category is a **projective object** if the functor $\operatorname{Hom}(P, \cdot)$ is exact. This can be stated in the following diagram.



In the category of A-modules, projective objects are called **projective modules**. Free modules are a notable example of projective modules.

projective space: Classically, **projective n-space** $\mathbb{P}_k^n$ over a field k is defined as the set of $(n+1)$ tuples $(x_0, \ldots, x_n) \neq (0, \ldots, 0)$ modulo nonzero scalars, so

$$(ax_0, \ldots, ax_n) = (x_0, \ldots, x_n).$$

In other words, this is the collection of lines passing through the origin in the affine space $\mathbb{A}_k^{n+1}$.

Scheme theoretically, we define **projective space over** A, $\mathbb{P}_A^n$, for any ring A by $\operatorname{Proj} A[x_0, \ldots, x_n]$, using the usual degree grading on $A[x_0, \ldots, x_n]$. Alternatively, we can construct the space by gluing $n+1$ copies of affine space $U_i \simeq \mathbb{A}_A^n \simeq \operatorname{Spec} A[x_{0/i}, \ldots, \widehat{x_{i/i}}, \ldots, x_{n/i}]$ along the map $x_{k/i} \mapsto x_{k/j}/x_{i/j}$.

Maps $X \to \mathbb{P}_{\mathbb{Z}}^n$ are determined by line bundles $\mathscr{L}$ on X with $n+1$ sections with no common zeros, up to isomorphism. Hence we can interpret $\mathbb{P}_{\mathbb{Z}}^n$ as the moduli space of this data. If Y is an arbitrary scheme, then $\mathbb{P}_Y^n$ is the moduli space of line bundles *on a Y-scheme* X with $n+1$ sections possessing no common zeros (up to isomorphism). This is seen to agree with $\mathbb{P}_Y^n = \mathbb{P}_{\mathbb{Z}}^n \otimes_{\mathbb{Z}} Y$ by the universal property of the fiber product.

proper: A morphism of schemes $\pi\colon X \to Y$ is **proper** if it is separated, finite type, and universally closed. A k-scheme X (or more generally an S-scheme X) is **proper** if the structure morphism $X \to \operatorname{Spec} k$ is proper (resp. the structure map $X \to S$ is proper).

pullback: Let $\pi\colon X \to Y$ be a map of schemes (so it comes with $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$) and $\mathscr{G}$ an $\mathscr{O}_Y$-module. We define the **pullback** $\pi^*\mathscr{G}$ to be

$$\pi^*\mathscr{G} = \mathscr{O}_X \otimes_{\pi^{-1}\mathscr{O}_Y} \pi^{-1}\mathscr{G},$$

which has a natural structure as an $\mathscr{O}_X$-module (here we're using the $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ version of the pullback map).

This defines a functor from $\mathscr{O}_Y$-modules to $\mathscr{O}_X$-modules which is seen to be (or defined to be) adjoint to pushforward:

$$\operatorname{Mor}_{\mathscr{O}_Y}(\pi^*\mathscr{G}, \mathscr{F}) = \operatorname{Mor}_{\mathscr{O}_X}(\mathscr{G}, \pi_*\mathscr{F}).$$

It's worth noting that since it's a left adjoint, we immediately have $\pi^*\mathscr{G}$ is right exact.

If further $\mathscr{G}$ is quasicoherent, then the pullback $\pi^*\mathscr{G}$ is also quasicoherent. To see this, take an open $\operatorname{Spec} B \subseteq Y$ on which $\mathscr{G}|_{\operatorname{Spec} B} \simeq \widetilde{M}$ and an affine open $\operatorname{Spec} A \subset X$ whose image is in $\operatorname{Spec} B$. This construction shows

$$\Gamma(\operatorname{Spec} A, \pi^*\mathscr{G}|_{\operatorname{Spec} A}, \operatorname{Spec} A) \simeq A \otimes_B M,$$

which is naturally an A-module. One further checks that this agrees on the appropriate distinguished open sets, so $\pi^*\mathscr{G}|_{\operatorname{Spec} A} \simeq \widetilde{A \otimes_B M}$.

pushforward: Let $\pi\colon X \to Y$ be a map of schemes. If $\mathscr{F}$ is a sheaf on X, then we define the **pushforward** $\pi_*\mathscr{F}$ as a sheaf on Y by

$$\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V)),$$

where $V \subset Y$ is an open set. If $U \subset V \subset Y$, the restriction maps $\pi_*\mathscr{F}(V) \to \pi_*\mathscr{F}(U)$ are precisely $\mathscr{F}(\pi^{-1}(V)) \to \mathscr{F}(\pi^{-1}(U))$, thus $\pi_*\mathscr{F}$ is seen to be a presheaf. The sheaf axioms for $\mathscr{F}$ imply those on $\pi_*\mathscr{F}$, which can be seen by recognizing that if $V = \cup V_i$, then

$\pi^{-1}(V) = \cup \pi^{-1}(V_i)$, so we can bootsrap the identity and gluing axioms from $\mathscr{F}$ to $\pi_* \mathscr{F}$. If $\pi(p) = q$, then we have an natural map on stalks $(\pi_* \mathscr{F})_q \to \mathscr{F}_p$.

Perhaps the most important example of a pushforward is that of the structure sheaf of a scheme. Namely, in giving a map of schemes $X \to Y$, one needs to provide a map of structure sheaves $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$. This is sometimes called the "pullback" map, because it tells us how to pull back functions on Y to functions on X. More precisely, if $s \in \Gamma(V, \mathscr{O}_Y)$, we should think of s as a function on V, so composing with $X \to Y$ should give us a function on $\pi^{-1}(V)$. This is precisely the data of $\mathscr{O}_Y(V) \to \pi_* \mathscr{O}_X(V) = \mathscr{O}_X(\pi^{-1}(V))$. Note that this also gives a map on stalks $\mathscr{O}_{Y,q} \to (\pi_* \mathscr{O}_X)_q \to \mathscr{O}_{X,p}$ if $p \mapsto q$, so "local functions" on Y pull back to local functions on X.

The pushforward retains any additional structure of $\mathscr{F}$ (i.e. sheaf of ab. groups, rings, etc.). Of particular interest, if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\pi_* \mathscr{F}$ can be endowed with the structure of an $\mathscr{O}_Y$-module. A priori, it has the structure of a $\pi_* \mathscr{O}_X$-module, then we use the map $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$ to view it as an $\mathscr{O}_Y$-module. When $X \to Y$ is quasicompact and quasiseparated and $\mathscr{F}$ is a quasicoherent sheaf of $\mathscr{O}_X$-modules, the pushforward $\pi_* \mathscr{F}$ is a quasicoherent $\mathscr{O}_Y$-module.

qcqs: A map (or a scheme?) is **qcqs** if it is both quasiseparated and quasicompact.

The reason this definition is useful is that if a scheme X is qcqs, it has a finite cover by affine opens (by quasicompactness) whose intersections are covered by finitely many affine opens (by quasiseparatedness).

quasicoherent: A sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is **quasicoherent** if there is a cover of X by affine opens $U_i \simeq \operatorname{Spec} A_i$ such that $\mathscr{F}|_{U_i}$ is isomorphic to $\widetilde{M_i}$ for an A_i-module M_i.

We say $\mathscr{F}$ is **coherent** if it is quasicoherent and each M_i is a finitely generated A_i-module.

quasicompact: A scheme X is **quasicompact**, abbreviated **qc**, if every open cover $X = \cup_{i \in I} U_i$ reduces to a finite subcover $X = \cup_{i \in S} U_i$, $S \subseteq I$ finite.

We say a *map* $\pi \colon X \to Y$ of schemes is **quasicompact** if for every affine open $U \subseteq Y$, the preimage $\pi^{-1}(U)$ is quasicompact.

quasifinite: A morphism $\pi \colon X \to Y$ is **quasifinite** if it is of finite type and for all $q \in Y$, the preimage $\pi^{-1}(q)$ is a finite set.

quasiseparated: A scheme X is **quasiseparated** if for any two quasicompact opens $U, V \subseteq X$, the intersection $U \cap V$ is also quasicompact.

We say a *map* $\pi \colon X \to Y$ of schemes is **quasiseparated** if for every affine open $U \subseteq Y$, the preimage $\pi^{-1}(U)$ is a quasiseparated scheme.

rational normal curve: A **rational normal curve** is a curve of degree n in $\mathbb{P}^n$ given explicitly by the parameterization

$$[x : y] \mapsto [x^n : x^{n-1}y : \cdots : xy^{n-1} : y^n].$$

Equivalently, it is the curve in $\mathbb{P}^n$ given by $|\mathscr{O}_{\mathbb{P}^1}(n)|$ for $n \geq 1$, as one checks that $\mathscr{O}(\mathbb{P}^1)$ is very ample and $h^0(\mathbb{P}^1, \mathscr{O}(n)) = n + 1$, so this gives an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. The explicit coordinates above come from choosing a basis of $H^0(\mathbb{P}^1, \mathscr{O}(n))$; one could obtain different coordinates by choosing a different basis, but these are all the same up to automorphism of $\mathbb{P}^n$.

reduced: A *ring* is **reduced** if it has no nonzero nilpotent elements. A *scheme* X is **reduced** if the ring of sections $\mathscr{O}_X(U)$ is reduced for all U. This can in fact be checked on *stalks* – that is X is reduced if $\mathscr{O}_{X,p}$ has no nonzero nilpotents for all points p.

regular embedding: Let M be an A-module, and $x_1, ..., x_r \in A$. We call this a **regular sequence for** M if

(i) For all i, x_i is not a zerodivisor for $M/(x_1, ..., x_{i-1})M$ (and x_1 is not a zerodivisor for M).

(ii) The inclusion $(x_1, ..., x_r)M \subsetneq M$ is proper.

If $\pi\colon X \to Y$ is a locally closed embedding, we say π is a **regular embedding (of codimension** r**) at** p for a point p in X, if the ideal of X, viewed in $\mathscr{O}_{Y,p}$ is generated by a regular sequence of length r. We say π is a **regular embedding (of codimension** r**)** if it is regular (of codimension r) at all $p \in X$.

regular: A Noetherian local ring $(A, \mathfrak{m})$ is a **regular local ring** if and only if $\dim A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. A Noetherian ring A is a **regular ring** if $A_\mathfrak{p}$ is a regular local ring for all primes $\mathfrak{p} \in \operatorname{Spec} A$.

A locally Noetherian scheme X is **regular at a point** p (or **nonsingular at** p if its stalk $\mathscr{O}_{X,p}$ (which is a Noetherian local ring) is a regular local ring. If such X is regular at all points p, then we say X is **regular**.

A point $p \in X$ for which $\mathscr{O}_{X,p}$ is not a regular local ring is called a **singularity**.

relative proj: Let $\mathscr{S}_\bullet$ be a graded quasicoherent sheaf of algebras over X, finitely generated in degree 1, which we take to mean

$$\operatorname{Sym}^\bullet_{\mathscr{O}_X} \mathscr{S}_1 \twoheadrightarrow \mathscr{S}_\bullet.$$

Then we define **relative proj**, $\mathcal{P}roj\ \mathscr{S}_\bullet$ to be the scheme we obtain by gluing together $\operatorname{Proj} \mathscr{S}_\bullet(\operatorname{Spec} A) \to \operatorname{Spec} A$ along the affine opens $\operatorname{Spec} A \subseteq X$ (this is nontrivial, see Exercise 17.2.B).

In particular, we see that for $\mathscr{S}_\bullet$ on an affine scheme $\operatorname{Spec} A$, we have $\mathcal{P}roj\ \mathscr{S}_\bullet \simeq \operatorname{Proj} \mathscr{S}_\bullet(\operatorname{Spec} A)$, so our construction agrees with the usual one.

relative spec: Let X be a scheme and $\mathscr{A}$ a quasicoherent sheaf of $\mathscr{O}_X$-algebras. There exists an X-scheme $\mathcal{S}pec\ \mathscr{A} \to X$ called **relative spec of** $\mathscr{A}$ satisfying the universal property that giving a map of X-schemes $T \to \mathcal{S}pec\ \mathscr{A}$ is the same giving a map of $\mathscr{O}_X$-algebras $\mathscr{A} \to \pi_*\mathscr{O}_T$, where $\pi\colon T \to X$ is the structure map. That is,

$$\operatorname{Hom}_X(T, \mathcal{S}pec\ \mathscr{A}) \simeq \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{A} \to \pi_*\mathscr{O}_T)$$

or equivalently the functor $T \mapsto \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{A} \to \pi_*\mathscr{O}_T)$ from X-schemes to sets is representable, by $\mathcal{S}pec\ \mathscr{A}$.

When the base is affine, say $X = \operatorname{Spec} B$, $\mathscr{A}$ is a sheaf $\widetilde{A}$ associated to a B-algebra A. In this case, we see $\mathcal{S}pec\ \widetilde{A} \simeq \operatorname{Spec} A$, using the fact that

$$\operatorname{Hom}_B(T, \operatorname{Spec} A) \simeq \operatorname{Hom}_B(A, \Gamma(T, \mathscr{O}_T)),$$

i.e. giving a map $T \to \operatorname{Spec} A$ is the same as giving an A-algebra structure to the global sections of T. One can then show that this construction glues to exist in general.

representable: A contravariant functor $F\colon \mathcal{C} \to \text{Sets}$ is **representable** if there exists an object $Y \in \mathcal{C}$ such that $F(X) = \operatorname{Mor}(X, Y)$. That is, F is isomorphic to h_Y, where h_Y is the functor of points. We say the functor F is **represented by** Y.

ringed space: A **ringed space** is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of rings on X, called the *structure sheaf of* X.

Two ringed spaces $(X, \mathscr{O}_x)$ and $(Y, \mathscr{O}_Y)$ are *isomorphic* if they come with a homeomorphism $\pi\colon X \to Y$ of underlying topological spaces and an isomorphism of structure sheaves $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$.

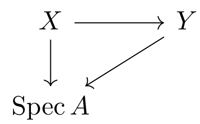
A ringed space is a **locally ringed space** if its stalks $\mathscr{O}_{X,p}$ are local rings for all points $p \in X$.

section: If $\mathscr{F}$ is a (pre)sheaf on X and $U \subseteq X$ is open, we may refer to $\mathscr{F}(U)$ as the **sections** of U. The sections of X itself are called *global sections*.

scheme: A **scheme** is a ringed space $(X, \mathscr{O}_X)$ which is "locally affine" in the sense that for every point $p \in X$ there exists an open neighborhood U such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

scheme over A: If A is a ring (often we are interested in $A = k$ a field), a **scheme over** A is a scheme X such that the sections of the structure sheaf $\mathscr{O}_X$ are A-algebras, and the restriction maps are maps of A-algebras.

This notion is equivalent to the data of a scheme X with a map to $\operatorname{Spec} A$, which makes it easier to define maps of schemes over A to be maps of schemes $X \to Y$ such that the diagram



commutes. This makes $\mathbf{Sch}_A$ into a category.

scheme-theoretic closure: The **scheme-theoretic closure** of a locally closed embedding $\pi\colon X \to Y$ is defined to be the scheme-theoretic image of π.

scheme-theoretic image: Given a map $\pi\colon X \to Y$, and a closed subscheme $i\colon Z \to Y$, we say the image lies in Z if the composition $\mathscr{I}_{Z/Y} \to \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is zero. The **scheme-theoretic image** of π is defined to be the intersection of all such Z, which is a closed subscheme of Y.

Informally, the image is the scheme cut out by functions which vanish when pulled back to X.

Segre embedding: The closed embedding $\mathbb{P}^m_A \times_A \mathbb{P}^n_A \hookrightarrow \mathbb{P}^{mn+m+n}_A$ given by mapping

$$([x_0, ..., x_m], [y_0, ..., y_n]) \mapsto [x_0y_0, x_0y_1, ..., x_0y_n, ..., x_my_n]$$

is called the **Segre embedding**. If $A = k$ is a field, the image is the **Segre variety**.

separated: A morphism of schemes $\pi\colon X \to Y$ is **separated** if the diagonal morphism $\delta_\pi\colon X \to X \times_Y X$ is a closed embedding. A scheme over A is **separated over** A if the structure map $X \to \operatorname{Spec} A$ is separated. We call a scheme itself **separated** if the map $X \to \operatorname{Spec} \mathbb{Z}$ is separated.

A (generically) finite morphism of integral schemes $X \to Y$ is said to be (generically) **separable** if it is dominant and the induced extension of function fields $K(Y) \to K(X)$ is a separable extension of fields. This is automatic in characteristic zero, but comes up as a hypothesis in Hurwitz's theorem.

sheaf: Let X be a topological space. A **sheaf** $\mathscr{F}$ on X with values in a category $\mathcal{C}$ is a presheaf satisfying two additional properties, which we call *identity* and *gluability*:

identity: Let $U = \cup U_i$ and suppose $f, g \in \mathscr{F}(U)$ satisfy $f|_{U_i} = g|_{U_i}$ for all i. Then $f = g$.

gluability: Let $U = \cup U_i$ and suppose we have $f_i \in \mathscr{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j. Then there exists $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for all i.

sheaf associated to an A-module M: Let $X = \operatorname{Spec} A$ be an affine scheme and M an A-module. We define the **sheaf associated to** M, denoted $\widetilde{M}$, to have sections

$$\widetilde{M}(U) = \{\, s\colon U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p} \mid (i) \text{ and } (ii) \,\}$$

where

(i) $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ for all $\mathfrak{p}$ and
(ii) for each $\mathfrak{p} \in U$, there exists a neighborhood $V \subseteq U$ and $m \in M$, $f \in A$, such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f$.

That is, the sections of $\widetilde{M}$ are the locally constant functions from U to the disjoint union of the localizations $M_{\mathfrak{p}}$. The restriction maps are taken to be the obvious ones, which restrict s to a smaller neighborhood $V \subseteq U$.

While not a definition, it's worth mentioning here that $\widetilde{M}$ is an $\mathscr{O}_X$-module, its stalks are localizations $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$, and its sections on distinguished opens are also localizations $\widetilde{M}(D(f)) = M_f$.

There's a projective version of this, where instead $X = \operatorname{Proj} S_\bullet$ is a projective A-scheme for $A = S_0$ and S_+ finitely generated. If $M_\bullet$ is a ($\mathbb{Z}$-)graded $S_\bullet$-module, we define the **associated sheaf** $\widetilde{M_\bullet}$ as follows.

For f homogeneous of postive degree, $\operatorname{Spec}((S_\bullet)_f)_0 \subset \operatorname{Proj} S_\bullet$, so we take $\widetilde{M_\bullet}|_{\operatorname{Spec}((S_\bullet)_f)_0}$ to be the sheaf on $\operatorname{Spec}((S_\bullet)_f)_0$ associated to $((M_\bullet)_f)_0$. One then needs to check that these glue appropriately, e.g. by taking generators $(f_1, \ldots, f_r) = S_+$ and verifying that the cocycle condition is satisfied on triple intersections.

As one might expect, if $\mathfrak{p}$ is a homogeneous prime of $S_\bullet$, we have the stalk $(\widetilde{M_\bullet})_{\mathfrak{p}} = ((M_\bullet)_{\mathfrak{p}})_0$, which we can see by further localizing $((M_\bullet)_f)_0$ for $f \notin \mathfrak{p}$.

sheaf cohomology: Let X be a topological space. The **sheaf cohomology functors** $H^i(X, \cdot)$, from the category of sheaves of abelian groups on X to the category of abelian groups, are the right derived functors of the global section functor $\Gamma(X, \cdot)$. For a sheaf of abelian groups $\mathscr{F}$ on X, we call $H^i(X, \mathscr{F})$ the i^{th} **cohomology groups** of $\mathscr{F}$.

sheaf hom: Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. We define the **sheaf hom**, denoted $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$, to be the sheaf given by

$$U \mapsto \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

This is also an $\mathscr{O}_X$-module.

sheaf of ideals: A **sheaf of ideals** on X is a subsheaf of $\mathscr{O}_X$, in that $\mathscr{I}(U) \subseteq \mathscr{O}_X(U)$ is an ideal for all open $U \subseteq X$.

sheaf of $\mathscr{O}_X$-modules: Let $(X, \mathscr{O}_X)$ be a scheme. A **sheaf of $\mathscr{O}_X$-modules** (or just an $\mathscr{O}_X$-module) is a sheaf of abelian groups $\mathscr{F}$ on X, such that for every open $U \subseteq X$, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module. Moreover, the module structure is compatible with restriction, in that whenever $V \subseteq U$ we have a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_X(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \\ \downarrow{\scriptstyle \text{res} \times \text{res}} & & \downarrow{\scriptstyle \text{res}} \\ \mathscr{O}_X(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \end{array}$$

where the horizontal maps are the module map.

A morphism of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{G}$ is a morphism of (pre)sheaves such that each map on sections $\mathscr{F}(U) \to \mathscr{G}(U)$ is an $\mathscr{O}_X(U)$-module homomorphism. We let $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ denote the group of $\mathscr{O}_X$-module morphisms from $\mathscr{F}$ to $\mathscr{G}$.

Kernels, cokernels, images, quotients, products, sums, limits, and colimits of $\mathscr{O}_X$-modules naturally carry the structure of an $\mathscr{O}_X$-module. This allows us to discuss exactness of sequences of $\mathscr{O}_X$-modules.

sheafification: Let $\mathscr{F}$ be a presheaf on X with values in $\mathcal{C}$. The **sheafification** $\mathscr{F}^+$ of $\mathscr{F}$ is a sheaf on X that is initial in the category of sheaves on X with maps from $\mathscr{F}$. That is, given a sheaf $\mathscr{G}$ and a map $\mathscr{F} \to \mathscr{G}$, there exists a unique map $\mathscr{F}^+ \to \mathscr{G}$ which commutes with the maps from $\mathscr{F}$.

smooth: Let X be a k-scheme of pure dimension d. We say X is **k-smooth of dimension d** if there exists a cover of X by open sets of the form $\operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ where the Jacobian matrix has corank d at all points. Equivalently, X is **k-smooth** if the (co)tangent sheaf $\Omega_{X/k}$ is locally free of rank d.

A morphism of schemes $X \to Y$ is **smooth of relative dimension n** if there exist open covers $X = \cup U_i$, $Y = \cup V_i$ with $\pi|_{U_i} : U_i \to V_i$, such that V_i is $\operatorname{Spec} B$, U_i is isomorphic to an open subscheme of $\operatorname{Spec} B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)$ (and $\pi|_{U_i}$ is induced by the obvious map of rings), and the determinant of the Jacobian matrix of the f_i's with respect to the first r x_i's is invertible on U_i.

Alternatively, we could equivalently define π to be **smooth of relative dimension n** if the cotangent sheaf $\Omega_\pi = \Omega_{X/Y}$ is locally finitely presented, flat of relative dimension n, and locally free of rank n.

stalk: Given a sheaf $\mathscr{F}$ on a topological space X, the **stalk** at a point $p \in X$ is the colimit

$$\mathscr{F}_p = \varinjlim \mathscr{F}(U)$$

where the colimit runs over all opens U containing p.

Of particular interest is the stalk of a point p in a scheme $(X, \mathscr{O}_X)$, which we take to be the stalk of the structure sheaf at p, $\mathscr{O}_{X,p}$. In the case where $X = \operatorname{Spec} A$, this turns out to be A_p, where here p is interpreted as a prime ideal of A.

support: The **support** of a sheaf of abelian groups $\mathscr{F}$ on X (or of rings or $\mathscr{O}$-modules) is the set of points $p \in X$ for which the stalk is nonzero,

$$\operatorname{Supp} \mathscr{F} = \{\, p \in X \mid \mathscr{F}_p \neq 0 \,\}.$$

If $\mathscr{F}$ is quasicoherent and finite type (e.g. locally free of finite rank), $\operatorname{Supp} \mathscr{F} \subseteq X$ is a closed subset. To see this, look on affine locally and show that the primes for which $\mathscr{F}_p = 0$ correspond to primes not containing a certain ideal.

symmetric algebra: Let M be an A-module. The **n-th symmetric power** $\operatorname{Sym}^n M$ is the quotient of $T^n M$ by the ideal generated by elements of the form $(m_1 \otimes \cdots \otimes m_n) - (m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)})$ where $\sigma \in S_n$ is any permutation. We can also form a **symmetric algebra** $\operatorname{Sym}^\bullet M$, similarly to the tensor algebra which is a graded ring.

As one might suspect, $\operatorname{Sym}^\bullet M$ satisfies a universal property: if $M \to B$ is a linear map and B is a *commutative* B-algebra, then we have a unique map $\operatorname{Sym}^\bullet M \to B$ commuting with the natural inclusion.

If $\mathscr{F}$ is a quasicoherent sheaf on X, we can define $\operatorname{Sym}^n \mathscr{F}$ by looking locally: if $\mathscr{F}|_{\operatorname{Spec} A} \simeq M$ we define $\operatorname{Sym}^n \mathscr{F}|_{\operatorname{Spec} A} = \operatorname{Sym}^n M$. One must check that this construction glues.

If $M \simeq A^{\oplus m}$ is a free module of rank m, we have $\operatorname{Sym}^n(A^m) \simeq A^{\binom{m-n-1}{n}}$. This can be seen by, e.g. computing the dimension of the kernel $T^n A^m \to \operatorname{Sym}^n A^m$. If $\mathscr{F}$ is a locally free sheaf of rank m, this implies $\operatorname{Sym}^n \mathscr{F}$ has rank $\binom{m-n-1}{n}$.

If $s \in \Gamma(X, \mathscr{F})$ the **support of** s is the set of $p \in X$ for which $s_p \neq 0$ in the stalk $\mathscr{F}_p$.

tangent space: The **Zariski cotangent space of a local ring** $(A, \mathfrak{m})$ is defined to be the quotient $\mathfrak{m}/\mathfrak{m}^2$. This is a vector space over the field $A/\mathfrak{m}$. The dual vector space $\left(\mathfrak{m}/\mathfrak{m}^2\right)^\vee = \operatorname{Hom}_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2, A/\mathfrak{m})$ is called the **Zariski tangent space** of $(A, \mathfrak{m})$.

If X is a scheme, the **Zariski cotangent space at** $p \in X$, denoted $T^\vee_{X,p}$, is the Zariski cotangent space of $\mathscr{O}_{X,p}$, i.e. $\mathfrak{m}_p/\mathfrak{m}_p^2$ as an $\mathscr{O}_{X,p}/\mathfrak{m}_p = \kappa(p)$ vector space. Similarly, $T_{X,p}$ is the dual of $T^\vee_{X,p}$. Elements of $T^\vee_{X,p}$ are called **differentials** or **cotangent vectors**, while elements of $T_{X,p}$ are called **tangent vectors**.

(co)tangent sheaf: Let $X \to Y$ be a map of schemes. Let $\delta\colon X \hookrightarrow X \times_Y X$, which is always a locally closed embedding. We define the **cotangent sheaf** $\Omega_{X/Y}$ to be the conormal sheaf of the diagonal,

$$\Omega_{X/Y} = \mathscr{N}^\vee_{X/X\times_Y X}.$$

One checks that affine locally, $\Omega_{X/Y}$ agrees with the module of differentials associated with the appropriate ring map.

As one might expect, the **tangent sheaf** $\mathscr{T}_{X/Y}$ is taken to be the dual

$$\mathscr{T}_{X/Y} = \Omega^\vee_{X/Y}.$$

tensor algebra: Let M be an A-module. The **n-th tensor algebra** T^nM is defined to be the n-fold tensor product of M over A with itelf, $M^{\otimes n}$. Setting $T^0M = A$, we form a graded ring $T^\bullet M$, where multiplication is defined in the obvious way from $T^nM \times T^pM \to T^{n+p}M$, sending $((m_1 \otimes \cdots \otimes m_n), (m'_1 \otimes \cdots \otimes m'_p) \mapsto (m_1 \otimes \cdots \otimes m_n \otimes m'_1 \otimes \cdots \otimes m'_p)$ and extending linearly. Note this algebra is most certainly not commutative!

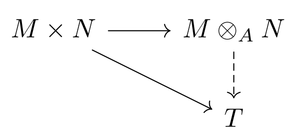
$T^\bullet M$ satisfies a universal property similar to that of a free object. If $M \to B$ is any linear map to an A-algebra B, there is a unique map $T^\bullet M \to B$ which commutes with the inclusion $M \hookrightarrow T^\bullet M$.

If $\mathscr{F}$ is a quasicoherent sheaf on X, we can define $T^n\mathscr{F}$ by looking locally: if $\mathscr{F}|_{\operatorname{Spec} A} \simeq M$ we define $T^n\mathscr{F}|_{\operatorname{Spec} A} = T^nM$. One must check that this construction glues.

If $M \simeq A^{\oplus m}$ is a free module of rank m, we have $T^n(A^m) \simeq A^{mn}$. To see this, take $e_1, \ldots, e_m$ a basis for A and see that $e_{i_1} \otimes \cdots \otimes e_{i_n}$ is a basis for $T^n(A^m)$. If $\mathscr{F}$ is a locally free sheaf of rank m, this implies $T^n\mathscr{F}$ has rank mn.

tensor product: Let $B \to A$ be a map of rings and M a B-module. Any A-module naturally has a B-module structure, but to endow M with an A-module structure, we tensor $M \otimes_B A$. The functor $\cdot \otimes_B A$ is a covariant functor from B-modules to A-modules.

If M and N are A-modules, the tensor product $M \otimes_A N$ satisfies a universal property: for any A-bilinear map $M \times N \to T$ where T is an A-modules, there exists a unique map of A-modules from the tensor product to T.



To show such an object exists, we construct it explictly as the A-linear combinations of simple tensors $m \otimes n$, where we identify

$$(a_1m_1 + a_2m_2) \otimes n = a_1(m_1 \otimes n) + a_2(m_2 \otimes n),$$
$$m \otimes (b_1n_1 + b_2n_2) = b_1(m \otimes n_1) + b_2(m \otimes n_2),$$
$$a(m \otimes n) = (am) \otimes n = m \otimes (an).$$

One can check this satisfies the universal property, with the map $(m, n) \mapsto m \otimes n$.

Generally tensor functors are right exact and left adjoints to an appropriate Hom functor.

Given sheaves of $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we define the **tensor product** $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$

We need to sheafify because tensoring is right exact, and does not (always) preserve the necessary limits in the sheaf axioms.

tor: If M is an A-module, the **Tor functors**, $\operatorname{Tor}_i^A(M,\cdot)$ for $i \geq 0$ are the left derived functors of the right exact functor $M \otimes_A \cdot$. $\operatorname{Tor}_i^A(M,\cdot)$ itself is left exact and induces a long exact sequence; if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of A-modules then

$$\cdots \to \operatorname{Tor}_A^{i+1}(M,N'') \to \operatorname{Tor}_A^{i}(M,N') \to \operatorname{Tor}_A^{i}(M,N) \to \operatorname{Tor}_A^{i}(M,N'') \to \operatorname{Tor}_A^{i-1}(M,N') \to \cdots$$

is a long exact sequence, terminating in

$$\operatorname{Tor}_A^1(M,N'') \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

i.e. $\operatorname{Tor}_A^0(M,N) \simeq M \otimes_A N$.

$\operatorname{Tor}_i^A(M,N)$ vanishes for all N if and only if $\operatorname{Tor}_i^A(M,N)$ vanishes for all $i > 0$ and N, and these are equivalent to M being flat. We can compute the Tor functors via free, or indeed projective module resolutions.

total quotient ring: Let A be a ring and S the multiplicative subset of elements which are not zerodivisors. The **total quotient ring of** A is the localization $S^{-1}A$. This is the "closest thing to the field of fractions" when A is not an integral domain.

Let X be a scheme. For each open $U \subseteq X$, let $S(U)$ denote the multiplicative subset of $\Gamma(U, \mathscr{O}_X)$ consisting of elements which are not zerodivisors in $\mathscr{O}_{X,p}$ for all $p \in U$. Then the sheaf associated to the presheaf $U \mapsto S(U)^{-1}\Gamma(U, \mathscr{O}_X)$ is called the **sheaf of total quotient rings** of $\mathscr{O}_X$. This is an analog of the function field for an integral scheme.

twist: A **twist** of a variety X/k is another variety T/k such that they become isomorphic upon base extension, i.e. $X \times_k K \simeq T \times_k K$ for a field extension K/k.

A prototypical example is a quadratic twist of an elliptic curve $E\colon y^2 = f(x)$. The twist is given by $E_d\colon dy^2 = f(x)$. If $\sqrt{d} \notin k$ then $E \not\simeq E_d$, but they are always isomorphic over $k(\sqrt{d})$ (or $\overline{k}$).

twisting sheaf: Consider the projective space $\mathbb{P}_k^m$ (or $\mathbb{P}_A^m$). We may define a sheaf $\mathscr{O}_{\mathbb{P}_k^m}(n)$ by taking it to be the degree n functions on each of the usual affine patches. The transition functions are then multiplying appropriately by n-th powers. These glue into a sheaf called $\mathscr{O}_{\mathbb{P}_k^m}(n)$. In particular, $\mathscr{O}_{\mathbb{P}_k^m}(1)$ is sometimes called the **twisting sheaf**.

It can be shown that $\mathscr{O}_{\mathbb{P}_k^m}(n) \otimes_{\mathscr{O}_{\mathbb{P}_k^m}} \mathscr{O}_{\mathbb{P}_k^m}(n') = \mathscr{O}_{\mathbb{P}_k^m}(n+n')$. When k is a field, all line bundles on $\mathbb{P}_k^m$ arise in this way, so we have $\mathbb{Z} \simeq \operatorname{Pic} \mathbb{P}_k^m$. Moreover, for $n \geq 0$, the global sections of $\mathscr{O}_{\mathbb{P}_k^m}(n)$ correspond to n-forms, allowing us to easily compute the dimension

$$\Gamma(\mathbb{P}_k^m, \mathscr{O}_{\mathbb{P}_k^m}(n)) = \binom{m+n}{n}.$$

If $X \hookrightarrow \mathbb{P}_k^m$ is a projective variety, we define $\mathscr{O}_X(n) = \mathscr{O}_X \otimes_{\mathscr{O}_{\mathbb{P}_k^m}} \mathscr{O}_{\mathbb{P}_k^m}(n)$, the pullback of $\mathscr{O}(n)$ to X.

This can be further generalized by considering a Cartier divisor D on X. We define $\mathscr{O}(D)$ to be the dual of the (invertible) ideal sheaf of D.

universal: The δ-functor $T = (T^i)\colon \mathscr{A} \to \mathscr{B}$ is **universal** if for any other δ-functor $T' = (T'^i)$ and any morphism $f^0\colon T^0 \to T'^0$, there exist unique $f^i\colon T^i \to T'^i$ for all i which commute with both sets of δ maps for all exact sequences.

unramified: If $\pi\colon X \to Y$ is a map of schemes, and $\Omega_\pi = \Omega_{X/Y}$ is the sheaf of relative differentials, we say π is **formally unramified** if $\Omega_\pi = 0$. We say π is **unramified** if it is formally unramified and locally of finite type.

The **ramification locus** is the support of Ω_π, which is a subset of X. The **branch locus** is the image in Y of the ramification locus.

valuation: Let K be a field. A **valuation** on K *with values in a totally ordered abelian group* G is a group homomorphism $v\colon K - 0 \to G$ satisfying $v(x+y) \geq \min(v(x), v(y))$. The **valuation ring** is the subring of K consisting of elements with nonnegative valuation and zero. We call a ring R a **valuation ring** if R is the valuation ring for a valuation v on $\operatorname{frac} R$, and we say a valuation is **discrete** if $G = \mathbb{Z}$.

If Y is a prime divisor of X, with generic point η then the stalk $\mathscr{O}_{X,\eta}$ is a discrete valuation ring with quotient field K, the function field of X. We call v_Y the **valuation of** Y.

vanishing scheme: Given a scheme Y and a global section $s \in \Gamma(Y, \mathscr{O}_Y)$, the **vanishing scheme** $V(s)$ is a closed subscheme. On an affine open $\operatorname{Spec} B \subseteq Y$, it corresponds to $\operatorname{Spec} B/(s_B)$ where $s_B = s|_{\operatorname{Spec} B}$. Given a set S of global sections, we can take $V(S)$ to be defined by $\operatorname{Spec} B/(S_B)$ on an affine open.

variety: An affine scheme over a field k which is reduced and of finite type is called an **affine** k**-variety**. A reduced (quasi)projective k-scheme is called a **(quasi)projective** k**-variety**.

More generally, a **variety** over a field k is defined to be a reduced, separeted scheme of finite type over k. Note that affine schemes and (quasi)projective schemes are always separated, so this additional hypothesis is not necessary.

vector bundle: A vector bundle is another name for a locally free sheaf. Classically, a vector bundle on X is a topological space with a map to X such that the fibers are vector spaces, "continuously varying" as we move along X. The trivial vector bundle is $X \times V$, and a general vector bundle is locally trivial, in that on some open cover it is isomorphic to a trivial bundle. As an nontrivial example, think of the Möbius strip as a rank one vector bundle on the circle — it is not globally trivial because it has a twist, but it is trivial if we remove a point on the circle.

Veronese: Given $\mathbb{P}^n$ and the (very ample) line bundle $\mathscr{O}(d)$, we have an embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ given by $|\mathscr{O}(d)|$, which we call a **Veronese embedding**. In particular,

$$N = h^0(\mathbb{P}^n, \mathscr{O}(d)) - 1 = \binom{n+d}{n} - 1.$$

In the special case of $n = 1$, the image curve is called a rational normal curve. As in that case, writing down a basis for $H^0(\mathbb{P}^n, \mathscr{O}(d))$ can give explicit coordinates for the map.

Weil divisor: Let X be a scheme satisfying $(*)$. A **prime divisor** on X is a closed integral subscheme Y of codimension one. A **Weil divisor** is an element of the free abelian group on prime divisors, $\operatorname{Div} X$. That is, a Weil divisor D is a formal integer linear combination of prime divisors, $\sum n_i Y_i$.

Vakil takes a **Weil divisor** to mean a (formal) $\mathbb{Z}$-linear combination of irreducible closed subsets of codimension 1 on a *Noetherian* scheme X. Note this doesn't use the full strength of condition (*), in that it doesn't assume integrality, separatedness, or regularity, though he quickly adds back in reducedness and regularity.

Zariski sheaf: A contravariant functor $F\colon \mathcal{C} \to \mathrm{Sch}$ is a **Zariski sheaf** if for any scheme Y, the assignment $U \mapsto F(U)$ forms a sheaf on Y.

zero: Let X be a scheme with function field K and let Y be a prime divisor. We say $f \in K^\times$ has a zero at Y if $v_Y(f) > 0$, where v_Y is the valuation at Y.

2. Results and discussions

Adjunction formula: Let X be a smooth variety over k and Z a smooth (closed) subvariety. Then we can compute the canonical bundle $\mathscr{K}_Z$ by

$$\mathscr{K}_Z \simeq \mathscr{K}_X|_Z \otimes \det \mathscr{N}_{Z/X}.$$

Furthermore, if Z has codimension 1 (i.e. a divisor) then we have $\mathscr{N}_{Z/X} \simeq \mathscr{O}_X(Z)|_Z$, so this is sometimes written

$$\mathscr{K}_Z \simeq (\mathscr{K}_X \otimes \mathscr{O}_X(Z))\,|_Z.$$

In the notation of divisors, we have

$$K_Z = (K_X + Z)|_Z$$

Proof sketch. Use the conormal exact sequence first, which is exact on the left by smoothness.

$$0 \to \mathscr{N}^\vee_{Z/X} \to i^*\Omega_{X/k} \to \Omega_{Z/k} \to 0.$$

This induces an alternating product of determinants

$$\det \mathscr{N}^\vee_{Z/X} \otimes \left(\det i^*\Omega_{X/k}\right) \otimes \det \Omega_{Z/k} \simeq \mathscr{O}_Z.$$

Dualizing appropriately and using the definition of $\mathscr{K}$, we have

$$\mathscr{K}_Z \simeq \mathscr{K}_X|_Z \otimes \det \mathscr{N}_{Z/X}.$$

□

Affine communication lemma: Let P be a property of affine open subsets of a scheme X. Suppose

(i) every affine open $\operatorname{Spec} A \subseteq X$ that has P implies $\operatorname{Spec} A_f \subseteq X$ also has P and
(ii) if $A = (f_1, \ldots, f_n)$ and $\operatorname{Spec} A_{f_i} \subseteq X$ has P for all i then $\operatorname{Spec} A \subseteq X$ also has P.

Then if $X = \cup \operatorname{Spec} A_i$ where $\operatorname{Spec} A_i$ has P for all i, then *every affine open* $\operatorname{Spec} A \subseteq X$ has P as well. The proof involves intersecting $\operatorname{Spec} A$ with $\operatorname{Spec} A_i$ and covering by affine opens simultaneously distinguished in both.

Morally, properties P satisfying (i) and (ii) above need only be checked on an open cover. Here are some examples of affine local conditions:

- Noetherianness (c.f. Proposition 5.3.3),
- finite typeness (c.f. Proposition 5.3.3),
- reducedness (and other stalk-local conditions)

A nonexample of such a property is integrality. For example, take $\operatorname{Spec}(A \times B)$ where A and B are arbitrary integral domains. This is a disjoint union (not irreducible) but can be covered by the integral schemes $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

Affine reduction iff affine: Let X be a scheme. Then X is affine if and only if the reduced subscheme X_{red} is affine.

Proof. If $X = \operatorname{Spec} A$, then the reduction $X_{\text{red}} = \operatorname{Spec} A/N$, where N is the ideal of nilpotents. There is surprising content in the converse.

Suppose X_{red} is affine. Recall we have a closed embedding $i\colon X_{\text{red}} \hookrightarrow X$, with the ideal sheaf of X_{red} given by $\mathscr{N}$, the ideal of nilpotents. Let $\mathscr{F}$ be a coherent sheaf of ideals on X. We have a filtration

$$\mathscr{F} \supset \mathscr{F}\mathscr{N} \supset \mathscr{F}\mathscr{N}^2 \supset \cdots.$$

Notice that for any $i \geq 0$ we have

$$0 \to \mathscr{F}\mathscr{N}^{i+1} \hookrightarrow \mathscr{F}\mathscr{N}^i \twoheadrightarrow \mathscr{F}\mathscr{N}^i/\mathscr{F}\mathscr{N}^{i+1} \to 0,$$

where the rightmost sheaf has the natural structure of an $\mathscr{O}_X/\mathscr{N} \simeq \mathscr{O}_{X_{\text{red}}}$-module. Since $H^1(X_{\text{red}}, \mathscr{F}\mathscr{N}^i/\mathscr{F}\mathscr{N}^{i+1}) = 0$ by Serre's cohomological criterion for affineness, it is enough to prove that $H^1(X, \mathscr{F}\mathscr{N}^i) = 0$ for *some* i, as then we have $H^1(X, \mathscr{F}\mathscr{N}^{i-1}) = 0$, since it's sandwiched between two vanishing H^1's. Going up the filtration, we have $H^1(X, \mathscr{F}) = 0$, and again by Serre's criterion we have X is affine.

To show that $\mathscr{F}\mathcal{N}^i$ has vanishing H^1 for some i, we argue instead that $\mathcal{N}^i = 0$ for $i \gg 0$. Covering X by finitely many affine opens $\operatorname{Spec} A$, we see that $\mathcal{N}|_{\operatorname{Spec} A} \simeq N$, where N is the nilradical of A. Let's further assume A is Noetherian. In the local ring $A_\mathfrak{p}$, where we know $\cap_{i=1}^\infty \mathfrak{p}^i = 0$, we have $\cap_{i=1}^\infty \mathfrak{p}^i = 0$ and the proof of this fact shows that $\mathfrak{p}^i = 0$ for some $i \gg 0$. Since $\operatorname{Spec} A$ has finitely many irreducible components (Noetherian), this means that we can take n to be the maximum such i for finitely many generic points. Then for $f \in N^n$, we have f^n vanishes in all local rings, so it must be that $f = 0$, and hence $N^n = 0$. Doing this for each of the affine opens covering X, we have $\mathcal{N}^i = 0$ for some $i \gg 0$, and as desired, $\mathscr{F}\mathcal{N}^i = 0$ as well, allowing us to use the previous argument. □

Bezout's theorem: Let $X \in \mathbb{P}^n_k$ be a projective scheme of degree $\deg X$ and $H \subset \mathbb{P}^n_k$ a hypersurface of degree d not containing a component of X. The (scheme theoretic) intersection $H \cap X$ has degree

$$\deg(H \cap X) = d(\deg X).$$

In the special case of curves in $\mathbb{P}^2$ (here curves are hypersurfaces) we have the classical result that the number of intersection points of curves C_1 and C_2 which don't overlap on an irreducible component is $(\deg C_1)(\deg C_2)$.

One way to prove this is with Hilbert polynomials. The key insight is that the ideal sheaf of $H \cap X$ in X is $\mathscr{O}_X(-d)$. Another is to intersect X with $\dim X$ many general hyperplanes and count intersection points to define degree, then use Bertini's theorem to make sense of this.

Closed embedding exact sequence of sheaves: Let $i\colon Z \to X$ be a closed embedding of schemes with ideal sheaf $\mathscr{I}$. We have an exact sequence of $\mathscr{O}_X$-modules

$$0 \to \mathscr{I} \to \mathscr{O}_X \to i_*\mathscr{O}_Z \to 0.$$

In the special case where X is a Noetherian scheme which is regular in codimension one (and possibly normal too) and Z is closed of codimension one — i.e. a divisor D — then we have $\mathscr{I} = \mathscr{O}(-D) = \mathscr{O}(D)^\vee$, giving us the exact sequence

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0.$$

Tensoring this exact sequence is often useful.

Conormal exact sequence: (affine version) Let $B \to A \to A/I$ be maps of rings and $\Omega_{A/B}$ the module of differentials. We can extend the cotangent exact sequence to the left (recognizing that $\Omega_{(A/I)/A} = 0$ by

$$I/I^2 \to \Omega_{A/B} \otimes_B A/I \to \Omega_{(A/I)/B} \to 0,$$

where the leftmost map sends $i + I^2 \mapsto di \otimes 1$, and extending (A/I)-linearly.

(global version) If $i\colon Z \hookrightarrow X$ is a closed embedding and $\pi\colon X \to Y$, then we have an exact sequence of sheaves on Z,

$$\mathscr{N}^\vee_{Z/X} \to i_*\Omega_{Z/Y} \to \Omega_{Z/X} \to 0.$$

Note the discrepancy with the subscripts of the cotangent exact sequence.

Cotangent exact sequence: (affine version) Let $C \to B \to A$ be maps of rings and $\Omega_{A/B}$ the module of A-differentials. We have a right-exact sequence

$$A \otimes_B \Omega_{B/C} \to \Omega_{A/C} \to \Omega_{A/B} \to 0.$$

(global version) Let $X \to Y \to Z$ be morphisms of schemes and $\Omega_{X/Y}$, etc. the module of relative differentials, i.e. the co(co)tangent sheaf. We have a right-exact sequence

$$\pi^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

where π denotes the map $X \to Y$.

Criteria for basepoint free and very ample on curves: Let X be a projective regular integral curve over an algebraically closed field $k = \overline{k}$. We record three observations:

(i) *Twisting by a point drops the dimension by* ≤ 1: Let $\mathscr{L}$ be a line bundle on X and p a closed point. Then

$$h^0(X, \mathscr{L}) - h^0(X, \mathscr{L}(-p)) \leq 1.$$

(ii) *Criteria for basepoint freeness*: A line bundle $\mathscr{L}$ on X is basepoint free if and only if for all closed points $p \in X$, the inequality in (i) is an equality,

$$h^0(X, \mathscr{L}) - h^0(X, \mathscr{L}(-p)) = 1.$$

(iii) *Criteria for very ampleness*: A line bundle $\mathscr{L}$ on X is very ample if and only if for all closed points $p, q \in X$, not necessarily distinct, the dimension drops maximally,

$$h^0(X, \mathscr{L}) - h^0(X, \mathscr{L}(-p-q)) = 2.$$

Proof sketch. For (i), we use the closed subscheme exact sequence

$$0 \to \mathscr{O}_X(-p) \to \mathscr{O}_X \to \mathscr{O}_X|_p \to 0,$$

suitably twisted by $\mathscr{L}$

$$0 \to \mathscr{L}(-p) \to \mathscr{L} \to \mathscr{L}|_p \to 0.$$

This gives the LES in cohomology,

$$0 \to H^0(X, L(-p)) \to H^0(X, \mathscr{L}) \to H^0(p, \mathscr{L}|_p) \to \cdots .$$

Since p is a degree one point and $\mathscr{L}$ is locally free, we have $\mathscr{L}|_p \simeq \mathscr{O}_p$, so $H^0(p, \mathscr{L}|_p) = k$. Therefore the map $H^0(X, \mathscr{L}) \to H^0(p, \mathscr{L}|_p)$ is either surjective, in which case the dimension drops by 1, or it's the zero map, in which case the dimensions are equal.

For (ii) we interpret $H^0(X, \mathscr{L}(-p))$ as the sections of $H^0(X, \mathscr{L})$ which vanish at p. If the dimension always drops by one, then we have a section *not* vanishing at p. If this occurs for all p, then $\mathscr{L}$ is basepoint free. Conversely, basepoint freeness implies the existence of such a section for each p, so the dimension must drop by one.

For (iii), we will show very ampleness by arguing that the map associated to the complete linear series $|\mathscr{L}|$ is injective on both points and tangent vectors (see Theorem 19.1.1). For injectivity on points, consider *distinct* points p, q. The dimension of sections drops as we go from $\mathscr{L}(-q)$ to $\mathscr{L}(-p-q)$, indicating the existence of a section vanishing on q, but not on p. Hence there is a hyperplane in $\mathbb{P}^{\dim |\mathscr{L}|-1}$ containing (the image of) q but not p. In particular, the map is injective on points.

To see injectivity on tangent vectors consider a closed point p. Dualizing, it suffices to show that the map on *co*tangent vectors is surjective. Since $\Omega_{X/k}$ is one dimensional, we need only see that the map is nonzero. Recall the (Zariski) cotangent space is $\mathfrak{m}_p/\mathfrak{m}_p^2$, so we need to find a section vanishing at p to degree precisely one. However, this exists because

$$h^0(X, \mathscr{L}(-p)) - h^0(X, \mathscr{L}(-2p)) = 1.$$

Thus such an $\mathscr{L}$ is very ample. Conversely, very ample $\mathscr{L}$ implies (i) and (ii), and because it separates points and tangent vectors, we find (iii) holds. □

Dimensions of Γ (and H^i) for $\mathscr{O}_{\mathbb{P}^n}$: Let $\mathbb{P}^n_k$ be projective A-space and $\mathscr{O}_{\mathbb{P}^n_k}(m) = \mathscr{O}(m)$. Then we can compute the dimensions of the (Cech) cohomology groups $H^i(\mathbb{P}^n_k, \mathscr{O}(m))$ as follows:

$$\begin{aligned} \dim_k H^0(\mathbb{P}^n_k, \mathscr{O}(m)) &= \binom{m+n}{m} = \binom{m+n}{n}, \\ \dim_k H^i(\mathbb{P}^n_k, \mathscr{O}(m)) &= 0, && 0 < i < n \\ \dim_k H^n(\mathbb{P}^n_k, \mathscr{O}(m)) &= \binom{-m-1}{-n-m-1} = \binom{-m-1}{n}, && m+1 \le -n \\ \dim_k H^j(\mathbb{P}^n_k, \mathscr{O}(m)) &= 0, && j > n. \end{aligned}$$

Note that the above also work over a ring A in place of k, where instead of dimension as a k-vector space, we take the rank of $H^\bullet$ as a free A-module.

The key idea here is to interpret sections of $\mathscr{O}(m)$ as homogeneous degree m polynomials. See e.g. the discussion in §14.1 of Vakil. This immediately allows us to compute the dimension of H^0 by counting degree m monomials in $n+1$ variables. If $U_i = D(x_i)$ is the usual affine patch, then $\Gamma(U_i, \mathscr{O}(m))$ may be identified with the degree m piece of $k[x_0, \ldots, x_n, \frac{1}{x_i}]$ (which in turn has a natural identification with $k[x_{0/i}, \ldots, \widehat{x_{i/i}}, \ldots, x_{n/i}]$). Moreover, if $I \subset \{0, \ldots, n\}$ the restriction maps $\Gamma(U_I, \mathscr{O}(m)) \to \Gamma(U_J, \mathscr{O}(m))$ with $I \subset J$ are interpreted as natural inclusions of polynomials where x_i appears with *nonnegative* exponent for all $i \notin I$.

Now we have the Cech complex of $\oplus \mathscr{O}(m)$, where taking the degree m piece gives the Cech complex of $\mathscr{O}(m)$.

$$0 \to k[x_0, \ldots, x_n, \frac{1}{x_0}] \times \cdots \times k[x_0, \ldots, x_n, \frac{1}{x_n}] \to k[x_0, \ldots, x_n, \frac{1}{x_0}, \frac{1}{x_1}] \times \cdots k[x_0, \ldots, x_n, \frac{1}{x_{n-1}}, \frac{1}{x_n}] \to \cdots$$
$$\cdots \to k[x_0, \ldots, x_n, \frac{1}{x_0}, \ldots, \frac{1}{x_{n-1}}] \times \cdots \times k[x_0, \ldots, x_n, \frac{1}{x_1}, \ldots, \frac{1}{x_n}] \to k[x_0, \ldots, x_n, \frac{1}{x_0}, \ldots, \frac{1}{x_n}] \to 0.$$

We can more neatly abbreviate this

$$0 \to \prod A_i \to \prod A_{ij} \to \cdots \to \prod_{\#I=n} A_I \to A_{0\ldots n} \to 0,$$

and extend it to

$$0 \to A \to \prod A_i \to \prod A_{ij} \to \cdots \to \prod_{\#I=n} A_I \to A_{0\ldots n} \to 0.$$

Note that in the above sequence, exactness at the first two places is equivalent to the computation of H^0. At each place thereafter, taking kernel mod image computes the desired cohomology groups.

Importantly, these inclusions preserve both degree and multidegree (i.e. degree of each factor x_i), allowing us to compute the cohomology monomial by monomial, following the strategy outlined for $\mathbb{P}^2$ in §18.3 of Vakil. We note here that this also shows the final assertion, that $H^j = 0$ for $j > n$ (in other words, by affine cover vanishing, since $\mathbb{P}^n$ is covered by $n+1$ affine opens).

(*all negative*) Let's consider monomials $x_0^{a_0} \cdots x_n^{a_n}$ where all $a_i < 0$. The exact sequence above becomes

$$0 \to 0_{H^0} \to \prod 0_i \to \cdots \prod_{\#I=n} 0_I \to A_{0\ldots n} \to 0.$$

It's easy to see that the sequence is exact everywhere except the final, n-th place.

(*one nonnegative*) Consider now the case that all $a_i < 0$ except for a_n. The sequence becomes

$$0 \to 0_{H^0} \to \prod 0_i \to \cdots \to \prod_{\#I=n-1} 0_I \to A_{0\ldots(n-1)} \times \prod_{\#I=n, n\in I} 0_I \to A_{0\ldots n} \to 0.$$

Since the second to rightmost map is the inclusion $A_{0\ldots(n-1)} \to A_{0\ldots n}$, we have exactness in the $(n-1)$-th place. This is seen to surject for monomials of the stated form, giving exactness of the entire sequence. Hence there is no cohomology here!

(*at least one negative*) Suppose $a_0 < 0$. The strategy above essentially works to show that the sequence is exact once again, though we have to take some care with the signs of the maps. For concreteness, consider a monomial where $a_0, \ldots, a_k < 0$ and $a_{k+1}, \ldots, a_n \geq 0$. It's straightforward to see that the leftmost map of

$$A_{0\ldots k} \to \prod_{k<i\leq n} A_{0\ldots ki} \to \prod_{k<i<j\leq n} A_{0\ldots kij}$$

is an injection, hence we have exactness in the k-th place. For exactness in the middle, we see that a (tuple of) monomial(s) can be mapped to zero if and only if it's in the image of $A_{0\ldots k}$. This argument continues to extend to the right, so we see the full sequence is exact in this case.

(*all nonnegative*) Finally, suppose $a_i \geq 0$ for all i. We can give a short exact sequence of complexes,

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A_n & \longrightarrow & A_{0n} \times \cdots A_{(n-1)n} & \longrightarrow & \cdots & \longrightarrow & A_{0\ldots n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A_{H^0} & \longrightarrow & A_0 \times A_{n-1} \times \cdots \times A_n & \longrightarrow & \prod A_{ij} & \longrightarrow & \cdots & \longrightarrow & A_{0\ldots n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A_{H^0} & \longrightarrow & A_0 \times \cdots \times A_{n-1} & \longrightarrow & \prod_{i,j\neq n} A_{ij} & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

The top row is exact, by the $a_n < 0$ case. The bottom row is exact by essentially the same argument — just think about what has to happen for all these maps to be exact. In particular, all cohomology groups of the top and bottom complexes vanish! The long exact sequence in cohomology induced by this SES of complexes implies that the middle row must also be exact.

All of this computation shows that $H^i(\mathbb{P}^n_k \mathscr{O}(m)) = 0$ for $0 < i < n$, since the Cech complex has no cohomology here. The n-th (final) Cech cohomology vanishes unless we're in the "all negative case." This means that for $\oplus\mathscr{O}(m)$, the Cech cohomology group is $x_0^{-1} \cdots x_n^{-1} k[x_0^{-1}, \ldots, x_n^{-1}]$, as this contains the only monomials which can appear. Thus the "degree m piece" corresponds to the degree $-m-n-1$ polynomials in the x_i^{-1}, giving a dimension count of $\binom{-m-1}{-m-n-1}$, which makes sense only when $m \leq -n-1$.

Finiteness of zeros and poles: Suppose X is an integral Noetherian scheme. Then

- any rational function $t \in K(X)^\times$ has finitely many zeros and poles (i.e. it vanishes on finitely many codimension one primes).
- If X is regular in codimension one, div t is well defined.
- If X is normal, having no poles is equivalent to being a regular function.

Proof. For the first statement, we can check on an affine cover of X by Spec A where A is a Noetherian integral domain. Suppose $t = \frac{f}{g}$ for $f, g \in A$. It suffices to check that f (hence also g) is contained in finitely many codimension one primes. To see this, we look at $A/(f)$, which is Noetherian and whose minimal prime ideals consist of the codimension one primes of A containing f (since $f \neq 0$).

Thus we need only show Noetherian rings have finitely many *minimal* prime ideals. This can be done geometrically by recognizing that a minimal prime ideal corresponds to a maximal irreducible closed subset of Spec A, i.e. an irreducible component. Since we also know Spec A is Noetherian as a topological space, it suffices to show Noetherian topological spaces have finitely many irreducible components. If X has a countably infinite collection of irreducible components $X_1, X_2, \ldots$ then we take

$$X \supseteq \cup_{i\geq 1} X_i \supsetneq \cup_{i\geq 2} X_i \supsetneq \cdots$$

which contradicts the Noetherianness.

Hence our Noetherian ring A has finitely many minimal primes, which means that any function f vanishes on finitely many codimension one primes. Thus $t = \frac{f}{g}$ has finitely many zeros and poles.

To make sense of the *order* of zeros and poles, we use the regular in codimension one condition. This ensures $A_\mathfrak{p}$ is a regular local ring when $\mathfrak{p}$ is height one, which implies $A_\mathfrak{p}$ is a DVR (see e.g. §12.5), hence the order of vanishing of a rational function t at $\mathfrak{p}$ is well defined by the valuation. Thus we can take

$$\text{div } t = \sum_{\mathfrak{p} \text{ codim. } 1} v_\mathfrak{p}(t)\mathfrak{p}.$$

Suppose X is normal, so the stalks If t has no poles, we claim it is regular. Again, we can see this on the level of affine schemes, using the fact that for a Noetherian integrally closed domain A, we have $A = \cap A_\mathfrak{p}$, where the intersection is taken over the codimension one primes. If t has no poles, then $t \in A_\mathfrak{p}$ for all $\mathfrak{p}$, hence in A. □

FHHF theorem: Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive functor of abelian categories. Suppose $C^\bullet$ is a complex in $\mathscr{A}$ (so $F(C^\bullet)$ is a complex in $\mathscr{B}$) and use $H^i(C^\bullet)$ to denote the i-th cohomology. Then

(i) If F is *right exact* then there is a natural morphism

$$F\left(H^i\left(C^\bullet\right)\right) \to H^i\left(F\left(C^\bullet\right)\right).$$

(ii) If F is *left exact* then there is a natural morphism *the other way*

$$F\left(H^i\left(C^\bullet\right)\right) \leftarrow H^i\left(F\left(C^\bullet\right)\right).$$

(iii) If F is *exact* then the natural morphism(s) from (i) and (ii) are isomomorphisms

$$F\left(H^i\left(C^\bullet\right)\right) \simeq H^i\left(F\left(C^\bullet\right)\right).$$

Proof sketch. We will give the idea of (ii) here. The proof of (i) is analogous after switching some arrows. (iii) follows from (i) and (ii), as we will see.

As $H^i(C^\bullet) = \ker \delta^i / \operatorname{im} \delta^{i-1}$ we have

$$0 \to \operatorname{im} \delta^{i-1} \to \ker \delta^i \to H^i \to 0.$$

Applying F, we see

$$0 \to F\left(\operatorname{im} \delta^{i-1}\right) \to F\left(\ker \delta^i\right) \to FH^i.$$

On the other hand, we have the natural exact sequence in $\mathscr{B}$ given by

$$0 \to \operatorname{im} F\delta^{i-1} \to \ker F\delta^i \to H^i F \to 0.$$

We need a quick intermediate: a natural map (a monomorphism in fact) $\operatorname{im} Ff \to F(\operatorname{im} f)$ for any map f in $\mathscr{A}$ *when F is left exact.* In fact, it's enough to observe a natural map $\operatorname{coker} F(f) \to F(\operatorname{coker} f)$, by

$$\begin{array}{ccccc} & & & & F(\operatorname{coker} f) \\ & \nearrow^{F(c\circ f)=0} & & \nearrow_{c} & \uparrow \\ F(A) & \xrightarrow[f]{} & F(B) & \longrightarrow & \operatorname{coker} F(f). \end{array}$$

Thus we have natural maps

$$\operatorname{im} F(f) = \ker(\operatorname{coker} F(f)) \to \ker(F(\operatorname{coker} F)) \simeq F(\ker \operatorname{coker} f) = F(\operatorname{im} f)$$

by left exactness — F commutes with kernels.

This gives the diagram below, inducing the dashed line by universal properties, completing the proof of (ii).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} F\delta^{i-1} & \longrightarrow & \ker F\delta^{i} & \longrightarrow & H^iF & \longrightarrow & 0 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & F\left(\operatorname{im}\delta^{i-1}\right) & \longrightarrow & F\left(\ker\delta^{i}\right) & \longrightarrow & FH^i & & \end{array}$$

To show (i), do the same thing, only using the sequence

$$0 \to H^i \to \operatorname{coker} \delta^{i-1} \to \operatorname{im} \delta^i \to 0$$

and applying the right exact functor. Similarly, we'll need to argue that $F(\operatorname{im}\delta) \to \operatorname{im} F(\delta)$.

If F is exact, it is both left and right exact, and hence commutes with kernels and cokernels. However, this means $F(\ker \delta^i) = \ker F(\delta^i)$ and $F(\operatorname{im} \delta^{i-1}) = \operatorname{im} F(\delta^{i-1})$ and finally

$$FH^i = F(\ker\delta^i/\operatorname{im}\delta^{i-1}) = F(\ker\delta^i)/F(\operatorname{im}\delta^{i-1}) = \ker F(\delta^i)/\operatorname{im} F(\delta^{i-1}) = H^iF$$

where "=" is used to indicate "unique isomorphism." □

Finite morphisms: There are several nice properties of finite morphisms. Below, $\pi\colon X \to Y$ is a finite morphism.

- finite implies projective (implies proper)
- finite = integral + finite type
- finite implies quasifinite
- finite = proper + quasifinite

Proof. We need to show that $X = \mathcal{P}roj\ \mathscr{S}_\bullet$ for some quasicoherent sheaf of Y-algebras. Ultimately, this boils down to the affine case: showing that a finite B-algebra A is in fact projective. To see this, let $S_\bullet$ be given by $S_0 = B$, $S_1 = A$, and $S_n = A$ for $n > 1$. The multiplication is given by multiplication in A, and this is clearly finitely generated as a B-module in degree 1.

To check $\operatorname{Spec} A \simeq \operatorname{Proj}_B S_\bullet$, we look at the affine opens $\operatorname{Spec}((S_\bullet)_f)_0$ for $f \in A$ considered as degree n; in fact we can assume $n = 1$. The ring $((S_\bullet)_f)_0$ is generated by elements of the form g/f^k where $g \in A$ has degree k. We see that two such elements being equal is the same as them being equal in A_f, so $\operatorname{Proj} S_\bullet$ is covered by affine opens that are isomorphic to distinguished opens in $\operatorname{Spec} A$, hence they are isomorphic.

We have seen projective implies proper, so we have finite morphisms are proper.

For finite is integral and finite type, we first observe that finite trivially implies finite type. Integrality follows from seeing that if $B \to A$ is a finite map of algebras, it is an integral one. The converse comes from using finite type to give algebra generators $a_1, \dots, a_n$, and recognize that integrality implies a (finite) power basis of the a_i's generates A as a B-module.

For finite implies quasifinite, consider a point $q \in \operatorname{Spec} B \subseteq Y$, i.e. a map $\operatorname{Spec} \kappa(q) \to Y$ coming from $B \to B_{\mathfrak{q}}/\mathfrak{q}$. Since finite morphisms are affine, the preimage of $\operatorname{Spec} B$ is $\operatorname{Spec} A \subseteq X$ and we have that the fiber is precisely

$$\pi^{-1}(q) = \operatorname{Spec} A \otimes_B B_{\mathfrak{q}}/\mathfrak{q}.$$

Now since $A \otimes_B B_{\mathfrak{q}}/\mathfrak{q}$ is finitely generated as a $\kappa(q)$ vector space, we have reduced to the case $B = k$ a field and A a finite k-algebra. Since A is Noetherian (rings are Noetherian and finitely generated modules are Noetherian) Spec A has finitely many irreducible components, i.e. it's covered by Spec A_i for integral domains A_i.

If A is an integral domain and a finite k-algebra, it is a field because if $f \neq 0$ then the multiplication by f map is an injection $A \to A$ of k-modules, hence surjective by linear algebra, giving an inverse. Thus A is a field and has only one prime. □

Grothendieck's vanishing theorem: Let X be a scheme of dimension n. Then

$$H^i(X, \mathscr{F}) = 0$$

for all $i > n$ and quasicoherent $\mathscr{F}$ on X.

Proof idea (projective case). Use affine cover cohomology vanishing. Find $n+1$ hypersurfaces in $\mathbb{P}^n$ whose intersections miss X, thus X is contained in the union of $n+1$ $D(f)$'s. Closed embeddings are affine, so the preimages of these are affine and cover X. Thus the cohomology vanishes for all $i \geq n+1$. □

Integral = reduced + irreducible: X is integral if and only if X is reduced and irreducible.

($\implies$) Suppose X is integral. Then $\mathscr{O}_X(U)$ is an integral domain for all U. In particular, all rings of sections are reduced, so X is reduced. Suppose X is reducible. Then $X = Z_1 \cup Z_1$ for nonempty sets Z_i. Assume $p_i \in Z_i - Z_j$, and $p_i \in \operatorname{Spec} A_i$ for affine opens in X. Moreover, we can assume $\operatorname{Spec} A_i \cap Z_j = \emptyset$. But then Spec A_i are disjoint, so $\mathscr{O}_X(\operatorname{Spec} A_1 \cup \operatorname{Spec} A_2) = A_1 \times A_2$, which is only an integral domain if one of them is the zero ring (it isn't because of nonemptyness!).

($\impliedby$) Suppose X is reduced and irreducible. Let's do the affine case first. Suppose $f, g \in A$ such that $fg = 0$. Then $V(fg) = V(f) \cup V(g) = \operatorname{Spec} A$. By irreducibility, one of $V(f)$ or $V(g)$ is zero, so assume it's f. Then f is nilpotent, but by reducedness, we have $f = 0$, therefore A is an integral domain.

For the general case, we'll need to cover X by affine opens Spec A_i. If $f, g \in \Gamma(X, \mathscr{O}_X)$ with restrictions f_i, g_i, suppose $fg = 0$, which implies $f_i g_i = 0$. By the previous paragraph (and the observation that a dense open set of an irreducible scheme is irreducible) we have $f_i g_i = 0 \implies f_i = 0$ or $g_i = 0$. If $f_i, g_j \neq 0$ for some i, j, then we have $V(f) \cup V(g) = X$ is a reduction. Conclude f (or g) must be zero, hence $\mathscr{O}_X(X)$ is an integral domain, and hence $\mathscr{O}_X(U)$ is for all U.

Lying over, going up, going down: This flavor of result relates primes on either side of ring maps. These are often useful in proving things about dimension.

- (lying over) Suppose $\phi \colon B \to A$ is an integral *extension* of rings. Then for all primes $\mathfrak{q} \subset B$, there exists a prime $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap B = \mathfrak{q}$, i.e. $\mathfrak{p}$ "lies over" $\mathfrak{q}$.

  Geometrically, this is saying that Spec $A \to$ Spec B is surjective, because the image of $\mathfrak{p} \in \operatorname{Spec} A$ is precisely its preimage under ϕ, which is intersection with B since the map is an inclusion of rings.

- (going up) Suppose $\phi \colon B \to A$ is an integral *map* of rings (not necessarily an extension). Then if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain in B and $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_m$ is a chain in A with $\mathfrak{p}_i$ lying over $\mathfrak{q}_i$, the chain upstairs can be extended to $\mathfrak{p}_n$ lying over $\mathfrak{q}_n$.

  We are "going up" in the sense that we are extending the chain of primes in A lying over those in B in the upward direction.

- (going down) Suppose $\phi \colon B \hookrightarrow A$ is a finite extension of integral domains with B integrally closed. Given a chain $\mathfrak{q} \subset \mathfrak{q}'$ in B and a prime $\mathfrak{p}'$ lying over $\mathfrak{q}'$, there exists $\mathfrak{p} \subset \mathfrak{p}'$

such that $\mathfrak{p}$ lies over $\mathfrak{q}$.

Here we are "going down" by going the other way along the chain of inclusions.

Map of proper curves is constant or surjective: Let C, C' be proper irreducible curves over k. Then any map $C \to C'$ over k is surjective or a constant map.

Proof idea. By "property P" arguments, such a map is proper, since C' is separated and closed embeddings are proper. The image of C must be an irreducible closed subset, hence either a closed point or all of C'. □

Maps to $\mathbb{P}^n$ using line bundles: Let X be a k-scheme and $V \subseteq \Gamma(X, \mathscr{L})$ where $\mathscr{L}$ is a line bundle on X. We could just as well take $V \subseteq \Gamma(X, \mathscr{O}_X(D))$ where D is a Weil divisor, if they are well defined. Let $\dim V = n+1$. In general, we have

$$X - B \to \mathbb{P}V,$$

where $\mathbb{P}V$ is the coordinate-free projectivization of V.

We'll prove this with coordinates. Let $s_0, \ldots, s_n$ be a basis of sections and take B to be the base locus, i.e. the scheme theoretic intersection of $V(s_i)$. (If V is base point free we may ignore this altogether.) We define

$$X - B \to \mathbb{P}^n, \qquad P \mapsto [s_0(P) : \cdots : s_n(P)]$$

informally. More precisely, consider the open subscheme $D(s_i) \subseteq X$ where s_i doesn't vanish. Then we have on any trivializing open subset U of X, $D(s_i) \to D(x_i)$, where $D(x_i)$ is the standard affine open patch of $\mathbb{P}^n$. There is a natural map $k[x_0, \ldots, x_n]_{x_i} \to \Gamma(D(s_i), \mathscr{O}_U)$ sending $x_j \mapsto s_j$, and thus we glue to obtain $U \to \mathbb{P}^n$. We can glue along the trivializing open sets as well to get $X - B \to \mathbb{P}^n$.

Some other useful comments:

- If $\mathscr{L}$ (or D) is very ample, then $|\mathscr{L}|$ (or $|D|$) defines an embedding of X into $\mathbb{P}^n$ where the image is a degree $\deg \mathscr{L}$ (or $\deg D$) variety. In this case, $\mathscr{L}$ is the pullback of $\mathscr{O}(1)$ along the embedding.
- If X is a curve and $\dim V = 2$ then the induced map $X \to \mathbb{P}^1$ has degree $\deg \mathscr{L}$ (or $\deg D$).
- In fact, all maps to $\mathbb{P}^n$ arise in this way, up to isomorphism (of the bundle and the sections). To see this, given $X \to \mathbb{P}^n$, we pull back the $n+1$ hyperplane sections to $s_i \in \Gamma(X, \pi^*\mathscr{O}(1))$ and check that the above construction gives back the map to $\mathbb{P}^n$.

Nakayama's lemma: There are several related results that go by the name "Nakayama's lemma." Here are some of them. Unless otherwise noted, A denotes a ring, I an ideal of A, and M an A-module.

(i) Suppose M is finitely generated and $M = IM$. Then there exists $a \equiv 1 \pmod{I}$ such that $aM = 0$. Equivalently, since $a = 1+i$ for $i \in I$, we have $(-i)m = m$ for all $m \in M$.
(ii) Suppose I is contained in all maximal ideals (i.e. $I \subseteq \operatorname{Jac} A$) and M is finitely generated, such that $M = IM$. Then $M = 0$.
(iii) Suppose I is contained in all maximal ideals (i.e. $I \subseteq \operatorname{Jac} A$) and M is finitely generated, with a submodule $N \subseteq M$. If $N/IN \to M/IM$ is a surjection, then $M = N$.
(iv) Suppose $(A, \mathfrak{m})$ is local, M is finitely generated, and (the images of) $f_1, \ldots, f_n \in M$ generate $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$-vector space. Then the f_i's generate M as an A-module.

Proof. To prove (i), we can use the determinant trick. Let $m_1, \ldots, m_n$ be generators. Then $m_i = \sum_j a_{ij} m_j$ for $a_{ij} \in I$. Thus (a_{ij}) times the vector $(m_1, \ldots, m_n)^T$ is precisely $(m_1, \ldots, m_n)^T$. Equivalently, $I - (a_{ij})$ induces the zero map $M \to M$. Multiplying by the adjoint matrix, we have $\det(I - (a_{ij}))$ induces the zero map as well, but upon inspection we see that the deteriminant comes out to an element of A which is $1 \pmod{I}$.

For (ii), we need only recognize that $a \equiv 1 \pmod{I}$ implies $a \in A^\times$ when I is contained in all maximal ideals. Since $a \equiv 1 \pmod{\mathfrak{m}}$ for all maximal $\mathfrak{m}$, we have $a \notin \mathfrak{m}$. All nonunits are in a maximal ideal, so a is invertible, and hence $M = 0$.

For (iii) we leverage (ii) as follows. Let L be the cokernel of the inclusion $M \to N$. Since taking quotients is right exact (tensoring by A/I) we have an exact sequence

$$N/IN \to M/IM \to L/IL \to 0.$$

The fact that $N/IN \to M/IM$ is a surjection menas that $L/IL = 0$, i.e. $L = IL$. Since L is finitely generated (it's the quotient of a f.g. module) and I is contained in all maximal ideals, by (ii) we have $L = 0$, so $M = N$.

(iv) is just (iii) applied to the case of $N = \langle f_1, \ldots, f_n \rangle$ and $I = \mathfrak{m}$, which is trivially contained in all maximal ideals. □

Pic $\mathbb{P}^n_k \simeq \mathbb{Z}$: The Picard group of projective n-space $\mathbb{P}^n_k$ is given by

$$\operatorname{Pic} \mathbb{P}^n_k \simeq \operatorname{Cl} \mathbb{P}^n_k \simeq \mathbb{Z}.$$

Explicitly, this group is generated by the class of a hyperplane, or equivalently the line bundle $\mathscr{O}(1)$.

Proof. First note that $\mathbb{P}^n$ is covered by copies of $\mathbb{A}^n$, and $k[x_1, \ldots, x_n]$ is a UFD, so its stalks are as well. Hence we are free to identify the Picard group with the Weil divisor class group.

Next we use the excision exact sequence,

$$0 \to \mathbb{Z} \to \operatorname{Div} X \to \operatorname{Div}(X - Z) \to 0,$$

where Z is an irreducible codimension one subscheme (i.e. a prime divisor) and the map sends $1 \mapsto Z$. For this problem, we take $X = \mathbb{P}^n$ and Z to be a hyperplane isomorphic to $\mathbb{P}^{n-1}$ (the hyperplane at infinity, if you like). Quotienting by principal divisors, we have

$$\mathbb{Z} \to \operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} \mathbb{A}^n \to 0.$$

Again, using that $\mathbb{A}^n \simeq \operatorname{Spec} k[x_1, \ldots, x_n]$ which is a UFD, the class group vanishes, so $Z \twoheadrightarrow \operatorname{Cl} \mathbb{P}^n$ is a surjection. Thus the class of the hyperplane, $[H]$ generates $\operatorname{Cl} \mathbb{P}^n$.

It remains to see that $n[H] \neq 0$ for all $n > 0$. Suppose $nH = \operatorname{div} t$ for $t \in K(\mathbb{P}^n)^\times$ and $n \geq 0$. Then since nH has nonnegative degree, t has no poles, and thus is regular. But the only regular functions on $\mathbb{P}^n$ are the constants, which have degree zero (when nonzero). Hence $[nH] \neq 0$ for $n > 0$, i.e. $\operatorname{Cl} \mathbb{P}^n \simeq \langle [H] \rangle \simeq \mathbb{Z}$.

It's useful to identify $[H]$ with the line bundle $\mathscr{O}(1)$, and thus $[nH]$ with $\mathscr{O}(n)$. To see this, consider the global section x_0 of $\mathscr{O}(1)$. Its image in the standard affine patch U_i is 1 for $i = 0$ and $x_{0/i}$ for $0 < i \leq n$. Thus for $i > 0$, the image of x_0 has a zero at the $V(x_{0/i})$ hyperplane, and we have $\operatorname{div} x_0 = H$. In fact, any two hyperplanes are linearly equivalent, a fact we can realize by passing through $\operatorname{div} x_0 = H_0$: if H is some other hyperplane, it is given by the vanishing of a linear form s, and hence $\operatorname{div} \frac{s}{x_0} = H - H_0$ is principal. □

$\operatorname{Pic} X$ isomorphic to $H^1(X, \mathscr{O}_X^\times)$: Let X be a quasicompact separated scheme (i.e. a scheme where Cech cohomology makes sense). Then

$$\operatorname{Pic} X \simeq H^1(X, \mathscr{O}_X^\times)$$

where $\mathscr{O}_X^\times$ is the sheaf of abelian groups given by taking units: $U \mapsto \Gamma(U, \mathscr{O}_X)^\times$.

Proof idea. Use Cech cohomology on some affine open cover $X = \cup \operatorname{Spec} A_i$, where the intersection $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j = \operatorname{Spec} A_{ij}$ is affine. Giving a line bundle involves giving transition functions, which will correspond to units in the rings A_{ij}. Then show these units sit in the kernel of the map in the Cech complex

$$\prod A_{ij}^\times \to \prod A_{ijk},$$

giving a map

$$\mathscr{L} \mapsto H^1(X, \mathscr{O}_X^\times).$$

Next, show that if these transition functions are coboundaries, i.e. in the image of $\prod A_i^\times$, then the original line bundle is trivial. Hence the map above is an isomorphism.

Note, this uses the fact that we can compute H^1 via Cech cohomology, which is nontrivial in this case since $\mathscr{O}_X^\times$ is not a quasicoherent sheaf. However, we have now seen that Pic X maps to the Cech H^1 for any open cover, so it has a map to the limit which must also be an isomorphism. An exercise in Hartshorne (III.4.4) shows that derived functor H^1 is isomorphic to the limit of Cech H^1's, completing the proof. □

Properties of cohomology: Let X be a scheme. Given a quasicoherent $\mathscr{O}_X$-module $\mathscr{F}$, recall $H^i(X, \mathscr{F})$ is the i-th sheaf cohomology group, which we may compute via Cech cohomology if X is quasicompact and separated.

(i) **functoriality:** $H^i(X, \cdot)$ is a covariant functor from $\mathrm{QCoh}_X \to \mathrm{Ab}$. If X is a scheme over A then it is a functor $\mathrm{QCoh}_X \to \mathrm{Mod}_A$. In particular, $H^i(X, \mathscr{F})$ is a vector space for a scheme over a field.

(ii) ***H*^{0} is global sections:** For any sheaf $\mathscr{F}$ on X, we have $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

(iii) **long exact sequence:** Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be a short exact sequences of sheaves on X. There is a long exact sequence in cohomology,

$$\begin{aligned} 0 \to & H^0(X, \mathscr{F}') \to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}'') \to H^1(X, \mathscr{F}') \to H^1(X, \mathscr{F}) \to \cdots \\ \cdots \to & H^n(X, \mathscr{F}') \to H^n(X, \mathscr{F}) \to H^n(X, \mathscr{F}'') \to H^{n+1}(X, \mathscr{F}') \to H^{n+1}(X, \mathscr{F}) \to \cdots \end{aligned}$$

(iv) **contravariant in space:** If $\pi \colon X \to Y$ is a map of schemes (possibly over a base) then there are natural maps

$$\begin{aligned} H^i(Y, \pi_* \mathscr{F}) &\to H^i(X, \mathscr{F}) \\ H^i(Y, \mathscr{G}) &\to H^i(X, \pi^* \mathscr{G}), \end{aligned}$$

allowing us to reasonably say that H^i is "contravariant in the space."

(v) **affine morphisms:** Suppose $\pi \colon X \to Y$ is an affine morphism. Then the natural map of (iv) is an isomorphism

$$H^i(Y, \pi_* \mathscr{F}) \xrightarrow{\sim} H^i(X, \mathscr{F}).$$

(vi) **affine cover vanishing:** Suppose X is covered by n affine open sets. Then

$$H^i(X, \mathscr{F}) = 0$$

for all $i \geq n$. In particular, if X is affine then all higher cohomology vanishes for all quasicoherent $\mathscr{F}$.

(vii) ***H*^{i} commutes with (filtered) colimits:** Let X be an A-scheme. Then for a filtered direct system $\mathscr{F}_j$, we have

$$\varinjlim_j H^i(X, \mathscr{F}_j) \simeq H^i(X, \varinjlim_j \mathscr{F}_j).$$

In particular, cohomology commutes with direct sums

$$\oplus_j H^i(X, \mathscr{F}_j) \simeq H^i(X, \oplus_j \mathscr{F}_j).$$

(viii) **Dimensions of Γ (and H^i) for $\mathscr{O}_{\mathbb{P}^n}$:** Consider $\mathbb{P}^n_k$. Then

$$h^0(\mathbb{P}^n, \mathscr{O}(m)) = \binom{m+n}{n}$$
$$h^n(\mathbb{P}^n, \mathscr{O}(m)) = \binom{-m-1}{n}$$
$$h^i(\mathbb{P}^n, \mathscr{O}(m)) = 0 \text{ for all } i \neq 0, n.$$

(ix) **H^i finitely genereated:** Let X be a projective A-scheme for a Noetherian ring A and $\mathscr{F}$ a coherent sheaf on X. Then $H^i(X, \mathscr{F})$ is finitely generated for all i. In particular, if $A = k$ then $H^i(X, \mathscr{F})$ is a finite dimensional vector space over k.

(x) **Serre vanishing:** Let X be a projective A-scheme for a Noetherian ring A and $\mathscr{F}$ a coherent sheaf on X. Then for $m \gg 0$, we have $H^i(X, \mathscr{F}(m)) = 0$ for all $i > 0$. This holds without Noetherian hypotheses as well.

(xi) **base change:** Let X be a quasicompact and separated over a field k. Then for any field extension K/k, we have

$$H^i(X, \mathscr{F}) \otimes_k K \simeq H^i(X \times_k \operatorname{Spec} K, \mathscr{F} \otimes_k K).$$

(xii) **dimensional vanishing, aka. ??**: Let X be a projective k-scheme. Then $H^i(X, \mathscr{F}) = 0$ for all $i > \dim X$ and any quasicoherent $\mathscr{F}$.

(xiii) **Serre's cohomological criterion for affineness**: X is affine if and only if $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ and all quasicoherent $\mathscr{F}$ on X. This is a converse to (vi) when $n = 1$.

Proof sketch(es). (i) This is obvious from the derived functor setup. If you setup with Cech cohomology, a map of sheaves gives a map on Cech complexes, which admits maps on cohomology.

(ii) In the derived functor setup, this is by definition, as $H^0(X, \cdot) = \Gamma(X, \cdot)$. For Cech cohomology, this is precisely the sheaf axiom.

(iii) Again, if you take the derived functor approach this is implied by the LES for derived functors. In general, if we have an exact sequence of complexes $0 \to C'_\bullet \to C_\bullet \to C''_\bullet \to 0$ we can take any two "rows"

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_i & \longrightarrow & C_i & \longrightarrow & C''_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'_{i+1} & \longrightarrow & C_{i+1} & \longrightarrow & C''_{i+1} & \longrightarrow & 0 \end{array}$$

and quotient the top row by the image, while taking the kernel of the bottom, because these are complexes. This removes left exactness on top and right exactness on the bottom:

$$\begin{array}{ccccccccc} & & C'_i/\operatorname{im}(C'_{i-1}) & \longrightarrow & C_i/\operatorname{im}(C_{i-1}) & \longrightarrow & C''_i/\operatorname{im}(C''_{i-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker \delta'_{i+1} & \longrightarrow & \ker \delta_{i+1} & \longrightarrow & \ker \delta''_{i+1} & & \end{array}$$

One observes the kernels of the vertical maps are H^i's while the cokernels are H^{i+1}'s. Applying the snake lemma gives the LES.

(iv) Using Cech cohomology, it's straightforward to see there's a map on Cech complexes $C^\bullet(Y, \pi_* \mathscr{F}) \to C^\bullet(X, \mathscr{F})$, by choosing covers appropriately and taking restriction maps. For the pullback, if $\mathscr{G}$ is a quasicoherent sheaf on Y, there is a natural map $\mathscr{G} \to \pi_* \pi^* \mathscr{G}$ by adjointness, so by the above and (i) we have

$$H^i(Y, \mathscr{G}) \to H^i(Y, \pi_* \pi^* \mathscr{G}) \to H^i(X, \pi^* \mathscr{G}).$$

(v) When computing the natural map on Cech cohomology in (iv), observe that if $\cup U_i = Y$ is a finite affine cover of Y, then $\cup \pi^{-1}(U_i)$ is a finite affine cover of X. Thus the Cech complexes are identical.

(vi) Computing via Cech cohomology, the complex stops at the n-th position, so the n-th and higher cohomology vanish *if $H^i(X, \mathscr{F})$ is independent of the cover we compute with.* In order to prove this, we actually need to first show the $n = 1$ case, that $H^i(X, \mathscr{F})$ vanishes for $i > 0$ when X is affine (for any cover).
The proof goes as follows. First assume that $\operatorname{Spec} A$ itself is in the cover. Then the Cech complex for this cover (with $\Gamma(X, \mathscr{F})$ appended) sits in the middle of an exact sequence with the complex forced to contain $\operatorname{Spec} A$ on top and the cover with $\operatorname{Spec} A$ removed on the bottom. The top and bottom are identical, but shifted, and the maps on cohomologies between them are isomorphisms, forcing $H^i(X, \mathscr{F}) = 0$ for $i > 0$.
In general, if $X = \operatorname{Spec} A$ and $\cup U_i$ is an affine cover, we choose a distinguished open cover of X such that each distinguished open is contained in a U_i. This allows us to compute locally on our distinguished cover, which is in the previous case by our choices, so the cohomology vanishes.
We can then show that adding an open set to a cover doesn't change the Cech cohomology, so any two affine covers compute the same H^i's.

(vii) This is a consequence of the fact that filtered colimits are exact in Mod_A and the FHHF theorem, which states that exact functors commute with cohomology. See Exercises 1.6.H and 1.6.L.

(viii) See Dimensions of Γ (and H^i) for $\mathscr{O}_{\mathbb{P}^n}$.

(ix) We're free to use (v) to compute $\pi_* \mathscr{F}$ on $\mathbb{P}^n_A$ instead, allowing us to use (viii) above. We'll also need that since $\mathscr{F}$ is coherent, $\mathscr{F}(m)$ is globally generated for some $m \gg 0$ (this has to do with (very) ampleness). What we need is that

$$0 \to \mathscr{G} \to \mathscr{O}(m)^{\oplus j} \to \mathscr{F} \to 0$$

is exact for some m and j, and coherence implies that $\mathscr{G}$ is also coherent. By (vi) we have vanishing of $H^i(\mathbb{P}^n, \mathscr{F})$ for $i > n$. The LES of (iii) ends in

$$\cdots \to H^n(\mathbb{P}^n, \mathscr{G}) \to H^n(\mathbb{P}^n, \mathscr{O}(m))^{\oplus j} \to H^n(\mathbb{P}^n, \mathscr{F}) \to 0.$$

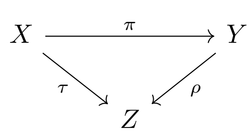
Thus $H^n(\mathbb{P}^n, \mathscr{F})$ is a quotient of a finitely generated A-module, and hence is finitely generated itself. This holds for $H^n(\mathbb{P}^n, \mathscr{G})$ as well, as we haven't used anything about $\mathscr{F}$ here. Now we see $H^{n-1}(\mathbb{P}^n, \mathscr{F})$ is sandwiched between finitely generated modules in the LES, so it too must be finitely generated. Inducting downwards, we are done.

(x) Repeat the above proof, but twist by $\mathscr{O}(N)$ for N sufficiently large that $H^n(\mathbb{P}^n, \mathscr{O}(m + N)) = 0$, which is possible by (viii). Then the argument from (ix), combined with the fact that $H^i(\mathbb{P}^n, \mathscr{O}(m + N)) = 0$ always for $0 < i < n$ by (viii), we find that $H^i(\mathbb{P}^n, \mathscr{F}(N)) = 0$ for $0 < i < n$.

(xi) This follows from the FHHF theorem, since the functor $\otimes_k K$ is exact on vector spaces. This can be extended to flat base change by the same argument.

□

Property P arguments: Let P be some class of morphisms which is preserved under base change and composition. Suppose



such that τ is in P and the diagonal of ρ, $\delta\colon Y \to Y \times_Z Y$, is in P. Then π is in P.

As a consequence, we have

- If locally closed embeddings are in P then τ in P *always* implies π in P;
- If closed embeddings are in P, then τ in P plus $\rho\colon Y \to Z$ *separated* implies π in P;
- If quasicompact morphisms are in P, then τ in P plus $\rho\colon Y \to Z$ *quasiseparated* implies π in P.

As a useful example, consider P to be proper morphisms, which are preserved by base change and composition (this can be proven via e.g. the valuative criterion). If $\pi\colon X \to Y$ is a morphism of proper k-schemes, we find π itself is proper. To see this, we take $Z = \operatorname{Spec} k$ in the diagram above, with the structure morphisms τ, ρ both proper. We know closed embeddings are proper, and proper includes separated, so by the second bullet point above, π is proper.

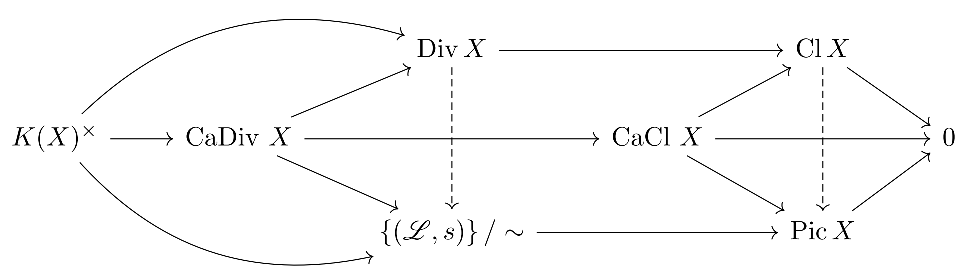
Proof sketch. Define the graph of π to be the map $X \to X \times_Z Y$ given by (id, π). Then check that

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{id},\pi)} & X \times_Z Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle (\pi,\mathrm{id})} \\ Y & \xrightarrow[\delta]{} & Y \times_Z Y \end{array}$$

is Cartesian (it's a magic square, if that's meaningful to you). Since property P is preserved by base change, the top arrow, i.e. the graph morphism, has property P. The projection $X \times_Z Y \to Y$ also has property P because it is the base change of τ by ρ. Therefore the composition $X \to X \times_Z Y \to Y$, which is π itself, has property P.

The bullet points follow directly from the characterizations of the diagonal: it is always locally closed, it's closed — hence in P — if ρ is separated, and likewise it's quasicompact and in P if ρ is quasiseparated. □

Relationship(s) between line bundles, Weil divisors, Cartier divisors: Let X be a scheme. In order for the group of Weil divisors its class group (top row of the diagram) to make sense, we ask for X to be Noetherian and regular in codimension one. Then the solid arrows form a commutative diagram.



If X is Noetherian and factorial (note this is stronger than normal, since UFDs are integrally closed) then the dotted arrows exist and

$$\operatorname{CaCl} X \simeq \operatorname{Cl} X \simeq \operatorname{Pic} X.$$

Proof idea. First we define the appropriate maps.

$$\operatorname{CaDiv} X \to \operatorname{Div} X$$

is obtained by taking the closed subscheme associated to an (effective) invertible ideal sheaf $\mathscr{I}$ on X. Since $\mathscr{I}$ is locally principal (part of the definition of a Cartier divisor) the subscheme it cuts out is codimension one.

In the other direction, we get

$$\operatorname{CaDiv} X \to \{(\mathscr{L}, s)\} / \sim$$

by sending $\mathscr{I} \mapsto (\mathscr{I}, 1)$ and extending linearly.

The dotted arrow is $D \mapsto (\mathscr{O}(D), 1)$ in one direction, and $(\mathscr{L}, s) \mapsto \operatorname{div} s$ in the other. The remaining content is to show that the maps on class groups exist, and when X is locally factorial, they are isomorphisms.

To see the maps on class groups exist, we need only check that for a rational function $t \in K(X)^\times$, the associated Cartier divisor, Weil divisor, and line bundle + section all agree. This is easy to see, because if $t = \frac{f}{g}$ on an affine open neighborhood $\operatorname{Spec} A$, the Cartier divisor is the difference of the ideals (f) and (g), which is invertible by construction. The associated line bundle is $(\mathscr{O}_X, t)$, and the associated Weil divisor is simply div t, so these all agree.

Finally, we comment that div : $(\mathscr{L}, s) \to \operatorname{Div} X$ always makes sense. Since locally $\mathscr{L} \simeq \mathscr{O}$, and s is a rational section, we can interpret the divisor accordingly, and moreover, it is always *locally principal*, i.e. locally coming from a rational function. Indeed, given an isomorphism of pairs $(\mathscr{L}, s) \sim (\mathscr{L}', s')$ the image is the same Weil divisor. There exist examples of X (necessarily not factorial) with Weil divisors which are not locally principal.

When X is factorial, $\mathscr{O}(D)$ is invertible for all Weil divisors D. This is seen for prime D by covering X by $X - D$ and an open set on which a function generating the ideal for D in $\mathscr{O}_{X,p}$ doesn't vanish. One can then see these glue to give a line bundle on X, and that this is inverse to div . □

Riemann–Hurwitz: Let $\pi\colon X \to Y$ be a separable morphism of projective regular curves of degree n, with ramification divisor R. Then

$$\deg \Omega_{X/k} = (\deg \pi)(\deg \Omega_{Y/k}) + \deg R.$$

Equivalently, this can be written in terms of canonical divisors since we are working with curves:

$$\deg K_X = n \cdot \deg K_Y + \deg R.$$

Recalling Riemann–Roch, we know $\deg K_X = 2g_X - 2$, giving a relation on the genera of X and Y,

$$(2g_X - 2) = n(2g_Y - 2) + \deg R.$$

Moreover, if π is tamely ramified (trivial if char $k = 0$) then R may be interpreted as

$$R = \sum_{P \in X} e_P - 1,$$

where e_P, known as the ramification index at P, is the valuation of the image of the uniformizer of the image of P.

Proof idea. Use (generically) separable to argue that the cotangent sequence is *left* exact (see Proposition 21.7.2 or IV.2.1 in Hartshorne)

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

Then since degree of coherent sheaves is additive on exact sequences, we have

$$\deg \Omega_{X/k} = \deg \pi^*\Omega_{Y/k} + \deg \Omega_{X/Y}.$$

Since degree is well behaved under pullback (see e.g. Exercise 18.4.F), we need only interpret the degree of $\Omega_{X/Y}$ to get the first statement.

$\Omega_{X/Y}$ is supported (by definition) on the ramification locus, and we can compute it exactly by tensoring with $\mathscr{O}_{Y,q}$ for q in the branch locus. Given $\mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$, with uniformizers t and s respectively, we take $e_p = v(\operatorname{im}(t))$, i.e. $t \mapsto us^{e_p}$ for some unit u. This allows us to compute $\Omega_{X/Y}$ at p and to make sense of its degree, which is $\sum_{p \text{ ram.}} e_p - 1$ if all ramification is tame. All that remains is to argue that this agrees with the ramification divisor R. ☐

Riemann–Roch: Let X be a curve and D a divisor on X. Then

$$\chi(X, \mathscr{O}_X(D)) = \deg D + \chi(X, \mathscr{O}_X).$$

Equivalently,

$$h^0(X, \mathscr{O}_X(D)) - h^1(X, \mathscr{O}_X(D)) = \deg D + 1 - p_a(X).$$

This motivates the definition of the degree $\deg \mathscr{L}$ for line bundles and even quasicoherent sheaves more generally.

Assuming X is smooth over k, Serre duality relates $h^1(X, \mathscr{O}_X(D))$ to the canonical sheaf and identifies the arithmetic and geometric genera,

$$h^0(X, \mathscr{L}) = h^0(X, \mathscr{K} \otimes \mathscr{L}^\vee) + \deg \mathscr{L} + 1 - g.$$

This is often written with divisors more simply as

$$h^0(D) = h^0(K - D) + \deg D + 1 - g.$$

Proof idea. Use the closed subscheme exact sequence

$$0 \to \mathscr{O}_X(-P) \to \mathscr{O}_X \to \mathscr{O}_P \to 0.$$

Tensor by $\mathscr{O}_X(D + P)$ and use the additivity of χ to see that for any D, we have

$$\chi(X, \mathscr{O}_X(D + P)) = \chi(X, \mathscr{O}_X(D)) + h^0(P, \mathscr{O}_P(D + P)).$$

Using the fact that $\mathscr{O}_X(D + P)$ is locally free, we have $\mathscr{O}_P(D + P) \simeq \mathscr{O}_P$, so the rightmost term above is 1. This shows the statement is true for both $D + P$ and D so long as it's true for one of them. The base case of $D = 0$ is trivial, so by induction we are done. ☐

Right adjoints preserve limits (RAPL), LAPC, and why we care: Let (F, G) be an adjoint pair between categories in which limits and colimits exist (i.e. abelian categories). We claim that right adjoints "preserve" limits, i.e.

$$G(\varprojlim B_i) = \varprojlim G(B_i).$$

To see this, we will show that $G(\varprojlim B_i)$ satisfies the universal property of limits.

First, observe that if we have a diagram with objects B_i and morphisms between them, then applying G to objects and morphisms yields another diagram $G(B_i)$. Since we have assumed limits to exist in both categories, $\varprojlim G(B_i)$ makes sense. Since $\varprojlim B_i$ has a natural map to the diagram, $G(\varprojlim B_i)$ does as well, yielding a unique map

$$G(\varprojlim B_i) \to \varprojlim G(B_i).$$

Now suppose we have a map T to the diagram $G(B_i)$. Applying F and using the naturality of the adjunction, whenever $G_i \to G_j$ is a map in the diagram, we have

$$\begin{array}{ccc} \mathrm{Mor}(T, G(B_i)) & \longrightarrow & \mathrm{Mor}(F(T), B_i) \\ \downarrow & & \downarrow \\ \mathrm{Mor}(T, G(B_j)) & \longrightarrow & \mathrm{Mor}(F(T), B_j). \end{array}$$

This means that $F(T)$ maps to the diagram B_i, and hence we have a unique $F(T) \to \varprojlim B_i$ (the content of the above is checking that things commute). Doing this again, we find that this map is identified with a unique map $T \to G(\varprojlim B_i)$, which commutes with the map from $G(\varprojlim B_i)$ to the diagram $G(B_i)$. Thus the limit is "preserved" under G!

On the other hand, suppose we start with a diagram A_i and its colimit $\varinjlim A_i$. Running this argument the other way, we find that $F(\varinjlim A_i) = \varinjlim F(A_i)$. Briefly, we take some T together with a map from the diagram $F(A_i)$ and recognize that adjointness gives us a map from the diagram A_i to $G(T)$, hence from $\varinjlim A_i \to G(T)$. This we parlay back into a map $F(\varinjlim A_i) \to T$, as desired.

Why does this matter? There are some pretty useful and prevalent limits and colimits, as well as adjoint pairs. This fact alone can often tell us useful information. In particular, we can use this to prove that as functors from A-modules to A-modules, $\cdot \otimes_A N$ is right exact, while $\operatorname{Hom}_A(N, \cdot)$ is left-exact. Recall that these functors are an adjoint pair, with tensor the left adjoint and Hom the right adjoint.

To see $\operatorname{Hom}_A(N, \cdot)$ is left exact, we only need RAPL! If

$$0 \to M' \to M \to M'' \to 0,$$

we have that $M' \simeq \ker(M \to M'')$. Taking Homs, we find $\operatorname{Hom}(N, M') \simeq \operatorname{Hom}(N, \ker(M \to M'')$, but since a kernel is just a specific case of a limit, this means that $\operatorname{Hom}(N, M') \simeq \ker(\operatorname{Hom}(N, M), \operatorname{Hom}(N, M''))$, i.e.

$$0 \to \operatorname{Hom}(N, M') \to \operatorname{Hom}(N, M) \to \operatorname{Hom}(N, M'')$$

is exact.

On the other hand, $\cdot \otimes_A N$ is right exact by LAPC, for essentially the same reasons. We have the natural map $M' \otimes_A N \to M \otimes_A N$, and to compute the cokernel, we recognize that it's a colimit, so

$$\operatorname{coker}(M' \otimes_A N \to M \otimes_A N) \simeq \operatorname{coker}(M' \to M) \otimes_A N \simeq M'' \otimes_A N,$$

or in other words

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is exact.

Serre's cohomological criterion for affineness: The following are equivalent for a Noetherian scheme.

(i) X is affine,
(ii) $H^i(X, \mathscr{F}) = 0$ for all $i > 0$ and all quasicoherent sheaves $\mathscr{F}$ on X,
(iii) $H^1(X, \mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$ on X.

Proof idea (Noetherian case). (i $\implies$ ii) is true by Cech cohomology; see Properties of cohomology. (ii $\implies$ iii) is clear. The content is to show (iii $\implies$ i).

Suppose X satisfies (iii). One shows that there exist sections $f_1, \ldots, f_r \in \Gamma(X, \mathscr{O}_X)$ for which $D(f_i)$ are all affine and the f_i's generate the unit ideal in $\Gamma(X, \mathscr{O}_X)$. Then $X = \operatorname{Spec} A$. This amounts to checking that the natural map $X \to \operatorname{Spec} \Gamma(X, \mathscr{O}_X)$ is an affine morphism, hence X is affine.

Let $p \in X$ be a closed point (which exist for Noetherian schemes!) and $Y = X - U$ for an affine open U containing p. Then we have an exact sequence

$$0 \to \mathscr{I}_{Y \cup \{p\}} \to \mathscr{I}_Y \to \kappa(p) \to 0.$$

Interpret this as "the functions vanishing on both Y and p inject into Y, the quotient of which is $\kappa(p)$, i.e. the skyscraper sheaf on p. Taking the long exact sequence on cohomology, using hypothesis (iii), we see that

$$\Gamma(X, \mathscr{I}_Y) \to \kappa(p) \to 0,$$

so there exists a function f on X which doesn't vanish at p. Since $D(f) \subseteq U$ and f is also a function on U, we have that $D(f)$ is affine, containing p. By quasicompactness, we obtain a finite open cover $\cup D(f_i) = X$.

It remains to show these generate the unit ideal. Consider

$$0 \to \mathscr{F} \to \mathscr{O}_X^r \to \mathscr{O}_X \to 0,$$

where the map sends $1 \mapsto f_i$ on the basis of $\mathscr{O}_X^r$. We need to show that this is surjective on global sections, i.e. $H^1(X, \mathscr{F}) = 0$. While this isn't true a priori, as (iii) applies only to coherent sheaves of ideals, we filter by

$$\mathscr{F} \supseteq F \cap \mathscr{O}_X^{r-1} \supseteq \cdots \supseteq \mathscr{F} \cap \mathscr{O}_X.$$

The rightmost sheaf in the sequence is indeed a coherent sheaf of ideals, hence it has vanishing cohomology, and each of the successive quotients may be interpreted as a coherent sheaf of ideals, allowing us to "walk up" the filtration (much like how we did to show X_{red} affine $\iff X$ affine!). □

Serre duality: (*existence*) Let X be a projective scheme over k. Then a dualizing sheaf, ω, exists. That is, for all coherent $\mathscr{F}$, we have a pairing

$$\operatorname{Hom}(\mathscr{F}, \omega) \times H^n(X, \mathscr{F}) \to k$$

with

$$\operatorname{Hom}(\mathscr{F}, \omega) \simeq H^n(X, \mathscr{F})^\vee.$$

When X is Cohen–Macaulay and of equidimension n and $\mathscr{F}$ is locally free, we have

$$H^i(X, \mathscr{F}) \simeq H^{n-i}(X, \omega \otimes \mathscr{F}^\vee)^\vee.$$

(*coincidence with canonical sheaf*) When X is smooth, we have that the dualizing sheaf ω coincides with the canonical bundle/sheaf $\mathscr{K}_X = \det \Omega_{X/k}$. In this case, Serre duality refers to the perfect pairing

$$H^i(X, \mathscr{F}) \times H^{n-i}(X, \mathscr{K}_X \otimes \mathscr{F}^\vee) \to H^n(X, \mathscr{K}_X) \simeq k,$$

which implies

$$h^i(X, \mathscr{F}) = h^{n-i}(X, \mathscr{K}_X \otimes \mathscr{F}^\vee).$$

(*curves*) In the special case of X a smooth curve ($n = 1$), we have the oft-used formula

$$h^1(X, \mathscr{O}_X(D)) = h^0(X, \mathscr{O}_X(K - D)),$$

used in a common formulation of Riemann–Roch. This also allows us to that the arithmetic genus $p_a = 1 - \chi(X, \mathscr{O}_X)$ agrees with the geometric genus $g = h^0(X, \omega_X)$,

$$p_a = 1 - \chi(X, \mathscr{O}_X) = 1 - h^0(X, \mathscr{O}_X) + h^1(X, \mathscr{O}_X) =^{(SD)} h^0(X, \omega_X) = g.$$

Sheaves which are not quasicoherent: Let $X = \operatorname{Spec} k[x]_{(x)}$, which has a generic point η and a closed point corresponding to (x). The open sets consist of $\emptyset, U = \{\eta\}, X$. Consider the sheaf of abelian groups $\mathscr{F}$ obtained by assigning

$$\Gamma(X, \mathscr{F}) = k(x), \quad \Gamma(U, \mathscr{F}) = 0,$$

where the restriction map is the obvious: the zero map. This is an $\mathscr{O}_X$-module, because each open set is a $k[x]_{(x)}$-module (we can make x act trivially on $k(x)$) and this agrees with restriction.

If this were quasicoherent, we'd have $\Gamma(X, \mathscr{F})_x = \Gamma(U, \mathscr{F})$, since $U = D(x)$. But of course, $k(x)_x = k(x) \neq 0$. Therefore this isn't quasicoherent!

A similar strategy can be used to show the pushforward of the sheaf $k(x)$ on the closed point at the origin (a.k.a. the skyskraper sheaf at the origin) is not quasicoherent on $\operatorname{Spec} k[x]$. Here the idea is that $\Gamma(\operatorname{Spec} k[x], i_* k(x)) = k(x)$, and $\Gamma(D(x), i_* k(x)) = 0$. Thus the powers of x must annihilate the module $k(x)$ if this were to be quasicoherent; this is clearly seen to be false!

Smoothness characterizations: Let X be a (necessarily) finite type k-scheme. The following conditions are equivalent for X to be k-smooth, of (pure) dimension n.

(i) X has a cover by affine open sets $\operatorname{Spec} k[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ where the Jacobian matrix has corank n at every point (we took this to be the definition of smooth).

(ii) The cotangent bundle $\Omega_{X/k}$ is locally free of rank n.
(iii) If k is a perfect field, X is regular and finite type.

Note that for (iii), we always have that smooth schemes are regular. When k is perfect, it turns out that regularity implies smoothness.

Consider now a morphism of schemes $\pi\colon X \to Y$. The following conditions about smoothness (of some relative dimension n) are equivalent.

(i) X and Y have covers by open sets such that π locally looks like the map induced by $B \to B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)$, where the Jacobian in the first r variables is invertible (we took this to be the definition).
(ii) π is locally finitely presented, flat of relative dimension n, and $\Omega_\pi = \Omega_{X/Y}$ is locally free of rank n.
(iii) π is locally finitely presented, flat of relative dimension n, and the fibers are smooth k-schemes of pure dimension n.
(iv) π is locally finitely presented, flat of relative dimension n, and the *geometric* fibers are smooth k-schemes of pure dimension n.

Recalling that a morphism is étale if it is smooth of relative dimension zero, we find that etaleness is equivalent to flat + loc. fin. pres. + unramified, or simply smooth and unramified. Conditions (iii) and (iv) above imply that the fibers of etale morphisms look like a disjoint union of copies of $\operatorname{Spec} K$, where $K/\kappa(p)$ is a finite separable extension (and in fact this is enough to *be* etale).

Proof idea/sketch. For smoothness of a k-scheme, the idea of (i) $\iff$ (ii) is to show that $\Omega_{\operatorname{Spec} k[x_1,\dots,x_m]/(f_1,\dots,f_r)/k}$ computes the cokernel of the Jacobian matrix (see Exercise 21.2.E). Thus if X satisfies (i), then the stalks of $\Omega_{X/k}$ have rank n, and since constant rank implies locally free (for finite type quasicoherent sheaves on a reduced scheme, see Exercise 13.7.K) we have (ii). Conversely, if $\Omega_{X/k}$ is locally free, we have that after covering by affine open sets of the form in (i), the module of differentials — which computes the cokernel of the Jacobian — has the correct rank.

For regularity implies smoothness, first we use that X is regular implies $X_{\overline{k}}$ is regular. See Exercise 12.2.O; the idea is that regular local rings are preserved under base extension, provided the residue field is separated over k (hence the perfection hypothesis!). We then recognize that if $X_{\overline{k}}$ is regular at its closed points, it must be $\overline{k}$-smooth, as the points *failing* to satisfy the Jacobian criterion are in the vanishing set of a certain determinant, hence this set contains closed points. In fact this is an if and only if. Finally, we have that $\Omega_{X/k} \otimes_k \overline{k} \simeq \Omega_{X_{\overline{k}}/\overline{k}}$ by pullback of differentials; this preserves rank, so one is locally free if and only if the other is.

For smoothness implies regular, we don't need the perfect hypothesis on k. Smoothness means we have X is an etale cover of $\mathbb{A}^n_k$, which is regular. Exercise 25.2.D shows that the preimage of a regular point under an etale map is regular. □

Useful adjoint pairs: Below are several useful left/right adjoint functor pairs.

- "–ify" (left adjoint) and "forget" (right adjoint), e.g.
  - sets to groups (here "–ify" is the functor producing the free group on a set),
  - presheaves to sheaves,
  - $\Gamma(X, \mathscr{O}_X)$-modules to $\mathscr{O}_X$-modules (here "–ify" is the $\widetilde{\cdot}$ functor), etc.
- Tensor $\cdot \otimes_A N$ (left adjoint) and $\operatorname{Hom}_A(N, \cdot)$ (right adjoint) as functors from A-modules to A-modules, for a fixed A-module N.
- Inverse image π^{-1} (left adjoint) and pushforward π_* (right adjoint), as functors to/from sheaves on X to sheaves on Y for a fixed map $\pi\colon X \to Y$.
- Pullback π^* (left adjoint) and pushforward π_* (right adjoint) as functors to/from $\mathscr{O}_X$-modules to $\mathscr{O}_Y$-modules, for fixed map $\pi\colon X \to Y$.

- Extension by zero $i_!$ (left adjoint) and inverse image i^{-1} (right adjoint), for an open embedding $i\colon U \hookrightarrow X$. This handily implies that i^{-1} is exact in this case.

Valuative criterion for properness: Let $\pi\colon X \to Y$ be a quasiseparated finite type map of schemes and K a valued field with valuation ring A. Then π is proper if and only if for every such K, A with outer diagram below, *there exists exactly one* diagonal arrow:

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} A & \longrightarrow & Y. \end{array}$$

If X and Y are locally Noetherian, then we need only consider discrete valuation rings A in the diagram above.

Note that the existence of a dashed arrow implies universal closedness. Since separatedness is part of the definition of properness, this combined with the uniqueness from the valuative criterion for separatedness gives properness.

Valuative criterion for separatedness: Let $\pi\colon X \to Y$ be a map of schemes and K a valued field with valuation ring A. Then π is separated if and only if for every such K, A with outer diagram below, *there exists at most one* diagonal arrow:

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & \overset{\leq 1}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} A & \longrightarrow & Y. \end{array}$$

If π is finite type and X and Y are locally Noetherian, then we need only consider discrete valuation rings A in the diagram above.

3. Practice questions

The following questions have been (or might be) asked by DZB in a qualifying exam. See also https://www.ocf.berkeley.edu/~mgsa/wiki/index.php/Algebraic_Geometry for qual questions from Berekeley over the years.

(1) What are the injective objects of Ab?

(2) Define cohomology via injective resolutions. Give an example of an injective object. Show that a short exact sequence gives rise to a long exact sequence in cohomology. Give examples of cohomology groups (one vanishing, on nonvanishing).

(3) State all of the theorems and properties about cohomology.

(4) Let X be an irreducible zero dimensional scheme. Show that the cohomology of any sheaf on X vanishes.

(5) Show that X is affine if and only if the reduction is affine.

(6) State Serre's cohomological criterion for affineness and prove the easy part (the cohomology of a quasicoherent sheaf on an affine scheme is trivial).

(7) Compute $H^1(\mathbb{P}^1, \mathscr{O})$. What about $H^1(\mathbb{A}^2 - 0)$?

(8) Prove that Pic is isomorphic to $H^1(\mathscr{O}^\times)$

(9) State and prove Grothendieck's vanishing theorem.

(10) State Serre duality.

(11) Prove that the cohomology of a flasque sheaf vanishes. What's the point of flasque sheaves?

(12) Compute $\operatorname{Pic} \mathbb{P}^n$.

(13) Prove that a morphism of complete curves is surjective or constant.

(14) Define and give an example of an ample divisor. Compute $\operatorname{Aut} \mathbb{P}^n$.

(15) Define the sheaf of differentials (in all the ways). Compute its global sections on $\mathbb{A}^1$ and for a field extension. Compute the global sections of Ω on $\mathbb{P}^n$.

(16) State all the theorems and properties about differentials.

(17) Let $X \to Y$ be an affine morphism and $\mathscr{F}$ a sheaf on X. Show that the cohomology $H^i(X, \mathscr{F}) = H^i(Y, \pi_* \mathscr{F})$ agree.

(18) What is a scheme? Give all the details. Why do we need schemes? What problems do they solve?

(19) Prove that a map of sheaves is an isomorphism if and only if the induced morphisms on stalks are all isomorphisms.

(20) What does (quasi)coherent mean? Give example(s) and an example of sheaf that is not quasicoherent.

(21) Name as many adjectives that you can. What does locally of finite type mean? Reduced? Integral? Show that integral iff irreducible and reduced.

(22) Show that the two definitions of locally of finite type are the same.

(23) What is a fiber product? Show that Spec of tensor is the product (in the category of schemes, not just affine schemes!).

(24) Define the residue field of a point and give examples. What is the residue field of the generic point of a curve in $\mathbb{A}^2$? Show that the fiber of a morphism is the same of the fiber product with the inclusion of the generic point.

(25) What is a Weil divisor? What about Pic? What is CaCl $\mathbb{P}^n$? Prove that these are isomorphic. Define div s for s a (rational) section of a line bundle. What adjectives do we need (on X or the bundle) for this to make sense?

(26) What is a line bundle? Invertible sheaf? Show that a locally free sheaf of rank one is invertible. Show that if every stalk of a sheaf is free, then the sheaf is locally free.

(27) Can you give an example of a surjection of sheaves which is not a surjection on global sections?

(28) What is $\mathscr{O}(d)$ on $\mathbb{P}^n$? What is the cohomology? State any other theorems about $\mathscr{O}(d)$.

(29) Prove using cohomology that a dimension 0 scheme is affine.

(30) Show that a map of sheaves is injective if and only if the induced morphism on stalks is injective.

(31) What does it mean for a divisor to be basepoint free? Very ample? If P is a base point for a linear system, what can we say about the divisors in the system?

(32) What is a hyperelliptic curve? What is its canonical divisor? Show that the canonical divisor on a hyperelliptic curve is basepoint free, but not very ample.

(33) Prove for a smooth curve X that $h^0(X, \mathscr{O}(D)) - h^0(X, \mathscr{O}(D - P)) \leq 1$