Math 524 - Problem Set 1
Due Wednesday, Feb 2

1. Describe the map $f : \operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{R}[x]$ induced by the inclusion $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$.

2. For each ring A and point $q \in \operatorname{Spec} A$, calculate the dimension of the tangent space to $\operatorname{Spec} A$ at q.
   (a) $A = \mathbb{C}[x, y, z]/(x^2 - y^3), q = (x, y)$,
   (b) $A = \mathbb{C}[x, y]/(x, y^3), q = (x, y)$,
   (c) $A = \mathbb{Z}[x], q = (3)$.

3. Let R, S be rings.
   (a) Check that the Zariski topology on $\operatorname{Spec} R$ is indeed a topology and that the distinguished open sets
   $$D(f) := \operatorname{Spec} R\backslash V(f)$$
   form a basis.
   (b) If $\phi : R \to S$ is a homomorphism, verify that the induced map $\operatorname{Spec} S \to \operatorname{Spec} R$ is continuous.

4. Let R be a ring and $I \subset R$ an ideal. Show that the natural map $f : \operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ is a homeomorphism onto its image $V(I)$.[1]

5. Let R be a ring. Show that a nonzero element $f \in R$ is a zerodivisor if and only if there are closed sets $X, Y \subseteq \operatorname{Spec} R$ with $Y \neq \operatorname{Spec} R$ such that $\operatorname{Spec} R = X \cup Y$ and f evaluated[2] at x is zero for all $x \in X$.

6. (a) For each open set $U \subseteq \mathbb{R}$ (given the standard topology), define $\mathcal{F}(U)$ to be the group of continuous functions $f : U \to \mathbb{R}$ that are nondecreasing. That is, for $x, y \in U$ with $x < y$, we require $f(x) \leq f(y)$. Is $\mathcal{F}$ a presheaf? Is it a sheaf?
   (b) What if you replace the word "nondecreasing" with "even"? That is, if $x, -x \in U$, then $f(x) = f(-x)$. In this case is $\mathcal{F}$ a (pre)sheaf?

7. Let X be a topological space and $\mathcal{F}$ and $\mathcal{G}$ presheaves on X. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism. Verify the following claim from class: there is an induced map on stalks $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ for each $P \in X$. Make sure you explicitly describe the map and check that it is well-defined.

8. Let X be a topological space. Let $\mathcal{F}$ be the constant presheaf, defined $\mathcal{F}(U) = \mathbb{Z}$ for every nonempty $U \subseteq X$ open, where the restriction maps are the identity, and let $\mathcal{F}'$ be the sheaf of locally constant functions described in class.
   (a) Check that $\mathcal{F}'$ is indeed a sheaf.
   (b) Show that $\mathcal{F}'$ is the sheafification[3] of $\mathcal{F}$.

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[1] The closed sets of $X \subseteq \operatorname{Spec}(R)$ are those of the form $V(J) \cap X$ for J an ideal in R.
[2] Via the natural evaluation map $A \to k(x)$ described in class.
[3] We will get to this in a couple weeks – see Prop-def 1.2 in Hartshorne for a definition.