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Dynamic coloring and list dynamic coloring of planar graphs

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Abstract

A *dynamic chromatic number* $\chi_d(G)$ of a graph G is the least number k such that G has a proper k-coloring of the vertex set $V(G)$ so that for each vertex of degree at least 2, its neighbors receive at least two distinct colors. We show that $\chi_d(G) \leq 4$ for every planar graph except C_5, which was conjectured in [5].

The *list dynamic chromatic number* $ch_d(G)$ of G is the least number k such that for any assignment of k-element lists to the vertices of G, there is a dynamic coloring of G where the color on each vertex is chosen from its list. Based on Thomassen's result [12] that every planar graph is 5-choosable, an interesting question is whether the list dynamic chromatic number of every planar graph is at most 5 or not. We answer this question by showing that $ch_d(G) \leq 5$ for every planar graph.

1 Introduction

A *dynamic coloring* of a graph G is a proper coloring of the vertex set $V(G)$ such that for each vertex of degree at least 2, its neighbors receive at least two distinct colors. A dynamic k-coloring of a graph is a dynamic coloring with k colors. A dynamic k-coloring is also called a *conditional* $(k,2)$*-coloring*. The smallest integer k such that G has a dynamic k-coloring is called the *dynamic chromatic number* $\chi_d(G)$ of G.

The relationship between the chromatic number $\chi(G)$ and the dynamic chromatic number $\chi_d(G)$ of a graph G has been studied in several papers (see [2], [7], [8], [11]). The gap $\chi_d(G) - \chi(G)$ could be infinitely large for some graphs. An interesting problem is to study which graphs have small values of $\chi_d(G) - \chi(G)$.

One of the most interesting problems about dynamic chromatic numbers is to find upper bounds of $\chi_d(G)$ for planar graphs G. It was showed in [5, 9] that $\chi_d(G) \leq 5$ if G is a planar graph, and it was conjectured in [5] that $\chi_d(G) \leq 4$ if G is a planar graph other than C_5. Note that the conjecture is an extension of Four Color Theorem except C_5. As a partial answer, Meng–Miao–Su–Li [10] showed that the dynamic chromatic number of Pseudo-Halin graphs, which are planar graphs, are at most 4, and the first and third author [6] showed that $\chi_d(G) \leq 4$ if G is a planar graph with girth at least 7. In this paper we settle the conjecture in [5] by showing the following theorem.

Theorem 1. *If G is a planar graph with $G \neq C_5$, then $\chi_d(G) \leq 4$.*

We also study the corresponding list coloring called a *list dynamic coloring*. For every vertex $v \in V(G)$, let $L(v)$ denote a list of colors available at v. An L-coloring is a proper coloring ϕ such that $\phi(v) \in L(v)$ for every vertex $v \in V(G)$. A graph G is called *k-choosable* if it has an L-coloring whenever all lists $L(v)$ of L have size at least k. The *list chromatic number* $ch(G)$ of G is the least integer k such that G is k-choosable. A dynamic L-coloring of G is a dynamic coloring

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of G which is an L-coloring of G. A graph G is called *dynamic k-choosable* if it has a dynamic L-coloring whenever all lists $L(v)$ have size at least k. The *list dynamic chromatic number* $ch_d(G)$ of G is the least integer k such that G is dynamic k-choosable.

The list dynamic chromatic number has been studied at several papers [1, 3, 6] for some classes of graphs. One of particular interests is to find upper bounds on $ch_d(G)$ for planar graphs G. The first and third author [6] showed that $ch_d(G) \leq 6$ for every planar graph G and $ch_d(G) \leq 4$ if G is a planar graph of girth at least 7, which is sharp since there is a planar graph with $ch_d(G) = 5$ with girth 6. Based on the result by Thomassen [12] that every planar graph is 5-choosable, a natural interesting question is whether every planar graph is dynamic 5-choosable or not. In this paper we answer this question. We show the following theorem which is an extension of the result of Thomassen [12].

Theorem 2. *If G is a planar graph, then $ch_d(G) \leq 5$.*

2 Proof of the main results

In order to show Theorems 1 and 2, we will prove the technical lemma that for every planar graph G other than odd cycles, there exists a planar graph H with $G \subset H$ and $V(G) = V(H)$ satisfying that a proper coloring of H gives a dynamic coloring of G (see Lemma 2). To prove Lemma 2, we first prove the case when planar graph G is 2-connected in Lemma 1. We shall then invoke induction to obtain Lemma 2 in full. In Lemma 1 the following propositions will be used.

Proposition 1. *Let G be a 2-connected plane graph. The boundary of each face in G is a cycle.*

The proof of Proposition 1 is given in Diestel [4, Proposition 4.2.6, page 89].

Proposition 2. *Let G be a 2-connected plane graph. Each vertex of degree d in G is incident with d faces.*

Proof. Let G be a 2-connected plane graph and let u be a vertex of G with degree d. Then $G - u$ is a connected plane graph. Let v, e and f (v', e' and f') be the number of vertices, edges and faces of G ($G - u$), respectively. Hence we have $v' = v - 1$ and $e' = e - d$. Using Euler's formula, we infer $f' = v - e + f - v' + e' = f - (d - 1)$ which implies that vertex u is incident with d faces in G. □

Now we are ready to prove Lemma 1.

Lemma 1. If G is a 2-connected planar graph other than odd cycles, then there exists a planar graph H with $G \subset H$ and $V(G) = V(H)$ such that for every vertex v of degree at least 2 in G, there are two vertices in $N_G(v)$ that are adjacent in H.

Proof. Let G be a 2-connected planar graph other than odd cycles. Fix a planar embedding of G, and for simplicity, denote the embedding by G. (Now G is a plane graph.) Note that since G is 2-connected, every vertex of G has degree at least 2. It suffices to show the following statement:

(*) There exists a plane multigraph H with $G \subset H$ (as plane embeddings) and $V(G) = V(H)$ such that for every vertex $v \in V(G)$, there are two vertices in $N_G(v)$ that are adjacent in H.

To this end, we introduce several definitions and notation. For a vertex $v \in V(G)$, if vertices v and $x, y \in N_G(v)$ are incident with a common face F of G, then we can add the edge xy in face F to G so that the resulting multigraph $G' = G + xy$ is still a plane graph. We call such edge xy an *addible* edge of v in G. For each vertex v, let A_v be the set of all addible edges of v in G. Note that since G is 2-connected, it follows from Proposition 2 that $|A_v| = d_G(v)$ where $d_G(v)$ is the degree of v in G.

Let $R := \{a_v : v \in V(G)\}$ be a set of addible edges obtained by choosing an arbitrary addible edge a_v from A_v for each vertex $v \in V(G)$. Let $H = G \cup R$ be the multigraph drawn on the plane by adding all edges in R to G. We call the edge a_v in R a *red edge* of v in H. Note that H may have multiple edges and edge crossings.

We define $\mathcal{F}(G)$ as the family of such multigraphs H. Note that each $H \in \mathcal{F}(G)$ satisfies the conditions in statement (*) except the condition that H is a plane multigraph. Let $cr(H)$ be the number of edge crossings in H. Let H_{min} be a multigraph in $\mathcal{F}(G)$ such that

$$cr(H_{min}) = \min\{cr(H) : H \in \mathcal{F}(G)\}. \tag{1}$$

Observe that $cr(H_{min}) = 0$ if and only if statement (*) holds. We will show that $cr(H_{min}) = 0$.

For a proof by contradiction, we suppose that $cr(H_{min}) > 0$. Then we will show that there is a multigraph $H' \in \mathcal{F}(G)$ such that $cr(H') < cr(H_{min})$, which contradicts to the minimality of $cr(H_{min})$.

Under the assumption $cr(H_{min}) > 0$, there are two adjacent vertices of G whose red edges in H_{min} cross each other. Let F be the face where the crossing occurs. Note that the boundary of F is a cycle by Proposition 1. Let $V(F) = \{v_1, v_2, \ldots, v_f\}$ be the set of all (distinct) vertices in the boundary of face F in G in a counterclockwise direction, where f is the degree of face F.

Now we are going to obtain $H' \in \mathcal{F}(G)$ such that $cr(H') < cr(H_{min})$ as follows. Delete all red edges of $v_1, v_2, \ldots, v_f$ from H_{min} and denote the resulting graph by W. Then we show that we can add red edges of $v_1, v_2, \ldots, v_f$ to W so that each new red edge of v_i does not cross each other and any other red edges in W. Hence the resulting graph H' satisfies $cr(H') < cr(H_{min})$. Now we describe how to add red edges of v_i in H'. We consider two cases. The first case is when the degree of F is even, and the second case is when the degree of F is odd.

Case 1. When the degree f of face F is even.

Draw the red edges v_1v_3,$v_3v_5, \ldots, v_{f-1}v_1$ to W inside face F to be the red edges of $v_2, v_4, \ldots, v_f$ in H'. Next, for each $v_i \in V(F)$, where i is odd, we will draw the red edge of v_i in a face which is incident with v_i other than F and does not contain any red edges of all neighbors of v_i. Now we claim that such a face exists for each v_{odd}, where v_{odd} denotes a vertex in $V(G)$ with odd index.

Let d_i be the degree of vertex v_i in G. Let $u_1, u_2, \ldots, u_{d_i-2}$ be all neighbors of v_i in G other than v_{i-1} and v_{i+1}. Note that from Proposition 2, there are d_i faces incident with v_i in G. Let $F_1, F_2, \ldots, F_{d_i-1}$ be the faces incident to v_i in G other than F. Since the red edges of v_{i-1} and v_{i+1} are inside face F, the set of all red edges of $u_1, u_2, \ldots, u_{d_i-2}$ can be contained in at most $d_i - 2$ faces among $F_1, F_2, \ldots, F_{d_i-1}$. Hence there is at least one face F_j which does not contain any red edges of all neighbors of v_i.

Let H' be the resulting graph after adding the new red edge of $v_i \in V(F)$ to W for all i. Now we justify that the red edge of each $v_i \in V(F)$ in H' does not cross any other red edges in H'. First note that the red edge of each v_{even} in H' does not cross any other red edges in H', where v_{even} denotes a vertex in $V(G)$ with even index. This is because the red edges of v_{even} in H' are only red edges inside F in H' and they do not cross each other. Next, let us consider the red edges of v_{odd}. Observe that if the red edge of v crosses the red edge of u, then u is a neighbor of v in G. Since the red edge of each v_{odd} is placed in a face which does not contain any red edges of all neighbors of v_{odd} in G, we infer that the red edge of each v_{odd} in H' does not cross any other red edges in H'.

Case 2. When the degree f of face F is odd.

Since G is not an odd cycle, there is at least a vertex in $V(F)$ whose degree at least 3 in G. Without loss of generality, let v_1 be a vertex in $V(F)$ with $d_G(v_1) \geq 3$.

If we try to draw the red edges of $v_i \in V(F)$ in the way in Case 1, there are two adjacent vertices v_1 and v_f with odd indices. Since v_1 and v_f are adjacent, the choice of the red edges of v_1 and v_f depend on each other. So we first consider the red edges of v_1 and v_f.

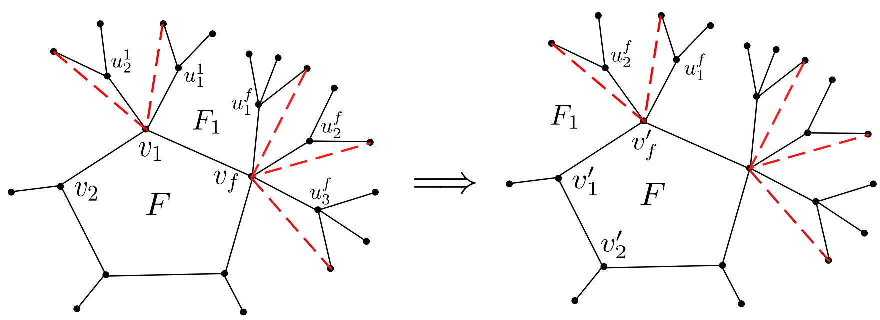

Figure 1: Relabeling in Subcase 2.2. Dashed lines represent red edges.

First we define several notation about vertices v_1 and v_f. Let d_1 and d_f be the degrees of v_1 and v_f in G, respectively. Let $u_1^1, u_2^1, \dots, u_{d_1-2}^1$ be all neighbors of v_1 in G other than v_f and v_2. Similarly, let $u_1^f, u_2^f, \dots, u_{d_f-2}^f$ be all neighbors of v_f in G other than v_1 and v_{f-1}. From Proposition 1, each edge in G is incident with 2 faces in G. Let F_1 be the face in G that is incident with edge v_1v_f other than F. Let $F_1^1(= F_1), F_2^1, \dots, F_{d_1-1}^1$ be all faces in G that are incident with v_1 other than F. Similarly, let $F_1^f(= F_1), F_2^f, \dots, F_{d_f-1}^f$ be all faces in G that are incident with v_f other than F.

We consider the positions of all red edges of $u_1^1, \dots, u_{d_1-2}^1, u_1^f, \dots, u_{d_f-2}^f$. Since all red edges of $u_1^1, \dots, u_{d_1-2}^1$ can be contained in at most $d_1 - 2$ faces among $F_1^1, F_2^1, \dots, F_{d_1-1}^1$, there is at least a face which does not contain any red edges of $u_1^1, \dots, u_{d_1-2}^1$. If there is a face that does not contain any red edges of $u_1^1, \dots, u_{d_1-2}^1$ and the face is different from F_1^1, we select one of the faces and say F_*^1. Otherwise, we set $F_*^1 = F_1^1 = F_1$. Similarly, we define F_*^f for vertex v_f.

If either $F_*^1 \neq F_1$ or $F_*^f \neq F_1$ occurs, we can draw the red edges of v_1 and v_f in F_*^1 and F_*^f, respectively, so that the red edges of v_1 and v_f do not cross each other and any other red edges in W. Otherwise, we cannot draw the red edges of v_1 and v_f without edge crossings, because face $F_1 = F_1^1 = F_1^f$ is the only face in which we can draw the red edges of v_1 and v_f. Hence we consider two subcases: one is when either $F_*^1 \neq F_1$ or $F_*^f \neq F_1$ occurs and the other is when $F_*^1 = F_*^f = F_1$.

Subcase 2.1: When $F_*^1 \neq F_1$ or $F_*^f \neq F_1$.

We describe how to draw the red edges of all $v_i \in V(F)$ in H'. First we draw the red edges of v_1 and v_f in faces F_*^1 and F_*^f, respectively. Then we draw the red edges v_1v_3,$v_3v_5, \dots, v_{f-2}v_f$ inside face F to be the red edges of $v_2, v_4, \dots, v_{f-1}$ in H'. Next, for each v_i, where i is odd and $i \neq 1, f$, we draw the red edge of v_i by the same way as in Case 1.

Now we explain that the red edge of each $v_i \in V(F)$ in H' does not cross any other red edges in H'. Clearly, the red edges of v_1 and v_f do not cross each other and any other red edges in W. With the argument in Case 1, the red edge of each $v_i \in V(F) \setminus \{v_1, v_f\}$ does not cross any other red edges in H'.

Subcase 2.2: When $F_*^1 = F_*^f = F_1$.

We relabel the vertices $v_1, v_2, \dots, v_f \in V(F)$ so that $v_1 = v_f'$ and $v_i = v_{i-1}'$ for $2 \leq i \leq f$. Note that $(v_1, v_2, \dots, v_f) = (v_f', v_1', \dots, v_{f-1}')$ (see Figure 1). Under the assumption $d_G(v_f') = d_G(v_1) \geq 3$, we have $F_*^f \neq F_1$ with the new label and hence we have Subcase 2.1. By the argument in Subcase 2.1, we can draw the red edges of all $v_i' \in V(F)$ in H' without edge crossings. □

Now we show that the 2-connected condition in Lemma 1 can be removed so that the same conclusion holds for arbitrary planar graphs except odd cycles.

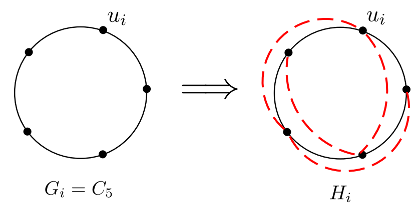

Figure 2: H_i when $G_i = C_5$

Lemma 2. If G is a planar graph other than odd cycles, then there exists a planar graph H with $G \subset H$ and $V(G) = V(H)$ such that for every vertex v of degree at least 2 in G, there are two vertices in $N_G(v)$ that are adjacent in H.

Proof. We use induction on k, the number of edges in G. First, Lemma 2 holds trivially for $k = 1$. Now suppose that every planar graph with at most k edges other than odd cycles has a supergraph satisfying all properties of H in Lemma 2. Let G be a planar graph with $k+1$ edges other than odd cycle. Lemma 1 gives that if G is 2-connected, then G has a supergraph satisfying all properties of H in Lemma 2. Hence we may assume that G is not 2-connected, that is, G has a cut-vertex u.

Let $\widetilde{G}_1, \widetilde{G}_2 \ldots, \widetilde{G}_l$, where $l \geq 2$, be the components of $G - u$. Set $\widehat{G}_1 = \widetilde{G}_1$ and $\widehat{G}_2 = \bigcup_{2 \leq i \leq l} \widetilde{G}_i$. For each $i \in \{1, 2\}$ let G_i be the supergraph of $\widehat{G}_i$ with $V(G_i) = V(\widehat{G}_i) \cup \{u_i\}$ such that $N_{G_i}(u_i) = \{w \in V(\widehat{G}_i) : uw \in E(G)\}$. In other words, vertices $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ are two copies of $u \in V(G)$. Let $G_1 * G_2$ denote the graph obtained from $G_1 \cup G_2$ by identifying $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ as vertex u. Note that $G = G_1 * G_2$.

Since G is a planar graph with $k+1$ edges, both G_1 and G_2 are planar graphs with at most k edges. For $i \in \{1, 2\}$, if G_i is not an odd cycle, then by the induction hypothesis there exists a planar graph H_i with $G_i \subset H_i$ and $V(G_i) = V(H_i)$ such that for every vertex v of degree at least 2 in G_i, there are two vertices in $N_{G_i}(v)$ that are adjacent in H_i.

On the other hand, if G_i is an odd cycle, then there exists a planar graph H_i with $G_i \subset H_i$ and $V(G_i) = V(H_i)$ such that for every vertex v in G_i except one vertex u_i, there are two vertices in $N_{G_i}(v)$ that are adjacent in H_i. For example, when an odd cycle C_{2k+1} is denoted by $v_1 v_2 \ldots v_{2k+1} v_1$, for $i \in \{1, \ldots, k\}$, draw $v_{2i-1} v_{2i+1}$ on the unbouned face of the cycle and draw $v_{2i} v_{2i+2}$ inside the cycle where indices are taken modulo $2k+1$. Denote the resulting plane graph by H. Then every vertex v of C_{2k+1} except v_1 has adjacent neighbors in H. (See Figure 2 for C_5.)

Since both H_1 and H_2 are planar graphs, there are planar embeddings of H_1 and H_2 such that vertices u_1 and u_2 are on the outer face the embeddings of H_1 and H_2. Hence there exist planar embeddings H_1' of H_1 and H_2' of H_2 satisfying the followng property:

(a) Vertex $u_1 \in V(H_1')$ is the rightmost part of H_1', that is, there is no other part of H_1' to the right side of u_1.

(b) Vertex $u_2 \in V(H_2')$ is the leftmost part of H_2', that is, there is no other part of H_2' to the left side of u_2.

Let $H_1 * H_2$ denote the graph obtained from $H_1 \cup H_2$ by identifying $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$ as vertex u. We will show that $H_1 * H_2$ satisfies all properties of H in Lemma 2.

Case 1. For some $i \in \{1, 2\}$, G_i is not an odd cycle and $d_{G_i}(u_i) \geq 2$.

One can easily check that $V(G) = V(G_1 * G_2) = V(H_1 * H_2)$ and $G = G_1 * G_2 \subset H_1 * H_2$. Also one can easily check that $H_1 * H_2$ has a planar embedding by using the plane graphs H_1' and H_2'. Hence $H_1 * H_2$ is a planar graph.

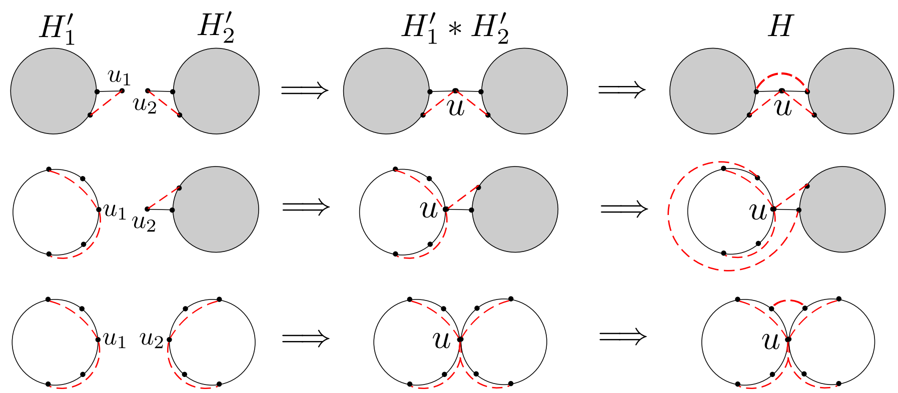

Figure 3: Adding a red edge of u to $H_1' * H_2'$

By the property of H_1 and H_2, it is clear that for every vertex $v \in V(G)$ of degree at least 2 in G, except u, there are two vertices in $N_G(v)$ which are adjacent in $H_1 * H_2$. Next we will show that there are two vertices in $N_G(u)$ which are adjacent in $H_1 * H_2$. Without loss of generality, we may assume that G_1 is not an odd cycle and $d_{G_1}(u_1) \geq 2$. Note that the supergraph $H_1 \supset G_1$ has a red edge of u_1 which are incident with two vertices in $N_{G_1}(u_1)$. Therefore for every vertex $v \in V(G)$ of degree at least 2 in G there are two vertices in $N_G(v)$ which are adjacent in $H_1 * H_2$. Thus $H_1 * H_2$ satisfies all properties of H in Lemma 2.

Case 2. For each $i \in \{1, 2\}$, G_i is an odd cycle or $d_{G_i}(u_i) = 1$.

For $i \in \{1, 2\}$, denote a planar embedding of H_i satisfying the property (a) and (b) by H_i'. Let $H_1' * H_2'$ be a planar embedding of $H_1 * H_2$ using the planar embedding H_1' and H_2'. Then by the argument in Case 1, the graph $H_1 * H_2$ satisfies all properties of H in Lemma 2 but the property that for vertex u, there are two vertices in $N_G(u)$ which are adjacent in $H_1 * H_2$.

We intend to find a plane graph $H \supset H_1' * H_2'$ which satisfies all properties of H in Lemma 2 by adding a red edge of u to $H_1' * H_2'$ without edge crossings. For each $i \in \{1, 2\}$, if $d_{G_i}(u_i) = 1$, then there is only one red edge incident with $u_i \in V(G_i)$ in the outer face of H_i'. Also if G_i is an odd cycle, then H_i can be drawn on the plane so that there is only one red edge incident with $u_i \in V(G_i)$ in the outer face of H_i'. Hence there are only two red edges incident with $u \in V(G)$ in the outer face of $H_1' * H_2'$. Moreover, there are two neighbors of u which are incident to outer face of $H_1' * H_2'$. One can easily check that the red edge of u can be added to $H_1' * H_2'$ without edge crossings (see Figure 3). In Figure 3, a shaded disk represents a planar graph that is not an odd cycle and a white disk represents an odd cycle. Dashed lines represent red edges.

Therefore, every planar graph G with $k+1$ edges other than odd cycles has a desired supergraph satisfying all properties of H in Lemma 2, completing the proof of Lemma 2. □

We shall apply Lemma 2 together with the Four Color Theorem and Thomassen's result [12] that every planar graph is 5-choosable in order to prove Theorems 1 and 2. Since Lemma 2 does not consider the case when G is an odd cycle, we first state previous results in [1, 7, 11] on $\chi_d(C_n)$ and $ch_d(C_n)$ for (odd) cycles C_n as follows:

$$\chi_d(C_n) = ch_d(C_n) = \begin{cases} \leq 4 & \text{if } n \neq 5 \\ = 5 & \text{if } n = 5 \end{cases} \tag{2}$$

Now we are ready to prove Theorems 1 and 2 in Introduction.

Theorem 1. If G is a planar graph with $G \neq C_5$, then $\chi_d(G) \leq 4$.

Proof. From (2) we have that every cycle C_n with $n \neq 5$ satisfies that $\chi_d(C_n) \leq 4$. Hence we assume that G is a planar graph that is not a cycle. From Lemma 2, there is a planar graph H with $G \subset H$ and $V(G) = V(H)$ such that for every vertex v of degree at least 2 in G, there exist two vertices in $N_G(v)$ that are adjacent in H.

Since H is planar, H has a proper 4-coloring f by the Four Color Theorem. Hence the coloring f of H is also a proper 4-coloring of G. Note that for every vertex v of degree at least 2 in G, there are two vertices in $N_G(v)$ that are adjacent in H. Hence the 4-coloring f of H is a dynamic 4-coloring of G. Therefore $\chi_d(G) \leq 4$. □

Theorem 2. If G is a planar graph, then $ch_d(G) \leq 5$.

Proof. From (2) we have that every cycle C_n is dynamic 5-choosable. Hence we assume that G is a planar graph that is not a cycle. For each vertex v in G, let $L(v)$ denote the list of colors available at v with $|L(v)| \geq 5$. We are going to show that G has a dynamic L- coloring ϕ.

From Lemma 2, there exists a planar graph H with $G \subset H$ and $V(G) = V(H)$ such that for every vertex v of degree at least 2 in G, there exist two vertices in $N_G(v)$ that are adjacent in H. Since every planar graph is 5-choosable, H has a proper L-coloring ϕ with 5 colors. Hence, the coloring ϕ of H is also a proper L-coloring of G. Note that for every vertex v of degree at least 2 in G, there are two vertices in $N_G(v)$ that are adjacent in H. Hence, the L-coloring ϕ of H is a dynamic L-coloring of G with 5 colors. Therefore $ch_d(G) \leq 5$. □

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