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Multigrid Analysis for the Time Dependent Stokes Problem

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MULTIGRID ANALYSIS FOR THE TIME DEPENDENT STOKES PROBLEM.

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Abstract. Certain implicit time stepping procedures for the incompressible Stokes or Navier-Stokes equations lead to a singular-perturbed Stokes type problem at each type step. The paper presents a convergence analysis of a geometric multigrid solver for the system of linear algebraic equations resulting from the disretization of the problem using a finite element method. Several smoothing iterative methods are considered: a smoother based on distributive iterations, the Braess-Sarazin and inexact Uzawa smoother. Convergence analysis is based on smoothing and approximation properties in special norms. A robust (independent of time step and mesh parameter) estimate is proved for the two-grid and multigrid W-cycle convergence factors.

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$ with $d = 2$ or $d = 3$, be a bounded polygonal domain. Consider the Stokes type problem given by:

$$\begin{aligned} -\Delta \mathbf{u} + \alpha\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= g \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

The mean value condition $\int_\Omega g \, d\mathbf{x} = \int_\Omega p \, d\mathbf{x} = 0$ should be imposed to make the problem well-posed for all $\alpha \geq 0$. The system (1.1) often appears as the auxiliary one for certain implicit time stepping procedures for the incompressible Stokes or Navier-Stokes equations, see e.g. [24]. The parameter α is typically proportional to the inverse of the time step scaled with viscosity parameter. This results in large values of α making the problem singular-perturbed. On the other hand, for slow flows the value of α can be modest or small. Discretization of (1.1) with finite element method or other conventional methods leads to a system of linear algebraic equations of saddle-point type with symmetric indefinite matrix. Hence one is interested in solvers for such a system which are robust with respect to the variation of α.

Among various existing solvers for discrete saddle-point systems, resulting from discretizations of PDEs, one may distinct between iterative methods with block preconditioners and direct multigrid methods, see e.g. [3]. This paper deals with direct (coupled) multigrid methods for (1.1). The well known and efficient multigrid techniques include the one based on distributive smoothing iterations [22, 29], coupled saddle-point smoothers [6, 31] and block Gauss-Seidel type smoothers (Vanka multigrid) [28], see also the overview in [30]. While the analysis of robust block preconditioners for the time-dependent Stokes problem can be found in [7, 16, 19, 21], we are not aware of any studies proving the efficiency of multigrid methods for (1.1) in the range of $\alpha \in [0, \infty)$. The analysis of various multigrid methods for the Stokes problem ($\alpha = 0$) can be found in several papers, see [5, 8, 18, 22, 26, 29]. The smoothing analysis from [6, 23, 31] can be also merged with the approximation property from [29] to establish the convergence of the two-grid method for the case of $\alpha = 0$.

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The major obstacle for extending existing analyses for the case of $\alpha > 0$ is the lack of an appropriate approximation property. Such approximation property is established in this paper. In general, the proof follows the simplified Verfürth's pattern from [17]. The handling of $\alpha > 0$ via special norms involves some equivalence results and representations from [16, 19, 21] for the discrete and continuous pressure Schur complement operators. Further we consider smoothing properties in appropriate norms of distributive iterations and coupled iterations similar to the methods of Braess-Sarazin [6] and Bank et al. [2]. From these results the convergence of the *two-grid* method follows immediately. To establish the *multigrid* convergence we additionally prove the stability of prolongation operator and smoothing iterations.

The mesh-dependent norms introduced here to prove approximation property (theorem 5.1) seem to be a natural extension for $\alpha > 0$ of the norms used in [29]. However to prove some specific norm equivalence results (lemma 2.1) we need the assumption on pressure finite element space to be a subspace of $H^1(\Omega)$. Not all stable discretizations of (1.1) satisfy this assumption, but many popular discretizations do satisfy, e.g. the family of Tailor-Hood elements or MINI element. All other assumptions which are used in proving approximation and smoothing properties are quite standard and collected in the next section. From approximation and smoothing properties the uniform convergence of the two-grid method follows. No extra assumptions are needed to pass from two-grid to multigrid convergence result.

The remainder of the paper is organized as follows. Section 2 introduces necessary spaces and norms. An important technical result is given by lemma 2.1. In section 3 we prove a priori estimates and error bounds for the solution of (1.1) and its finite element counterpart. Section 4 provides an algebraic framework for multigrid analysis. Based on results of section 3, the approximation property is proved in section 5. In section 6 we deduce smoothing and stability properties for the distributive, Braess-Sarazin and inexact Uzawa smoothing iterations. Finally, section 7 contains multigrid convergence estimates.

2. Preliminaries.

Throughout the paper we use the notation $(\cdot,\cdot)$ and $\|\cdot\|$ for the scalar product and norm in $L^2(\Omega)$ and $L^2(\Omega)^d$. Define the following spaces

$$\begin{aligned}\mathbf{V} &:= \{\mathbf{v}\in H^1(\Omega)^d \mid \mathbf{v}=0 \ \text{ on } \partial\Omega\},\\ \mathbb{Q} &:= \{q\in L^2(\Omega) \mid \int_\Omega q\,d\mathbf{x}=0\}.\end{aligned}$$

On $\mathbf{V}$ and $\mathbb{Q}$ we introduce the norms:

$$\|\mathbf{v}\|_{\mathbf{V}} := \left(\|\nabla\mathbf{v}\|^2+\alpha\|\mathbf{v}\|^2\right)^{\frac12}, \qquad \|q\|_Q := \sup_{\mathbf{v}\in\mathbf{V}}\frac{(\operatorname{div}\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbf{V}}}.$$

By $\mathbf{V}^{-1}$ we define the dual space to $\mathbf{V}$. Consider the operator $S := \operatorname{div}(\Delta-\alpha I)^{-1}\nabla$, where $-(\Delta-\alpha I)^{-1}$ is the solution operator to the following elliptic problem:

$$\begin{aligned}-\Delta\mathbf{u}+\alpha\mathbf{u}&=\mathbf{f} \quad \text{in } \Omega\\ \mathbf{u}&=0 \quad \text{on } \partial\Omega\end{aligned}$$

It is easy to check that S is a self-adjoint positive definite operator on $\mathbb{Q}$ and

$$(1+\alpha)^{-1}\|q\|^2 \lesssim (S\,q,q)=\|q\|_Q^2 \quad \text{for } q\in\mathbb{Q}. \tag{2.1}$$

The estimate in (2.1) follows with the help of the Nechas inequality: $\|q\|\lesssim\|\nabla q\|_{-1}$, $q\in\mathbb{Q}$. In order to avoid the repeated use of generic but unspecified constants, here and further by $x \lesssim y$ we mean that there is a constant c such that $x \le c\,y$, and c does not depend of the parameters which x and y may depend on, e.g. α and mesh size. Obviously, $x \gtrsim y$ is defined as $y \lesssim x$, and $x \backsimeq y$ when both $x \lesssim y$ and $y \lesssim x$.

By $\|\cdot\|_{Q^{-1}}$ we denote the dual norm to $\|\cdot\|_Q$ with respect to the L^2-duality. Clearly, it holds

$$\|q\|_{Q^{-1}} = (S^{-1}\,q, q)^{\frac{1}{2}} \quad \text{for} \quad q \in \mathbb{Q}.$$

On the product space $\mathbf{V} \times \mathbb{Q}$ we define the product norm and the bilinear form:

$$\begin{aligned} \|[\mathbf{v}, q]\| &= \left(\|\mathbf{v}\|_{\mathbf{V}}^2 + \|q\|_Q^2\right)^{\frac{1}{2}}, \\ a(\mathbf{u}, p; \mathbf{v}, q) &= (\nabla\mathbf{u}, \nabla\mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}). \end{aligned}$$

The weak formulation of the Stokes type problem (1.1) reads: Given $\mathbf{f} \in \mathbf{V}^{-1}$ and $g \in \mathbb{Q}$ find $\mathbf{u} \in \mathbf{V}$ and $p \in \mathbb{Q}$ such that

$$a(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + (g, q) \quad \forall\ \mathbf{v} \in \mathbf{V},\ q \in \mathbb{Q}. \tag{2.2}$$

Bilinear form $a(\cdot;\cdot)$ satisfies the following stability and continuity estimates [10]:

$$\|[\mathbf{u}, p]\| \lesssim \sup_{\mathbf{v}, q \in \mathbf{V} \times \mathbb{Q}} \frac{a(\mathbf{u}, p; \mathbf{v}, q)}{\|[\mathbf{v}, q]\|} \quad \forall\ \{\mathbf{u}, p\} \in \mathbf{V} \times \mathbb{Q} \tag{2.3}$$

$$a(\mathbf{u}, p; \mathbf{v}, q) \lesssim \|[\mathbf{u}, p]\|\,\|[\mathbf{v}, q]\| \quad \forall\ \{\mathbf{u}, p\},\ \{\mathbf{v}, q\} \in \mathbf{V} \times \mathbb{Q} \tag{2.4}$$

We will also assume to the following *H^2-regularity condition*: The domain Ω is such that the Stokes problem (1.1) with $\alpha = 0$ and $g = 0$ is H^2-regular, i.e., there is a constant c_R such that for any $\mathbf{f} \in L^2(\Omega)^d$ the solution $\{\mathbf{u}, p\}$ is an element of $H^2(\Omega)^d \times H^1(\Omega)$ and satisfies

$$\|\mathbf{u}\|_{H^2(\Omega)} + \|\nabla p\| \le c_R \|\mathbf{f}\|. \tag{2.5}$$

The condition is satisfied for convex domains [12].

For the discretization of (1.1) we introduce a quasi-uniform family of nested triangulations of Ω (triangles in 2D, tetrahedra in 3D) based on *global regular refinement*. We use conforming finite elements with piecewise polynomial functions. This results in a hierarchy of nested finite element spaces for velocity and pressure

$$\begin{aligned} \mathbf{V}_0 \subset \mathbf{V}_1 \subset \cdots \subset \mathbf{V}_k \subset \cdots \subset \mathbf{V}, \\ \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \cdots \subset \mathbb{Q}_k \subset \cdots \subset \mathbb{Q}. \end{aligned}$$

The corresponding mesh size parameter is denoted by h_k and satisfies $h_k/h_0 \backsimeq 2^{-k}$. We assume the discrete LBB condition to be valid

$$\sup_{\mathbf{u}_k \in \mathbf{V}_k} \frac{(\operatorname{div}\mathbf{u}_k, p_k)}{\|\nabla\mathbf{u}_k\|} \gtrsim \|p_k\| \qquad \forall\ p_k \in \mathbb{Q}_k. \tag{2.6}$$

We will also refer to the following inequality known as a weak infsup condition for the case of $\mathbb{Q}_k \subset H^1(\Omega)$:

$$\sup_{\mathbf{u}_k \in \mathbf{V}_k} \frac{(\operatorname{div}\mathbf{u}_k, p_k)}{\|\mathbf{u}_k\|} \gtrsim \|\nabla p_k\| \qquad \forall\ p_k \in \mathbb{Q}_k. \tag{2.7}$$

The proof of the inequality (2.7) for the Taylor-Hood and isoP2-P1 elements can be found in [4, 21], another example is the Mini element proposed in [1].

Assume the following standard interpolation properties of the finite element spaces ($H^0 := L^2(\Omega)^d$):

$$\inf_{\mathbf{v}_k\in\mathbf{V}_k} \|\mathbf{v}-\mathbf{v}_k\|_{H^\ell} \lesssim h_k\|\mathbf{v}\|_{H^{\ell+1}} \quad \text{for} \;\; \mathbf{v}\in H^{\ell+1}(\Omega)^d\cap\mathbf{V}, \;\; \ell=0,1 \tag{2.8}$$

$$\inf_{q_k\in\mathbb{Q}_k} \|q-q_k\| \lesssim h_k\|q\|_{H^1} \quad \text{for} \;\; q\in H^1(\Omega)\cap\mathbb{Q}, \tag{2.9}$$

and for the case $\mathbb{Q}_k\subset H^1(\Omega)$:

$$\inf_{q_k\in\mathbb{Q}_k} \|q-q_k\|_{H^1} \lesssim h_k\|q\|_{H^2} \quad \text{for} \;\; q\in H^2(\Omega), \tag{2.10}$$

The discrete problem on grid level k is given by: Find $\mathbf{u}_k\in\mathbf{V}_k$, $p_k\in\mathbb{Q}_k$ such that

$$a(\mathbf{u}_k,p_k;\mathbf{v}_k,q_k) = (\mathbf{f},\mathbf{v}_k)+(g,q_k) \quad \forall\; \mathbf{v}_k\in\mathbf{V}_k,\;\; q_k\in\mathbb{Q}_k. \tag{2.11}$$

Due to (2.6) there exists a unique solution to (2.11).

Besides the product norm $|[\cdot,\cdot]|$ defined above we endow every finite element subspace pair $\mathbf{V}_k\times\mathbb{Q}_k$ with the level-dependent product norm:

$$|[\mathbf{v}_k,q_k]|_k = \left(\|\mathbf{v}_k\|^2_{\mathbf{V}}+\|q_k\|^2_{Q_k}\right)^{\frac12}, \qquad \text{with} \;\; \|q\|_{Q_k} := \sup_{\mathbf{v}_k\in\mathbf{V}_k}\frac{(\operatorname{div}\mathbf{v}_k,p_k)}{\|\mathbf{v}_k\|_{\mathbf{V}}}.$$

Note that the later relation defines a norm on $\mathbb{Q}_k$ due to the LBB condition (2.6). Again, $\|\cdot\|_{Q^{-1}_k}$ denotes the dual norm to $\|\cdot\|_{Q_k}$ with respect to the L^2-duality. The choice of the norm yields the stability estimate on $\mathbf{V}_k\times\mathbb{Q}_k$ similar to (2.3):

$$|[\mathbf{u}_k,p_k]|_k \lesssim \sup_{\mathbf{v}_k,q_k\in\mathbf{V}_k\times\mathbb{Q}_k}\frac{a(\mathbf{u}_k,p_k;\mathbf{v}_k,q_k)}{|[\mathbf{v}_k,q_k]|_k} \quad \forall\;\{\mathbf{u}_k,p_k\}\in\mathbf{V}_k\times\mathbb{Q}_k \tag{2.12}$$

In the following lemma we prove an important technical result.

Lemma 2.1. *Assume* $\mathbb{Q}_k\subset H^1(\Omega)$ *and* (2.7). *Then it holds*

$$\|p_k\|_Q \lesssim \|p_k\|_{Q_k} \lesssim \|p_k\|_Q \qquad \forall\; p_k\in\mathbb{Q}_k. \tag{2.13}$$

and

$$\|p_k\|_{Q^{-1}} \lesssim \|p_k\|_{Q^{-1}_k} \lesssim \|p_k\|_{Q^{-1}} \qquad \forall\; p_k\in\mathbb{Q}_k. \tag{2.14}$$

Proof. The upper bound in (2.13) immediately follows from the definition of the norms and the embedding $\mathbf{V}_h\subset\mathbf{V}$.

To prove the low bound we use the following two inequalities [19, 21]:

$$\|p\|^2_Q \lesssim \inf_{q\in H^1(\Omega)}(\|p-q\|^2+\alpha^{-1}\|\nabla q\|^2) \qquad \forall\; p\in\mathbb{Q}, \tag{2.15}$$

$$\|p_k\|^2_{Q_k} \gtrsim \inf_{q_k\in\mathbb{Q}_k}(\|p_k-q_k\|^2+\alpha^{-1}\sup_{\mathbf{v}_k\in\mathbf{V}_k}\frac{(q_k,\operatorname{div}\mathbf{v}_k)^2}{\|\mathbf{v}_k\|^2}) \qquad \forall\; p_k\in\mathbb{Q}_k. \tag{2.16}$$

In particular, (2.15)–(2.16) follows from (2.25) in [21] and further application of the analysis of Sects 2.4 and 2.5 from [21] to the Stokes and the discrete Stokes problems.

Now the low bound in (2.13) follows from (2.7), (2.15), (2.16), and the embedding $\mathbb{Q}_k \subset H^1(\Omega)$. To prove (2.14) we use the following results [16, 21]:

$$\|p\|^2_{Q^{-1}} \simeq \|p\|^2 - \alpha(\Delta^{-1}p, p) \qquad \forall\, p \in \mathbb{Q}, \tag{2.17}$$

$$\|p_k\|^2_{Q_k^{-1}} \simeq \|p_k\|^2 - \alpha(\Delta_k^{-1}p_k, p_k) \qquad \forall\, p_k \in \mathbb{Q}_k, \tag{2.18}$$

where Δ^{-1} and Δ_k^{-1} are the solution operators for the Poisson-Neumann problem and the finite element Poisson-Neumann problem, respectively. We remark that relation (2.18) follows from theorems 2.1 and 3.1 in [16] and (2.18) follows from theorem 4.1 and the analysis of § 4.1 in [21]. The proofs of (2.16)–(2.18) use the H^2-regularity assumption. For any $p_k \in \mathbb{Q}_k$ it holds

$$-(\Delta^{-1}p_k, p_k) = \sup_{q \in H^1(\Omega)} \frac{(p_k, q)^2}{\|\nabla q\|^2} \quad \text{and} \quad -(\Delta_k^{-1}p_k, p_k) = \sup_{q_k \in \mathbb{Q}_k} \frac{(p_k, q_k)^2}{\|\nabla q_k\|^2}. \tag{2.19}$$

Therefore, the upper bound in (2.14) immediately follows from (2.17), (2.18),(2.19) and the embedding $\mathbb{Q}_k \subset H^1(\Omega)$.
Further, denote by $P_k q \in \mathbb{Q}_k$ the L^2-projection of $q \in H^1(\Omega) \cap \mathbb{Q}$ on $\mathbb{Q}_k$. Given our assumptions on the triangulation one has $\|\nabla P_k q\| \lesssim \|\nabla q\|$, cf. [9]. Therefore

$$-(\Delta^{-1}p_k, p_k) = \sup_{q \in H^1(\Omega)} \frac{(p_k, q)^2}{\|\nabla q\|^2} \lesssim \sup_{q \in H^1(\Omega)} \frac{(p_k, P_k q)^2}{\|\nabla P_k q\|^2} = \sup_{q_k \in \mathbb{Q}_k} \frac{(p_k, q_k)^2}{\|\nabla q_k\|^2} = -(\Delta_k^{-1}p_k, p_k).$$

This estimate together with (2.17) and (2.18) yields the low bound in (2.16).
□

Remark 1. Note that (2.14) does not follow directly from (2.13), since the inverse of the L^2-projection of the operator S on $\mathbb{Q}_k$ is not necessarily equal to the L^2-projection of S^{-1} on $\mathbb{Q}_k$.

3. A priori and error estimates.

First we prove two useful a priori estimates for the solution of (1.1).

Lemma 3.1. *Let* $\mathbf{f} \in L^2(\Omega)^d$. *The following estimate holds for the solution of* (1.1)

$$\alpha\|\mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^2 + \|p\|^2_Q \lesssim (1+\alpha)^{-1}\|\mathbf{f}\|^2 + \|g\|^2_{Q^{-1}}. \tag{3.1}$$

Furthermore, if the H^2*-regularity condition holds and* $g = 0$ *then* $\mathbf{u} \in H^2(\Omega)^d$, $p \in H^1(\Omega)$ *and*

$$\|\mathbf{u}\|_{H^2} + \|p\|_{H^1} \lesssim \|\mathbf{f}\| \tag{3.2}$$

Proof. The stability estimate (2.3), identity (2.2) and the Friedreich inequality $(1+\alpha)\|\mathbf{v}\| \lesssim \|\mathbf{v}\|_{\mathbf{V}}$ on $\mathbf{V}$ imply:

$$\begin{aligned} |[\mathbf{u}, p]| &\lesssim \sup_{\mathbf{v}, q \in \mathbf{V}\times\mathbb{Q}} \frac{a(\mathbf{u}, p; \mathbf{v}, q)}{|[\mathbf{v}, q]|} = \sup_{\mathbf{v}, q \in \mathbf{V}\times\mathbb{Q}} \frac{(\mathbf{f}, \mathbf{v}) + (g, q)}{|[\mathbf{v}, q]|} \\ &\le \left(\|\mathbf{f}\|^2_{\mathbf{V}^{-1}} + (S^{-1}g, g)\right)^{\frac{1}{2}} \lesssim \left((1+\alpha)^{-1}\|\mathbf{f}\|^2 + \|g\|^2_{Q^{-1}}\right)^{\frac{1}{2}} \end{aligned}$$

Thus we prove (3.1).
Assume now $g = 0$ and consider $\tilde{\mathbf{f}} = (\mathbf{f} - \alpha\mathbf{u})$, then $\mathbf{u}, p$ solves the Stokes problem

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p = \tilde{\mathbf{f}}, \quad \operatorname{div}\mathbf{u} &= 0 \qquad \text{in } \Omega \\ \mathbf{u} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Since $\tilde{\mathbf{f}} \in L^2(\Omega)^d$ and thanks to (3.1) it holds: $\|\tilde{\mathbf{f}}\| \le \|\mathbf{f}\| + \alpha\|\mathbf{u}\| \lesssim \|\mathbf{f}\|$. Now applying the standard regularity result (2.5) for the Stokes problem proves (3.2).
$\square$

Further in this section we prove several finite element convergence results for the generalized Stokes problem.

LEMMA 3.2. *Assume* $\mathbb{Q}_k \subset H^1(\Omega)$ *and* (2.7). *Let* $\mathbf{u}, p$ *be a solution to* (2.2) *and* $\mathbf{u}_k, p_k$ *solves* (2.11), *then it holds*

$$|[\mathbf{u} - \mathbf{u}_k, p - p_k]| \lesssim \inf_{\mathbf{v}_k \in \mathbf{V}_k} \inf_{p_k \in \mathbb{Q}_k} |[\mathbf{u} - \mathbf{v}_k, p - q_k]| \tag{3.3}$$

Proof. Let $\mathbf{u}_I$ be the best possible approximation to $\mathbf{u}$ in $\mathbf{V}_k$ with respect to the $\|\cdot\|_{\mathbf{V}}$ norm and p_I be the best possible approximation to p in $\mathbb{Q}_k$ with respect to the $\|\cdot\|_Q$ norm. The norm equivalence (2.13), stability (2.12), continuity (2.4) estimates and the orthogonality property of finite element error function give:

$$\begin{aligned}
|[\mathbf{u}_I - \mathbf{u}_k, p_I - p_k]| \lesssim |[\mathbf{u}_I - \mathbf{u}_k, p_I - p_k]|_k &\lesssim \sup_{\mathbf{v}_k, q_k \in \mathbf{V}_k \times \mathbb{Q}_k} \frac{a(\mathbf{u}_I - \mathbf{u}_k, p_I - p_k; \mathbf{v}_k, q_k)}{|[\mathbf{v}_k, q_k]|_k} \\
&\lesssim \sup_{\mathbf{v}_k, q_k \in \mathbf{V}_k \times \mathbb{Q}_k} \frac{a(\mathbf{u}_I - \mathbf{u}_k, p_I - p_k; \mathbf{v}_k, q_k)}{|[\mathbf{v}_k, q_k]|} \\
&= \sup_{\mathbf{v}_k, q_k \in \mathbf{V}_k \times \mathbb{Q}_k} \frac{a(\mathbf{u}_I - \mathbf{u}, p_I - p; \mathbf{v}_k, q_k)}{|[\mathbf{v}_k, q_k]|} \lesssim |[\mathbf{u}_I - \mathbf{u}, p_I - p]| \,.
\end{aligned}$$

With the help of this estimate and the triangle inequality we get

$$|[\mathbf{u} - \mathbf{u}_k, p - p_k]| \lesssim |[\mathbf{u}_I - \mathbf{u}, p_I - p]| = \inf_{\mathbf{v}_k \in \mathbf{V}_k} \inf_{p_k \in \mathbb{Q}_k} |[\mathbf{u} - \mathbf{v}_k, p - q_k]| \,.$$

$\square$

Taking $\mathbf{v}_k = 0$ and $q_k = 0$ on the right-hand side of (3.3) leads to

$$|[\mathbf{u} - \mathbf{u}_k, p - p_k]| \lesssim |[\mathbf{u}, p]| \,. \tag{3.4}$$

With the help of a standard duality argument we prove the lemma below.

LEMMA 3.3. *Let* $\mathbf{u}, p$ *be a solution to* (2.2) *and* $\mathbf{u}, p$ *solves* (2.11), *then*

$$\|\mathbf{u} - \mathbf{u}_k\| \lesssim \min\{h_k, \alpha^{-\frac{1}{2}}\} \, |[\mathbf{u} - \mathbf{u}_k, p - p_k r]| \,. \tag{3.5}$$

Proof. Denote $\mathbf{e}_k = \mathbf{u} - \mathbf{u}_k$, $r_k = p - p_k$. Consider $\mathbf{w} \in H^2(\Omega)^d$, $q \in H^1(\Omega) \cap \mathbb{Q}$ solving the Stokes type problem

$$\begin{aligned}
-\Delta \mathbf{w} + \alpha \mathbf{w} - \nabla q = \mathbf{e}_k, \quad \operatorname{div} \mathbf{w} &= 0 \qquad \text{in } \Omega \\
\mathbf{w} &= 0 \qquad \text{on } \partial\Omega
\end{aligned}$$

Using the weak form of the problem and the orthogonality property for $\mathbf{e}_k, r_k$, we get

$$\|\mathbf{e}_k\|^2 = a(\mathbf{w} - \mathbf{w}_k, q - q_k; \mathbf{e}_k, r_k)$$

with arbitrary $\mathbf{w}_k \in \mathbf{V}_k$, $q_k \in \mathbb{Q}_k$. Thanks to (2.4), interpolation properties (2.8)–(2.9), and a priori estimates from lemma 3.1, we get

$$\begin{aligned}
\|\mathbf{e}_k\|^2 &\lesssim |[\mathbf{w} - \mathbf{w}_k, q - q_k]| \, |[\mathbf{e}_k, r_k]| \lesssim h_k \left(\|\mathbf{w}\|_{H^2}^2 + \alpha\|\nabla \mathbf{w}\|^2 + \|\nabla q\|^2\right)^{\frac{1}{2}} |[\mathbf{e}_k, r_k]| \\
&\lesssim h_k \|\mathbf{e}_k\| \, |[\mathbf{e}_k, r_k]| \,.
\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{e}_k\|^2 &\lesssim |[\mathbf{w}-\mathbf{w}_k, q-q_k]|\,|[\mathbf{e}_k, r_k]| \lesssim (\|\nabla \mathbf{w}\|^2 + \alpha\|\mathbf{w}\|^2 + \|q\|_Q^2)^{\frac{1}{2}}\,|[\mathbf{e}_k, r_k]| \\ &\lesssim \alpha^{-\frac{1}{2}}\|\mathbf{e}_k\|\,|[\mathbf{e}_k, r_k]|\,.\end{aligned}$$

□

Now we are in position to prove the main result of this section.

Theorem 3.4. *Let* $\mathbf{f} \in L^2(\Omega)^d$. *Assume* $\mathbb{Q}_k \subset H^1(\Omega)$ *and* (2.7). *Let* $\mathbf{u}, p$ *be a solution to* (2.2) *and* $\mathbf{u}_k, p_k$ *solves* (2.11), *then the following error estimate holds*

$$\|\mathbf{u}-\mathbf{u}_k\| + \min\{h_k, \alpha^{-\frac{1}{2}}\}\|p-p_k\|_Q \lesssim \min\{h_k^2, \alpha^{-1}\}\left(\|\mathbf{f}\| + \max\{h^{-1}, \alpha^{\frac{1}{2}}\}\|g\|_{Q^{-1}}\right). \tag{3.6}$$

Proof. For arbitrary $\mathbf{v} \in \mathbf{V}$, $q \in \mathbb{Q}$ we denote by $\tilde{\mathbf{v}}_k$ and $\tilde{q}_k$ a unique projection on $\mathbf{V}_k$, $\mathbb{Q}_k$ such that

$$a(\mathbf{v}-\tilde{\mathbf{v}}_k, q-\tilde{q}_k; \mathbf{w}_k, r_k) = 0 \qquad \forall\ \mathbf{w}_k \in \mathbf{V}_k,\ r_k \in \mathbb{Q}_k.$$

Below we consequently use (2.3), orthogonality properties, estimates (3.4) and (3.5) for the differences $\mathbf{v}-\tilde{\mathbf{v}}_k$ and $q-\tilde{q}_k$:

$$\begin{aligned}|[\mathbf{u}-\mathbf{u}_k, p-p_k]| &\lesssim \sup_{\mathbf{v},q\in\mathbf{V}\times\mathbb{Q}} \frac{a(\mathbf{u}-\mathbf{u}_k, p-p_k; \mathbf{v}, q)}{|[\mathbf{v}, q]|} \\ &= \sup_{\mathbf{v},q\in\mathbf{V}\times\mathbb{Q}} \frac{a(\mathbf{u}-\mathbf{u}_k, p-p_k; \mathbf{v}-\tilde{\mathbf{v}}_k, q-\tilde{q}_k)}{|[\mathbf{v}, q]|} \\ &= \sup_{\mathbf{v},q\in\mathbf{V}\times\mathbb{Q}} \frac{a(\mathbf{u}, p; \mathbf{v}-\tilde{\mathbf{v}}_k, q-\tilde{q}_k)}{|[\mathbf{v}, q]|} \\ &= \sup_{\mathbf{v},q\in\mathbf{V}\times\mathbb{Q}} \frac{(\mathbf{f}, \mathbf{v}-\tilde{\mathbf{v}}_k) + (g, q-\tilde{q}_k)}{|[\mathbf{v}, q]|} \\ &\le \sup_{\mathbf{v},q\in\mathbf{V}\times\mathbb{Q}} \frac{\|\mathbf{f}\|\,\|\mathbf{v}-\tilde{\mathbf{v}}_k\| + \|g\|_{Q^{-1}}\|q-\tilde{q}_k\|_Q}{|[\mathbf{v}, q]|} \\ &\lesssim \sup_{\mathbf{v},q\in\mathbf{V}\times\mathbb{Q}} \frac{\|\mathbf{f}\|\,\min\{h_k, \alpha^{-\frac{1}{2}}\}\,|[\mathbf{v}, q]| + \|g\|_{Q^{-1}}\,|[\mathbf{v}, q]|}{|[\mathbf{v}, q]|} \\ &= \min\{h_k, \alpha^{-\frac{1}{2}}\}\,\|\mathbf{f}\| + \|g\|_{Q^{-1}}.\end{aligned}$$

We proceed using (3.5):

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_k\| + \min\{h_k, \alpha^{-\frac{1}{2}}\}\|p-p_k\|_Q &\lesssim \min\{h_k, \alpha^{-\frac{1}{2}}\}\,|[\mathbf{u}-\mathbf{u}_k, p-p_k]| \\ &\lesssim \min\{h_k^2, \alpha^{-1}\}\left(\|\mathbf{f}\| + \max\{h_k^{-1}, \alpha^{\frac{1}{2}}\}\|g\|_{Q^{-1}}\right).\end{aligned}$$

□

Remark 2. In the proof of the theorem the extra assumption $\mathbb{Q}_k \subset H^1(\Omega)$ was involved only through the usage of the estimate (3.4). We conjecture, however, that (3.4) still holds for more general case of LBB stable elements.

4. Multigrid method and algebraic framework. For the approximate solution of the discrete problem (2.11) we apply a multigrid method. The method and its convergence analysis will be presented in a matrix-vector form as in Hackbush [14]. To this end consider a space $\mathbb{Q}_k^+ := \mathbb{Q}_k \oplus \mathrm{span}\{1\}$, i.e. a pressure finite element space without orthogonality condition. Denote by $\{\phi_i\}_{1\le i\le n_k}$ and $\{\psi_i\}_{1\le i\le m_k}$ the standard nodal bases in $\mathbf{V}_k$ and $\mathbb{Q}_k^+$. Consider the isomorphisms:

$$\begin{aligned} P_k:\ \mathbf{X}_k &:= \mathbb{R}^{n_k} \to \mathbf{V}_k, \quad P_k\mathrm{u} = \sum_{i=1}^{n_k} u_i\phi_i \\ R_k:\ \mathbb{Y}_k &:= \mathbb{R}^{m_k} \to \mathbb{Q}_k^+, \quad R_k\mathrm{p} = \sum_{i=1}^{m_k} p_i\psi_i. \end{aligned}$$

Both on $\mathbf{X}_k$ and $\mathbb{Y}_k$ we use a Euclidean scalar product scaled with h_k^d, e.g. on $\mathbf{X}_k$ we use $\langle \mathrm{u},\mathrm{v}\rangle = h_k^d\sum_{i=1}^{n_k} u_iv_i$ and corresponding norm denoted by $\|\cdot\|$. The following norm equivalences hold on $\mathbf{X}_k$ and $\mathbb{Y}_k$:

$$\|\mathrm{u}\| \ \lesssim\ \|P_k\mathrm{u}\| \lesssim\ \|\mathrm{u}\| \quad \forall\ \mathrm{u}\in\mathbf{X}_k, \tag{4.1}$$

$$\|\mathrm{p}\| \ \lesssim\ \|R_k\mathrm{p}\| \lesssim\ \|\mathrm{p}\| \quad \forall\ \mathrm{p}\in\mathbb{Y}_k. \tag{4.2}$$

Let the matrices $\mathrm{A}_k\in\mathbb{R}^{n_k\times n_k}$, $\mathrm{B}_k\in\mathbb{R}^{m_k\times n_k}$ and the velocity and pressure mass matrices $\mathrm{M}_u\in\mathbb{R}^{n_k\times n_k}$ and $\mathrm{M}_p\in\mathbb{R}^{m_k\times m_k}$ be given by

$$\begin{aligned} &\langle \mathrm{A}_k\mathrm{u},\mathrm{v}\rangle = (\nabla\mathbf{u}_k,\nabla\mathbf{v}_k)+\alpha(\mathbf{u}_k,\mathbf{v}_k) \quad \forall\ \mathrm{u},\mathrm{v}\in\mathbb{R}^{n_k},\ \mathbf{u}_k = P_k\mathrm{u},\ \mathrm{v}_k = P_k\mathrm{v}, \\ &\langle \mathrm{B}_k\mathrm{u},\mathrm{p}\rangle = (\mathrm{div}\,\mathbf{u}_k, p_k) \quad \forall\ \mathrm{u}\in\mathbb{R}^{n_k},\ \mathrm{p}\in\mathbb{R}^{m_k},\ \mathbf{u}_k = P_k\mathrm{u},\ p_k = R_k\mathrm{p}, \\ &\langle \mathrm{M}_u\mathrm{u},\mathrm{v}\rangle = (\mathbf{u}_k,\mathbf{v}_k) \quad \forall\ \mathrm{u},\mathrm{v}\in\mathbb{R}^{n_k},\ \mathbf{u}_k = P_k\mathrm{u},\ \mathrm{v}_k = P_k\mathrm{v}, \\ &\langle \mathrm{M}_p\mathrm{q},\mathrm{p}\rangle = (q_k,p_k) \quad \forall\ \mathrm{q},\mathrm{p}\in\mathbb{R}^{m_k},\ q_k = R_k\mathrm{q},\ p_k = R_k\mathrm{p}. \end{aligned} \tag{4.3}$$

Finite element formulation (2.11) can be written as a linear system of the form

$$\begin{pmatrix} \mathrm{A}_k & \mathrm{B}_k^T \\ \mathrm{B}_k & 0 \end{pmatrix}\begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \begin{pmatrix} \mathrm{f} \\ \mathrm{g} \end{pmatrix} \tag{4.4}$$

with f and g such that $\langle \mathrm{f},\mathrm{v}\rangle = (\mathbf{f},P_k\mathrm{v})$ for all $\mathrm{v}\in\mathbb{R}^{n_k}$ and $\langle \mathrm{g},\mathrm{q}\rangle = (g,R_k\mathrm{q})$ for all $\mathrm{q}\in\mathbb{R}^{m_k}$. By $\mathcal{A}_k$ and S_k we denote the stiffness and pressure Schur complement matrices of the finite element problem (2.11) on level k:

$$\mathcal{A}_k := \begin{pmatrix} \mathrm{A}_k & \mathrm{B}_k^T \\ \mathrm{B}_k & 0 \end{pmatrix}, \qquad \mathrm{S}_k := \mathrm{B}_k\mathrm{A}_k^{-1}\mathrm{B}_k^T. \tag{4.5}$$

Remark 3. Note that both S_k and $\mathcal{A}_k$ are singular matrices and have a one-dimensional kernel. Define the constant vector $\mathrm{e} := R_k^{-1}1 = (1,\dots,1)^T\in\mathbb{R}^m$. Then we have $\ker(\mathrm{S}) = \mathrm{span}\{\mathrm{e}\}$. Note that

$$(R_k\mathrm{p},1) = 0 \ \Leftrightarrow\ (R_k\mathrm{p},R_k\mathrm{e}) = 0 \ \Leftrightarrow \langle \mathrm{M}_p\mathrm{p},\mathrm{e}\rangle = 0 \ \Leftrightarrow\ \langle \mathrm{p},\mathrm{M}_p\mathrm{e}\rangle = 0$$

Thus the orthogonality condition in $\mathbb{Q}_k$ corresponds to the orthogonality to the vector $\mathrm{M}_p\mathrm{e}$ in $\mathbb{R}^{m_k}$. This can be written as $\mathbb{Q}_k = \{\,R_k\mathrm{p}\mid \mathrm{p}\in(\mathrm{M}_p\mathrm{e})^\perp\,\}$. Denote $\tilde{\mathbb{Y}}_k := (\mathrm{M}_p\mathrm{e})^\perp$. Below we always consider S_k on the subspace $\tilde{\mathbb{Y}}_k$ and $\mathcal{A}_k$ on the subspace $\mathbf{X}_k\times\tilde{\mathbb{Y}}_k$. Moreover, from the definition of the $\|\cdot\|_{Q_k}$ norm and S_k it follows

$$\langle \mathrm{S}_k\mathrm{p},\mathrm{p}\rangle = \|p_k\|_{Q_k}^2 \quad \forall\ \mathrm{p}\in\mathbb{R}^{m_k},\ p_k = R_k\mathrm{p}. \tag{4.6}$$

Furthermore, we define two product norms on $\mathbf{X}_k \times \tilde{\mathbb{Y}}_k$:

$$\begin{aligned} \|\mathrm{u},\mathrm{p}\|_{S_k} &:= \left(\|\mathrm{u}\|^2 + \min\{h_k^2, \alpha^{-1}\}\langle \mathrm{S}_k\mathrm{p},\mathrm{p}\rangle\right)^{\frac{1}{2}}, \\ \|\mathrm{u},\mathrm{p}\|_{S_k^{-1}} &:= \left(\|\mathrm{u}\|^2 + \max\{h_k^{-2}, \alpha\}\langle \mathrm{S}_k^{-1}\mathrm{p},\mathrm{p}\rangle\right)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Denote $\mathrm{D}_k = \mathrm{diag}(\mathrm{A}_k)$. Due to regular mesh refinement the following relations hold (cf. [20], [21])

$$(1+\alpha)\mathrm{I}_k \ \lesssim\ \mathrm{A}_k \ \lesssim\ \mathrm{D}_k \tag{4.8}$$

$$\mathrm{D}_k \ \backsimeq\ (h_k^{-2}+\alpha)\mathrm{I}_k, \tag{4.9}$$

$$\mathrm{S}_k^{-1} \ \backsimeq\ \mathrm{I}_k + \alpha(\mathrm{B}_k\mathrm{M}_u^{-1}\mathrm{B}_k^T)^{-1}. \tag{4.10}$$

Here and further I_k is the identity matrix for a corresponding vector space and $A \le B$ for two square matrices if $B - A$ is non-negative definite.

For the prolongation and restriction in the multigrid algorithm we use the canonical choice:

$$\begin{aligned} &\mathbf{p}_k : \mathbf{X}_{k-1} \times \tilde{\mathbb{Y}}_{k-1} \to \mathbf{X}_k \times \tilde{\mathbb{Y}}_k, \quad \mathbf{p}_k = P_k^{-1}P_{k-1} \times R_k^{-1}R_{k-1} \\ &\mathbf{r}_k : \mathbf{X}_k \times \tilde{\mathbb{Y}}_k \to \mathbf{X}_{k-1} \times \tilde{\mathbb{Y}}_{k-1}, \quad \mathbf{r}_k = P_{k-1}^*(P_k^*)^{-1} \times R_{k-1}^*(R_k^*)^{-1}. \end{aligned} \tag{4.11}$$

Note that both $\mathbf{p}_k$ and $\mathbf{r}_k$ keep the pressure vector in the right subspace.

In section 6 we consider several linear smoothing iterations of the form

$$\{\mathrm{u}^{\mathrm{new}},\mathrm{p}^{\mathrm{new}}\} = \{\mathrm{u}^{\mathrm{old}},\mathrm{p}^{\mathrm{old}}\} - \mathcal{W}_k^{-1}(\mathcal{A}_k\{\mathrm{u}^{\mathrm{old}},\mathrm{p}^{\mathrm{old}}\} - b) \quad \text{for } \{\mathrm{u}^{\mathrm{old}},\mathrm{p}^{\mathrm{old}}\}, b \in \mathbf{X}_k \times \mathbb{Y}_k$$

with corresponding iteration matrix denoted by

$$\mathcal{L}_k = \mathcal{I}_k - \mathcal{W}_k^{-1}\mathcal{A}_k. \tag{4.12}$$

With the components defined above a standard multigrid algorithm with ν pre-smoothing iterations can be formulated (cf. [14]) with an iteration matrix that satisfies the recursion

$$\begin{aligned} &\mathcal{M}_0 = 0, \\ &\mathcal{M}_k = \left(\mathcal{I}_k - \mathbf{p}_k(\mathcal{I}_k - \mathcal{M}_{k-1}^{\gamma})\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\right)\mathcal{L}_k^{\nu}, \quad k = 1, 2, \ldots. \end{aligned}$$

The choices $\gamma = 1$ and $\gamma = 2$ correspond to the V- and W-cycle, respectively. For the analysis of this multigrid method we use the framework of [14] based on the approximation and smoothing property. Below we derive these properties for the generalized Stokes problem.

5. Approximation property.

The theorem below states the necessary approximation property.

THEOREM 5.1 (Approximation property). *Let $\mathcal{A}_k$ be the stiffness matrix from* (4.5) *and $\mathbf{p}_k, \mathbf{r}_k$ the prolongation and restriction as in* (4.11). *Then under the assumptions of theorem 3.4 the following approximation property holds:*

$$\|\mathcal{A}_k^{-1} - \mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\|_{S_k^{-1}\to S_k} \lesssim \min\left\{h_k^2, \alpha^{-1}\right\}$$

Proof. Take $\{\mathrm{f}_k, \mathrm{g}_k\} \in \mathbf{X}_k \times \tilde{\mathbb{Y}}_k$. Let $\{\mathbf{u}, p\} \in \mathbf{V} \times \mathbb{Q}$, $\{\mathbf{u}_k, p_k\} \in \mathbf{V}_k \times \mathbb{Q}_k$, and $\{\mathbf{u}_{k-1}, p_{k-1}\} \in \mathbf{V}_{k-1} \times \mathbb{Q}_{k-1}$ be such that

$$\begin{aligned} a(\mathbf{u}, p; \mathbf{v}, q) &= ((P_k^*)^{-1}\mathrm{f}_k, \mathbf{v}) + ((R_k^*)^{-1}\mathrm{g}_k, r) \quad \forall\ \{\mathbf{v}, q\} \in \mathbf{V} \times \mathbb{Q}, \\ a(\mathbf{u}_k, p_k; \mathbf{v}, q) &= ((P_k^*)^{-1}\mathrm{f}_k, \mathbf{v}) + ((R_k^*)^{-1}\mathrm{g}_k, r) \quad \forall\ \{\mathbf{v}, q\} \in \mathbf{V}_k \times \mathbb{Q}_k, \\ a(\mathbf{u}_{k-1}, p_{k-1}; \mathbf{v}, q) &= ((P_k^*)^{-1}\mathrm{f}_k, \mathbf{v}) + ((R_k^*)^{-1}\mathrm{g}_k, r) \quad \forall\ \{\mathbf{v}, q\} \in \mathbf{V}_{k-1} \times \mathbb{Q}_{k-1}. \end{aligned}$$

Putting $\mathbf{f} = (P_k^*)^{-1}\mathrm{f}_k$ and $g = (R_k^*)^{-1}\mathrm{g}_k$ in theorem 3.4, we obtain

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_l\| + \min\{h_l, \alpha^{-\frac{1}{2}}\}\|p - p_l\|_Q \\ &\qquad \lesssim \min\{h_l^2, \alpha^{-1}\}\left(\|(P_k^*)^{-1}\mathrm{f}_k\| + \max\{h_l^{-1}, \alpha^{\frac{1}{2}}\}\|(R_k^*)^{-1}\mathrm{g}_k\|_{Q^{-1}}\right) \end{aligned}$$

with $l = k$ and $l = k-1$. Due to $h_{k-1} \lesssim h_k$ this yields

$$\begin{aligned} &\|\mathbf{u}_k - \mathbf{u}_{k-1}\| + \min\{h_k, \alpha^{-\frac{1}{2}}\}\|p_k - p_{k-1}\|_Q \\ &\qquad \lesssim \min\{h_k^2, \alpha^{-1}\}\left(\|(P_k^*)^{-1}\mathrm{f}_k\| + \max\{h_k^{-1}, \alpha^{\frac{1}{2}}\}\|(R_k^*)^{-1}\mathrm{g}_k\|_{Q^{-1}}\right) \end{aligned}$$

Now we use the result of lemma 2.1 to obtain

$$\begin{aligned} &\|\mathbf{u}_k - \mathbf{u}_{k-1}\| + \min\{h_k, \alpha^{-\frac{1}{2}}\}\|p_k - p_{k-1}\|_{Q_k} \\ &\qquad \lesssim \min\{h_k^2, \alpha^{-1}\}\left(\|(P_k^*)^{-1}\mathrm{f}_k\| + \max\{h_k^{-1}, \alpha^{\frac{1}{2}}\}\|(R_k^*)^{-1}\mathrm{g}_k\|_{Q_k^{-1}}\right) \end{aligned}$$

From the definition of $\mathcal{A}_k$ and (4.11) it follows that $\{\mathbf{u}_k, p_k\} = (P_k, R_k)^T\mathcal{A}_k^{-1}\{\mathrm{f}_k, \mathrm{g}_k\}$ and $\{\mathbf{u}_{k-1}, p_{k-1}\} = (P_{k-1}, R_{k-1})^T\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\{\mathrm{f}_k, \mathrm{g}_k\}$. Thus, using (4.1) and (4.6), we get

$$\begin{aligned} &\|(\mathcal{A}_k^{-1} - \mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k)\{\mathrm{f}_k, \mathrm{g}_k\}\|_{S_k} \\ &\qquad \simeq \|\mathbf{u}_k - \mathbf{u}_{k-1}\| + \min\{h, \alpha^{-\frac{1}{2}}\}\|p_k - p_{k-1}\|_{Q_k} \\ &\qquad \lesssim\ \min\{h_k^2, \alpha^{-1}\}\left(\|(P_k^*)^{-1}\mathrm{f}_k\| + \max\{h^{-1}, \alpha^{\frac{1}{2}}\}\|(R_k^*)^{-1}\mathrm{g}_k\|_{Q_k^{-1}}\right) \\ &\qquad \simeq \min\{h_k^2, \alpha^{-1}\}\|\mathrm{f}_k, \mathrm{g}_k\|_{S_k^{-1}}\ , \end{aligned}$$

which proves the theorem.

□

Based on the "inexact" Schur complement $\widehat{\mathrm{S}}_k = \mathrm{B}_k\mathrm{D}_k^{-1}\mathrm{B}_k^T$, we define two more product norms on $\mathbf{X}_k \times \tilde{\mathbb{Y}}_k$:

$$\begin{aligned} \|\mathrm{u}, \mathrm{p}\|_{\widehat{S}_k} &:= \left(\|\mathrm{u}\|^2 + \min\{h_k^2, \alpha^{-1}\}\langle\widehat{\mathrm{S}}_k\mathrm{p}, \mathrm{p}\rangle\right)^{\frac{1}{2}}, \\ \|\mathrm{u}, \mathrm{p}\|_{\widehat{S}_k^{-1}} &:= \left(\|\mathrm{u}\|^2 + \max\{h_k^{-2}, \alpha\}\langle\widehat{\mathrm{S}}_k^{-1}\mathrm{p}, \mathrm{p}\rangle\right)^{\frac{1}{2}}. \end{aligned} \tag{5.1}$$

Thanks to (4.8) it holds $\widehat{\mathrm{S}}_k \lesssim \mathrm{S}_k$. Therefore we get from theorem 5.1

Corollary 5.2. *Under the assumptions of theorem 5.1 the following approximation property holds:*

$$\|\mathcal{A}_k^{-1} - \mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\|_{\widehat{S}_k^{-1} \to \widehat{S}_k} \lesssim \min\left\{h_k^2, \alpha^{-1}\right\}$$

6. Smoothing property.

In this section we prove a smoothing property for several iterative methods (smoothers) known from the literature. This smoothing property will complement the approximation property from the previous section, resulting in the uniform estimate of the two-grid convergence. We also analyze stability of smoothing iterations, since this property is used for proving multigrid W-cycle convergence.

We will need the following result, cf. e.g. [15].

LEMMA 6.1. *Assume* $A, \widehat{A}, \widehat{S}$ *are symmetric positive definite and* $S = BA^{-1}B^T$. *Assume also that the inequalities*

$$\rho_1 \widehat{A} \;\le\; A \;\le\; \rho_2 \widehat{A}, \tag{6.1}$$

$$\mu_1 \widehat{S} \;\le\; S \;\le\; \mu_2 \widehat{S} \tag{6.2}$$

hold with positive constants $\rho_1, \rho_2, \mu_1, \mu_2$. *Then all eigenvalues of the problem*

$$\begin{array}{rcl} A\mathrm{u} \;+\; B^T\mathrm{p} & = & \lambda\, \widehat{A}\mathrm{u}, \\ B\mathrm{u} & = & \lambda\, \widehat{S}\mathrm{p} \end{array} \tag{6.3}$$

belong to

$$\begin{aligned} &[\rho_1, \rho_2] \bigcup \left[\frac{\rho_1 + \sqrt{\rho_1^2 + 4\rho_1\mu_1}}{2}, \frac{\rho_2 + \sqrt{\rho_2^2 + 4\rho_2\mu_2}}{2} \right] \\ &\qquad \bigcup \left[\frac{\rho_2 - \sqrt{\rho_2^2 + 4\rho_2\mu_2}}{2}, \frac{\rho_1 - \sqrt{\rho_1^2 + 4\rho_1\mu_1}}{2} \right]. \end{aligned} \tag{6.4}$$

REMARK 4. Similar eigenvalue bounds for (6.3) can be found in other papers, e.g. [25]. Further we will use the result of the lemma for the case, when $\widehat{S}$ and S are symmetric positive definite on the subspace $\tilde{\mathbb{Y}}_k$ and the problem (6.3) has a zero eigenvalue corresponding to the eigenvector $\{0, \mathrm{e}\}$.

6.1. Distributive iterations.

Writing the system (4.4) in a general form $\mathcal{A}\,\mathrm{x}_k = \mathrm{b}$ the idea behind the distributive smoothing iterations can be expressed as follows. One chooses matrices $\mathcal{B}$ and $\mathcal{C}$ and consider smoothing iterations of the form:

$$\mathrm{y}^{\nu+1} = \mathrm{y}^{\nu} - \mathcal{C}^{-1}(\mathcal{A}\,\mathcal{B}\,\mathrm{y}^{\nu} - b), \quad \mathrm{x}_k = \mathcal{B}\,\mathrm{y}. \tag{6.5}$$

One possibility is to set $\mathcal{B} = \mathcal{C}^{-1}\mathcal{A}$. If $\mathcal{C} = \mathcal{C}^T > 0$, $\mathcal{C}^{-1}\mathcal{A}\mathcal{B}$ is a positive definite matrix and self-adjoint in a proper scalar product. We consider a Jacobi type iterations, i.e. $\mathcal{C}$ is a block diagonal matrix defined below. Let N_k be a matrix of the preconditioner for the discrete pressure Neumman problem, such that

$$\mathrm{N}_k \;\backsimeq\; \mathrm{B}_k \mathrm{M}_u^{-1} \mathrm{B}_k^T \tag{6.6}$$

Define a block diagonal matrix $\mathcal{D}_k$ as

$$\mathcal{D}_k^{-1} = \begin{pmatrix} \mathrm{diag}(\mathrm{A}_k)^{-1} & 0 \\ 0 & \mathrm{I}_k + \alpha \mathrm{N}_k^{-1} \end{pmatrix}. \tag{6.7}$$

and set $\mathcal{C} = \omega \mathcal{D}_k$ with a parameter $\omega > 0$. The iteration matrix $\mathcal{L}_k$ in this case can be written in the form (4.12) with $\mathcal{W}_k^{-1} = \omega^2 \mathcal{D}_k^{-1} \mathcal{A}_k \mathcal{D}_k^{-1}$.

Theorem 6.2 (Smoothing property). *Assume $\omega > 0$ is small enough, but independent of α and k. It holds*

$$\|\mathcal{A}_k\mathcal{L}_k^{\nu}\|_{S_k\to S_k^{-1}} \lesssim (h_k^{-2}+\alpha)\frac{1}{\sqrt{2\nu+1}}. \tag{6.8}$$

Proof. With the auxiliary matrix

$$\mathcal{D}_s = \begin{pmatrix} \mathrm{I}_k & 0 \\ 0 & \min\{h_k^2,\alpha^{-1}\}\mathrm{S}_k \end{pmatrix}, \tag{6.9}$$

it holds

$$\begin{aligned}\|\mathcal{A}_k\mathcal{L}_k^{\nu}\|_{S_k\to S_k^{-1}} &= \|\mathcal{A}_k(\mathcal{I}_k-\omega^2\mathcal{D}_k^{-1}\mathcal{A}_k\mathcal{D}_k^{-1}\mathcal{A}_k)^{\nu}\|_{S_k\to S_k^{-1}} \\ &= \|\mathcal{D}_s^{-\frac{1}{2}}\mathcal{A}_k(\mathcal{I}_k-\omega^2\mathcal{D}_k^{-1}\mathcal{A}_k\mathcal{D}_k^{-1}\mathcal{A}_k)^{\nu}\mathcal{D}_s^{-\frac{1}{2}}\|.\end{aligned}$$

Denote $\bar{\mathcal{A}} = \omega\mathcal{D}_k^{-\frac{1}{2}}\mathcal{A}_k\mathcal{D}_k^{-\frac{1}{2}}$ and observe the equality

$$\|\mathcal{D}_s^{-\frac{1}{2}}\mathcal{A}_k(\mathcal{I}_k-\omega^2\mathcal{D}_k^{-1}\mathcal{A}_k\mathcal{D}_k^{-1}\mathcal{A}_k)^{\nu}\mathcal{D}_s^{-\frac{1}{2}}\| = \|\omega^{-1}\mathcal{D}_s^{-\frac{1}{2}}\mathcal{D}_k^{\frac{1}{2}}\bar{\mathcal{A}}(\mathcal{I}_k-\bar{\mathcal{A}}^2)^{\nu}\mathcal{D}_k^{\frac{1}{2}}\mathcal{D}_s^{-\frac{1}{2}}\|.$$

We get

$$\|\mathcal{A}_k\mathcal{L}_k^{\nu}\|_{S_k\to S_k^{-1}} \le \omega^{-1}\|\mathcal{D}_k\mathcal{D}_s^{-1}\|\|\bar{\mathcal{A}}(\mathcal{I}_k-\bar{\mathcal{A}}^2)^{\nu}\|.$$

Thanks to the eigenvalue estimate of lemma 6.1 and bounds in (4.8) and (6.6) one can choose such $\omega \gtrsim 1$ that $\mathrm{sp}(\bar{\mathcal{A}}) \in [-1,1]$. Hence

$$\|\bar{\mathcal{A}}(\mathcal{I}_k-\bar{\mathcal{A}}^2)^{\nu}\| \le \max_{x\in[-1,1]}|x(1-x^2)^{\nu}| \le \frac{1}{\sqrt{2\nu+1}}.$$

Finally we use (4.9), (4.10) and (6.6) to verify that

$$\|\mathcal{D}_k\mathcal{D}_s^{-1}\| \lesssim (h_k^{-2}+\alpha).$$

□

Theorem 6.3 (Stability of smoother). *With the same choice of ω as in theorem 6.2 it holds*

$$\|\mathcal{L}_k^{\nu}\|_{S_k\to S_k} \lesssim 1. \tag{6.10}$$

Proof. From (4.9), (4.10) and (6.6) we get $\max\{h_k^{-2},\alpha\}\mathcal{D}_s \simeq \mathcal{D}_k$ for the matrices $\mathcal{D}_s$ and $\mathcal{D}_k$ defined in (6.7) and (6.9). Therefore

$$\|\mathcal{L}_k^{\nu}\|_{S_k\to S_k} = \|\mathcal{D}_s^{\frac{1}{2}}\mathcal{L}_k^{\nu}\mathcal{D}_s^{-\frac{1}{2}}\| \simeq \|\mathcal{D}_k^{\frac{1}{2}}\mathcal{L}_k^{\nu}\mathcal{D}_k^{-\frac{1}{2}}\| = \|(\mathcal{I}_k-\bar{\mathcal{A}}^2)^{\nu}\| \tag{6.11}$$

with $\bar{\mathcal{A}} = \omega\mathcal{D}_k^{-\frac{1}{2}}\mathcal{A}_k\mathcal{D}_k^{-\frac{1}{2}}$. In the proof of theorem 6.2 we have shown that $\mathrm{sp}(\bar{\mathcal{A}}) \in [-1,1]$. Hence it holds

$$\|(\mathcal{I}_k-\bar{\mathcal{A}}^2)^{\nu}\| \le 1. \tag{6.12}$$

Inequalities (6.11)–(6.12) yield (6.10).

□

For the Stokes problem ($\alpha = 0$) the similar smoothing iterations were considered first in [29] and [22]. The smoother from [29] and [22] can be written in the form of (6.5) with

$$\mathcal{C} = \omega \begin{pmatrix} h_k^{-2}\mathrm{I}_k & 0 \\ 0 & \mathrm{I}_k \end{pmatrix}.$$

Clearly, its analysis fits in the framework given in this paper.

6.2. Braess-Sarazin and inexact Uzawa smoothers. In this section D_k is arbitrary symmetric matrix satisfying (4.8) and (4.9). One may still think of D_k as $\mathrm{D}_k = \mathrm{diag}(\mathrm{A}_k)$. Other reasonable choices are $\mathrm{D}_k = (h^{-2}+\alpha)\mathrm{I}_k$ or $\mathrm{D}_k = (h^{-2}+\alpha)\mathrm{M}_u$, where M_u is the velocity mass matrix or its diagonal approximation. Let ω be some given positive parameter. Consider iterations of the form:

$$\begin{pmatrix} \mathrm{u}^{\mathrm{new}} \\ \mathrm{p}^{\mathrm{new}} \end{pmatrix} = \begin{pmatrix} \mathrm{u}^{\mathrm{old}} \\ \mathrm{p}^{\mathrm{old}} \end{pmatrix} - \begin{pmatrix} \omega\mathrm{D}_k & \mathrm{B}_k^T \\ \mathrm{B}_k & 0 \end{pmatrix}^{-1} \left\{ \begin{pmatrix} \mathrm{A}_k & \mathrm{B}_k^T \\ \mathrm{B}_k & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u}^{\mathrm{old}} \\ \mathrm{p}^{\mathrm{old}} \end{pmatrix} - \begin{pmatrix} \mathrm{f} \\ \mathrm{g} \end{pmatrix} \right\} \tag{6.13}$$

At each iteration (6.13) one has to solve the auxiliary system:

$$\begin{pmatrix} \omega\mathrm{D}_k & \mathrm{B}_k^T \\ \mathrm{B}_k & 0 \end{pmatrix} \begin{pmatrix} \mathrm{v} \\ \mathrm{q} \end{pmatrix} = \begin{pmatrix} \mathrm{r}^{\mathrm{old}} \\ \mathrm{B}_k\,\mathrm{u}^{\mathrm{old}} - \mathrm{g} \end{pmatrix}. \tag{6.14}$$

To solve (6.14) one can eliminate v from the system (6.14) and obtain a problem for the q variable (we recall notation $\widehat{\mathrm{S}}_k = \mathrm{B}_k\,\mathrm{D}_k^{-1}\mathrm{B}_k^T$):

$$\widehat{\mathrm{S}}_k\mathrm{q} = \mathrm{B}_k\,\mathrm{D}_k^{-1}\mathrm{r}^{\mathrm{old}} - \omega(\mathrm{B}_k\mathrm{u}^{\mathrm{old}} - \mathrm{g}). \tag{6.15}$$

The upper bound in (4.8) yields $\lambda_{\max}(\mathrm{D}_k^{-1}\mathrm{A}_k) \lesssim 1$. Thus one can take ω satisfying

$$\omega > \lambda_{\max}(\mathrm{D}_k^{-1}\mathrm{A}_k) \quad \text{and} \quad \omega \backsimeq 1. \tag{6.16}$$

Remark 5. Smoothing iterations (6.13) were first proposed in [6] with $\mathrm{D}_k = \mathrm{I}_k$ for the case $\mathrm{g} = 0$, see also [5]. More general choice of D_k was analyzed in [31] and [17]. Considering a general $\mathrm{g} \in \tilde{\mathbb{Y}}_k$ causes no additional difficulties.

The method requires an *exact* solution of the problem (6.15) which can be interpret as a discrete pressure Poisson problem. Note that the distributive smoother from section 6.1 requires an *approximate* solution of the similar problem, cf. (6.6). Below we also consider a smoother closely related to (6.13), which avoids the exact solution of (6.15). Hence consider the block iterative method from [2], which can be seen as a variant of inexact Uzawa method. Let G_k be a preconditioner for $\widehat{\mathrm{S}}_k$ such that

$$\mathrm{G}_k < \widehat{\mathrm{S}}_k \le (1+\beta)\mathrm{G}_k, \qquad \beta > 0 \tag{6.17}$$

One step of the method can be divided in the following three substeps:

$$\omega\mathrm{D}_k(\mathrm{u}^{\mathrm{aux}} - \mathrm{u}^{\mathrm{old}}) = \mathrm{f} - \mathrm{A}_k\mathrm{u}^{\mathrm{old}} - \mathrm{B}_k\mathrm{p}^{\mathrm{old}}, \tag{6.18}$$

$$\mathrm{G}_k(\mathrm{p}^{\mathrm{new}} - \mathrm{p}^{\mathrm{old}}) = \mathrm{B}_k\mathrm{u}^{\mathrm{old}} - \mathrm{g}, \tag{6.19}$$

$$\omega\mathrm{D}_k(\mathrm{u}^{\mathrm{new}} - \mathrm{u}^{\mathrm{aux}}) = -\mathrm{B}_k(\mathrm{p}^{\mathrm{new}} - \mathrm{p}^{\mathrm{old}}). \tag{6.20}$$

The iteration matrix of the method (6.18)–(6.20) is written in the form (4.12) with

$$\mathcal{W}_k = \begin{pmatrix} \omega\mathrm{D}_k & \mathrm{B}_k^T \\ \mathrm{B}_k & \widehat{\mathrm{S}}_k - \mathrm{G}_k \end{pmatrix}.$$

Thus iterations (6.13) can be interpret as (6.18)–(6.20) with exact preconditioner for the "inexact" Schur complement $\widehat{S}_k$ (for the sake of analysis we need a strict low bound in (6.17), however). The smoothing property of (6.18)–(6.20) is based on the following lemma from [31].

Lemma 6.4. *Assume* (6.16) *and* (6.17). *Denote* $\widetilde{\mathcal{D}}_s = \begin{pmatrix} \omega \mathrm{D}_k - \mathrm{A}_k & 0 \\ 0 & \widehat{\mathrm{S}}_k - \mathrm{G}_k \end{pmatrix}$, *then the matrix* $\bar{\mathcal{L}}_k = \widetilde{\mathcal{D}}_s^{\frac12} \mathcal{L}_k \widetilde{\mathcal{D}}_s^{-\frac12}$ *is symmetric and*

$$\operatorname{sp}(\bar{\mathcal{L}}_k) \in [-\beta - \sqrt{\beta^2+\beta}, 1].$$

Moreover the identity $\mathcal{A}_k \mathcal{L}_k^{\nu} = \widetilde{\mathcal{D}}_s^{\frac12} (\mathcal{I}_k - \bar{\mathcal{L}}_k) \bar{\mathcal{L}}_k^{\nu-1} \widetilde{\mathcal{D}}_s^{-\frac12}$ *holds.*

Now lemma 6.4 leads us to the smoothing property for (6.13) and (6.18)–(6.20) which complements the approximation property from corollary 5.2:

Theorem 6.5 (Smoothing property). *Let* $\mathcal{L}_k$ *be the iteration matrix of* (6.18)–(6.20). *Assume* (6.16) *and* (6.17) *with* $\beta < \frac13$, *then*

$$\|\mathcal{A}_k \mathcal{L}_k^{\nu}\|_{\widehat{S}_k \to \widehat{S}_k^{-1}} \lesssim (h_k^{-2} + \alpha) \frac{1}{\nu - 1}, \qquad \nu > 1. \tag{6.21}$$

Proof. Define the auxiliary matrix $\widehat{\mathcal{D}}_s = \begin{pmatrix} \mathrm{I}_k & 0 \\ 0 & \min\{h_k^2, \alpha^{-1}\}\widetilde{\mathrm{S}}_k \end{pmatrix}$, then $\|\cdot\|_{\widehat{S}_k} = \langle \widehat{\mathcal{D}}_s \cdot, \cdot \rangle^{\frac12}$. Thanks to (4.9), (6.16) and (6.17) we obtain $\|\widehat{\mathcal{D}}_s^{-1} \widetilde{\mathcal{D}}_s\| \lesssim h_k^{-2} + \alpha$. Therefore lemma 6.4 and assumption $\beta < \frac13$ yield

$$\begin{aligned} \|\mathcal{A}_k \mathcal{L}_k^{\nu}\|_{\widehat{S}_k \to \widehat{S}_k^{-1}} &= \|\widehat{\mathcal{D}}_s^{-\frac12} \mathcal{A}_k \mathcal{L}_k^{\nu} \widehat{\mathcal{D}}_s^{-\frac12}\| = \|\widehat{\mathcal{D}}_s^{-\frac12} \widetilde{\mathcal{D}}_s^{\frac12} (\mathcal{I}_k - \bar{\mathcal{L}}_k) \bar{\mathcal{L}}_k^{\nu-1} \widetilde{\mathcal{D}}_s^{-\frac12} \widehat{\mathcal{D}}_s^{-\frac12}\| \\ &\le \|\widehat{\mathcal{D}}_s^{-1} \widetilde{\mathcal{D}}_s\| \|(\mathcal{I}_k - \bar{\mathcal{L}}_k) \bar{\mathcal{L}}_k^{\nu-1}\| \lesssim (h_k^{-2} + \alpha) \max_{x \in [-\beta - \sqrt{\beta^2+\beta}, 1]} |(1-x)x^{\nu-1}| \\ &\le \frac{h_k^{-2} + \alpha}{\nu - 1}. \end{aligned}$$

$\square$

By a simple closure argument for the case $\mathrm{G}_k = \mathrm{S}_k$, $\omega = \lambda_{\max}(\mathrm{D}_k^{-1}\mathrm{A}_k)$ we arrive at

Corollary 6.6. *Assume* $\omega \ge \lambda_{\max}(\mathrm{D}_k^{-1}\mathrm{A}_k)$ *and* $\omega \simeq 1$. *Then the iteration matrices* $\mathcal{L}_k$ *of* (6.13) *satisfy the smoothing property* (6.21).

Theorem 6.6, corollary 6.5 together with theorem 5.1 guarantee the uniform convergence estimates for the two-grid method with Braess-Sarazin or inexact Uzawa smoothings. To analyze multigrid convergence we need additional stability property from the theorem below.

Theorem 6.7 (Stability of smoother). *Assume* (6.16) *and* $\mathrm{G}_k \le \mathrm{S}_k \le (1+\beta)\mathrm{G}_k$, *with* $\beta \in [0, \frac13]$, *then it holds*

$$\|\mathcal{L}_k^{\nu}\|_{\widehat{S}_k \to \widehat{S}_k} \lesssim 1. \tag{6.22}$$

Proof. Define the following product norms on $\mathbf{X}_k \times \tilde{\mathbb{Y}}_k$:

$$|||\mathrm{u}, \mathrm{p}||| := \Big(\omega \langle \mathrm{D}_k \mathrm{u}, \mathrm{u} \rangle + \omega^{-1} \langle \widehat{\mathrm{S}}_k \mathrm{p}, \mathrm{p} \rangle\Big)^{\frac12}$$

Due to (4.9) and (6.16) we have $\min\{h_k^2, \alpha^{-1}\}|||\mathrm{u}, \mathrm{p}||| \backsimeq \|\mathrm{u}, \mathrm{p}\|_{\widehat{S}_k}$. This implies

$$\|\mathcal{L}_k^\nu\|_{\widehat{S}_k \to \widehat{S}_k} \backsimeq |||\mathcal{L}_k^\nu||| \tag{6.23}$$

The assumption $\omega \geq \lambda_{\max}(\mathrm{D}_k^{-1}\mathrm{A}_k)$ implies the eigenvalue bound $|\lambda(\mathrm{I}_k - \omega^{-1}\mathrm{D}_k^{-1}\mathrm{A}_k)| < 1$. Now we apply theorem 2.1 from [27] with $\widehat{A} = \omega \mathrm{D}_k$, $\widehat{G} = \mathrm{B}_k \widehat{A}^{-1}\mathrm{B}_k^T$, $\beta = \frac{1}{3}$, $c = 1$ to conclude that the right-hand side of (6.23) is less than 1.
□

7. Multigrid convergence. In this section we prove the convergence result for the multigrid method. The result is based on the approximation and smoothing properties from the previous sections. First, however, we prove the following technical lemma.

LEMMA 7.1. *Let* $\mathbf{p}_k$ *and* $\mathbf{r}_k$ *be the prolongation and restriction operators defined in* (4.11)*. For all* $\{\mathrm{u}, \mathrm{p}\} \in \mathbf{X}_{k-1} \times \tilde{\mathbb{Y}}_k$ *it holds*

$$\|\mathbf{p}_k\{\mathrm{u}, \mathrm{p}\}\|_{S_k} \backsimeq \|\mathrm{u}, \mathrm{p}\|_{S_{k-1}} \quad \text{and} \quad \|\mathbf{p}_k\{\mathrm{u}, \mathrm{p}\}\|_{\widehat{S}_k} \backsimeq \|\mathrm{u}, \mathrm{p}\|_{\widehat{S}_{k-1}}. \tag{7.1}$$

For the case of distributive smoothings it holds

$$\|\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^\nu\|_{S_k \to S_{k-1}} \lesssim 1 \tag{7.2}$$

and for smoothings (6.13) *or* (6.18)–(6.20)

$$\|\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^\nu\|_{\widehat{S}_k \to \widehat{S}_{k-1}} \lesssim 1. \tag{7.3}$$

Proof. For arbitrary $\mathrm{u} \in \mathbf{X}_{k-1}$, $\mathrm{p} \in \tilde{\mathbb{Y}}_{k-1}$ consider $\mathbf{u}_{k-1} = P_{k-1}\mathrm{u} \in \mathbf{V}_{k-1}$ and $p_{k-1} = R_{k-1}\mathrm{p} \in \mathbb{Q}_{k-1}$, then from the definition of $\mathbf{p}_k$ and thanks to (4.1), (4.6) we conclude

$$\begin{aligned} \|\mathrm{u}, \mathrm{p}\|_{S_{k-1}} &\backsimeq \|\mathbf{u}_{k-1}\| + \min\{h_{k-1}, \alpha^{-\frac{1}{2}}\}\|p_{k-1}\|_{Q_{k-1}}, \\ \|\mathbf{p}_k\{\mathrm{u}, \mathrm{p}\}\|_{S_k} &\backsimeq \|\mathbf{u}_{k-1}\| + \min\{h_k, \alpha^{-\frac{1}{2}}\}\|p_{k-1}\|_{Q_k}. \end{aligned} \tag{7.4}$$

With the help of (2.13) we obtain

$$\|p_{k-1}\|_{Q_{k-1}} \backsimeq \|p_{k-1}\|_{Q_k} \tag{7.5}$$

Since $h_k \backsimeq h_{k-1}$ relations (7.4) and (7.5) prove the first relation in (7.1).
Now consider the relations

$$\lim_{\alpha \to \infty} \alpha^{-1}\mathrm{S}_k = \mathrm{B}_k\mathrm{M}_u^{-1}\mathrm{B}_k^T \backsimeq \min\{h_k^2, \alpha^{-1}\}\widehat{\mathrm{S}}_k$$

Thus we prove the second equivalence in (7.1) passing to the limit in the first relation and applying the scaling argument.
Now we are going to prove (7.2). Thanks to (7.1) we have with distributive smoother:

$$\|\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^\nu\|_{S_k \to S_{k-1}} \lesssim \|\mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^\nu\|_{S_k \to S_k}.$$

Now observe the following identity

$$\mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^\nu = (\mathcal{A}_k^{-1} - \mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k)(\mathcal{A}_k\mathcal{L}_k^\nu) - \mathcal{L}_k^\nu.$$

Hence using the approximation, smoothing and stability properties we get

$$\begin{aligned}&\|\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^{\nu}\|_{S_k\to S_{k-1}}\\ &\qquad\le \|(\mathcal{A}_k^{-1}-\mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k)\|_{S_k^{-1}\to S_k}\|(\mathcal{A}_k\mathcal{L}_k^{\nu})\|_{S_k\to S_k^{-1}}+\|\mathcal{L}_k^{\nu}\|_{S_k\to S_k}\lesssim 1\end{aligned}$$

The estimate (7.3) is proved similarly.
□

The iteration matrix of the multigrid W-cycle with ν pre-smoothings satisfies the recursion

$$\mathcal{M}_0:=0,\quad \mathcal{M}_k=\mathcal{T}_k+\mathbf{p}_k(\mathcal{M}_{k-1})^2\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k\mathcal{L}_k^{\nu}\ , \tag{7.6}$$

where $\mathcal{T}_k=(\mathcal{I}_k-\mathbf{p}_k\mathcal{A}_{k-1}^{-1}\mathbf{r}_k\mathcal{A}_k)\mathcal{L}_k^{\nu}$ is the iteration matrix of the two-grid method. Approximation and smoothing properties yield the estimates

$$\|\mathcal{T}_k\|_{S_k\to S_k}\lesssim\frac{1}{\sqrt{2\nu+1}}, \tag{7.7}$$

if distributive smoothings are used and

$$\|\mathcal{T}_k\|_{\widehat{S}_k\to\widehat{S}_k}\lesssim\frac{1}{\nu-1},\qquad \nu>1 \tag{7.8}$$

if smoothings (6.13) or (6.18)–(6.20) are used.

THEOREM 7.2. *Assume that the number of smoothing steps on every grid level is sufficiently large, but independent of all relevant parameters. Then for the contraction number of the multigrid W-cycle with distributive smoothings the inequality*

$$\|\mathcal{M}_k\|_{S_k\to S_k}\le\xi^*\ ,\quad k>0\ ,$$

holds with a constant $\xi^*<1$ independent of k and α. For the contraction number of the multigrid W-cycle with smoothings (6.13) *or* (6.18)–(6.20) *the inequality*

$$\|\mathcal{M}_k\|_{\widehat{S}_k\to\widehat{S}_k}\le\xi^*\ ,\quad k>0$$

holds with a constant $\xi^*<1$ independent of k and α.

Proof. Consider the W-cycle with distributive smoothings. Define $\xi_k:=\|\mathcal{M}_k\|_{S_k\to S_k}$. Using the recursion relation (7.6) for $\mathcal{M}_k$ and (7.1), (7.2) it follows that

$$\begin{aligned}\xi_k&\le\|\mathcal{T}_k\|_{S_k\to S_k}+\|\mathbf{p}_k\|_{S_k\to S_k}\|\mathcal{A}_k\mathbf{r}_k\mathcal{A}_{k-1}^{-1}\mathcal{L}^{\nu}\|_{S_k\to S_k}\|\mathcal{M}_{k-1}^{\mathrm{mgm}}\|_{S_k\to S_k}^2\\ &\le\|\mathcal{T}_k\|_{S_k\to S_k}+C\xi_{k-1}^2\end{aligned}$$

with a positive constant $C>0$ independent of all parameters. Now use the two-grid bound given in (7.7) with sufficiently large ν and a fixed point argument. It is clear that the proof of the theorem for the case of smoothings (6.13) or (6.18)–(6.20) is literally the same with the only difference that instead (7.2) and (7.7) one should use (7.3) and (7.8). □

REFERENCES

[1] Arnold, D.N., Brezzi, F., Fortin, M., A stable finite element method for the Stokes equations, *Calcolo*, **21** (1984), pp. 337-344.

[2] Bank, R., Welfert, B.D., Yserentant, H., A class of iterative methods for solving saddle point problems, *Numer. Math.*, **56** (1990), pp. 645-666

[3] Benzi, M., Golub, G.H. Liesen, J., Numerical solution of saddle point problems, *Acta Numerica*, **14** (2005), pp. 1–137.

[4] Bercovier, M., Pironneau, O., Error estimates for finite element method solution of the Stokes problem in primitive variables *Numer. Math.*, **33** (1979), pp. 211–224.

[5] Braess, D., Dahmen, W., A cascadic multigrid algorithm for the Stokes equations, *Numer. Math.*, **82** (1999), pp. 179–191.

[6] Braess, D., Sarazin R., An efficient smoother for the Stokes problem, *Applied Numerical Mathematics*, **23** (1997), pp. 3–19.

[7] Bramble, J.H., Pasciak, J.E., Iterative techniques for time dependent Stokes problems, *Comput. Math. Appl.*, **33** (1997), pp. 13–30.

[8] Brenner, S., A nonconforming multigrid method for the stationary Stokes problem, *Math. Comp.*, **55** (1990), pp. 411–473.

[9] Bramble, J.H., Pasciak, J.E., Steinbach, O., On the stability of the L^2 projection in $H^1(\Omega)$, *Math. Comp.*, **71** (2001), pp. 147–156.

[10] Brezzi, F., Fortin, M., *Mixed and hybrid finite element methods*, New-York: Springer-Verlag, 1991.

[11] Ciarlet, P.G.: Basic Error Estimates for Elliptic Problems. *Handbook of Numerical Analysis*, Vol. II, P.G. Ciarlet and J.L. Lions, eds., Elsevier Science, 1991, pp. 17–351.

[12] Dauge, M., Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with dorners. Part I. Linearized Equations, *SIAM J. Math. Anal.*, **20** (1989), pp. 74–97

[13] Girault, V., Raviart, P.-A., *Finite Element Methods for Navier-Stokes Equations,* Springer, Berlin, 1986.

[14] Hackbusch, W., *Multi-grid Methods and Applications*, Berlin, Heidelberg: Springer, 1985.

[15] Iliash, Yu., Tuomo, R., Toivanen, J.: Two iterative methods for solving the Stokes problem, Laboratory of Scientific Computing, University of Jyvaskyla, Report 2, 1993.

[16] Kobelkov, G.M., Olshanskii, M.A., Effective Preconditioning of Uzawa Type Schemes for Generalized Stokes Problem, *Numer. Math.*, **86** (2000), pp. 443–470

[17] Larin, M., Reusken A., A comparative study of efficient iterative solvers for generalized Stokes problem, *Numer. Linear Algebra Appl.*, **15** (2008), pp. 13–34.

[18] Manservisi, S., Numerical Analysis of Vanka-Type Solvers for Steady Stokes and NavierStokes Flows, *SIAM J. Numer. Anal.* **44** (2006), pp. 2025–2056

[19] Mardal, K.-A., Winther, R.: Uniform preconditioners for the time dependent Stokes problem, *Numer. Math.* **98** (2004), pp. 305–327

[20] Olshanskii, M. A., Reusken A., On the convergence of a multigrid method for linear reaction-diffusion problems, *Computing*, **65** (2000), pp. 193–202

[21] Olshanskii, M. A., Peters, J., Reusken A., Uniform preconditioners for a parameter dependent saddle point problem with application to generalized Stokes interface equations, *Numer. Math.*, **105** (2006), pp. 159–191.

[22] Shaidurov V.V., *Multigrid methods for finite elements,* Moscow: Nauka, 1989. (in Russian)

[23] Schoeberl, J., Zulehner, W., On Schwarz-type Smoothers for Saddle Point Problems, *Numer. Math.*, **95** (2002), pp. 377–399.

[24] Turek, S., *Efficient Solvers for Incompressible Flow Problems. An Algorithmic and Computational Approach*, Lecture Notes in Computational Science and Engineering, Vol. 6, Springer-Verlag, Berlin/Heidelberg/New York, 1999.

[25] Silvester, D., Wathen, A, Fast iterative solution of stabilised stokes systems. part II: using general block preconditioners. *SIAM J. Numer. Anal.*, **31** (1994), pp. 1352–1367.

[26] Stevenson, R., Nonconforming finite elements and the cascadic multi-grid method, *Numer. Math.*, **91** (2002), pp. 351–387.

[27] Tong, Z., Sameh, Z., On an iterative method for saddle point problems, *Numer. Math.*, **79** (1998), pp. 643-646.

[28] Vanka, S.P., Block-implicit multigrid solution of Navier-Stokes equations in primitive variables, *J. Comput. Phys.*, **65** (1986), pp. 138-158.

[29] Verfürth, R., A multilevel algorithm for mixed problems, *SIAM J. Num. Anal.*, **21** (1984), pp. 264–271.

[30] Wesseling, P., Oosterlee, C.W., Geometric multigrid with applications to computational fluid dynamics, *J. Comput. Appl. Math.*, **128** (2001) pp. 311–334.

[31] Zulehner, W., A class of smoothers for saddle point problems, *Computing*, **65** (2000), pp. 227–246.