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A Note on the Consistency and Stability Properties of Yosida Fractional Step Schemes for the Unsteady Stokes Equations

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A NOTE ON THE CONSISTENCY AND STABILITY PROPERTIES OF YOSIDA FRACTIONAL STEP SCHEMES FOR THE UNSTEADY STOKES EQUATIONS

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Abstract. In this short note we give some improvements to the stability and consistency analysis of algebraic splitting methods for solving the incompressible fluid equations carried out in [1]. In particular, we generalize to any value of the approximation index $p \in \mathbb{N}$ Proposition 3.1 and Lemma 3.2 of [1] on the consistency error associated with the algebraic splitting, originally proved for $p = 0, 1, 2$. Moreover, we remove an assumption of Lemma 3.5 of [1].

Key words. Unsteady Stokes Equations, Fractional Step Methods, Stability

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1. Introduction. Yosida schemes are a family of methods for solving the primitive variables formulation of incompressible Stokes and Navier-Stokes equations by splitting the computation of velocity and pressure. The basic idea (see [4, 3]) stems from an algebraic inexact LU factorization of the matrix associated with the full discretization (in space and time) of the problem. The algebraic nature of the method has the advantage of versatility in implementing mixed boundary values problems. Convergence analysis for the first order method in this class has been carried out in [3]. Successively, a more accurate (in time) scheme has been proposed in [6], with the introduction of an appropriate *pressure correction step*. In [2] a further improvement has been proposed. We recall basic features of these methods in Sect. 1.1. An interesting nested formulation of these methods prone to time adaptivity is pointed out in Sect. 2.1. Convergence analysis of these methods coupled with spectral element space discretization has been recently carried out by P. Gervasio in [1], proving a time accuracy of order $p+3/2$ for the velocity and $p+1$ for the pressure being $p = 0, 1, 2$ a parameter selecting different methods of the family. The aim of this short note is to improve these results provided, by generalizing some properties of the method to any integer value of the parameter p (Sect. 2.2). Moreover, in the stability analysis carried out by P. Gervasio an assumption is postulated on the basis of numerical evidence. Here we give a rigorous proof and complete the stability analysis accordingly (Sect. 3).

1.1. Yosida schemes. Time and space discretizations of unsteady Stokes equations (or Navier-Stokes with a Lagrangian formulation of the time-derivative) lead to solve at each time step systems in the form

$$\mathcal{A}\mathbf{W}^{n+1} = \mathbf{G}^{n+1} \tag{1.1}$$

with $\mathbf{W}^{n+1,T} \equiv \left[\mathbf{U}^{n+1}, \mathbf{P}^{n+1}\right]^T$ and

$$\mathcal{A} \equiv \begin{bmatrix} \mathrm{C} & \mathrm{B}^T \\ \mathrm{B} & 0 \end{bmatrix}, \mathbf{G}^{n+1} = \begin{bmatrix} \mathbf{F}_1^{n+1} \\ \mathbf{F}_2^{n+1} \end{bmatrix}. \tag{1.2}$$

Here $\mathbf{U}^k$ (resp. $\mathbf{P}^k$) is the vector of the discrete velocity (resp. pressure) at time t^k, whose entries are typically the velocity (resp. pressure) nodal values. Vector $\mathbf{F}_1^k$ collects the contribution of boundary data, the forcing term and the time discretization,

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while $\mathbf{F}_2^k$ depends on the boundary data. Matrix B stems from the space discretization of the operator $-\nabla\cdot(\cdot)$. If M denotes the mass matrix, K the stiffness matrix associated with the Laplace operator then $\mathrm{C}\equiv\dfrac{\beta_{-1}}{\Delta t}\mathrm{M}+\nu\mathrm{K}$, where β_{-1} (following the notation of [1]) is the coefficient associated with the step $n+1$ in the discretization of the velocity time derivative and ν is the kinematic viscosity of the fluid. We assume that the space discretization is such that B is a full-rank matrix. As it is well known, matrix $\mathcal{A}$ is indefinite and typically large. For this reason it is often worth to solve system (1.1) by splitting it into a sequence of smaller problems. This can be done following different strategies, see e.g. [5]. In particular, Yosida schemes rely on an inexact LU block factorization of matrix $\mathcal{A}$, basically with the aim of splitting velocity and pressure computations. This can be done by exploiting the following factorization

$$\mathcal{A}=\begin{bmatrix}\mathrm{C}&0\\ \mathrm{B}&\Sigma\end{bmatrix}\begin{bmatrix}\mathrm{I}&\mathrm{C}^{-1}\mathrm{B}^T\\ 0&\mathrm{I}\end{bmatrix}\approx\begin{bmatrix}\mathrm{C}&0\\ \mathrm{B}&\mathrm{S}\end{bmatrix}\begin{bmatrix}\mathrm{I}&\mathrm{C}^{-1}\mathrm{B}^T\\ 0&\mathrm{Q}\end{bmatrix}\equiv\widehat{\mathcal{A}}. \tag{1.3}$$

Here I stands for the identity matrix (whose dimensions can be different in the different occurrences, as can be inferred by the context), $\Sigma\equiv-\mathrm{B}\mathrm{C}^{-1}\mathrm{B}^T$ is the Schur complement of $\mathcal{A}$, S is an approximation of the Σ. In particular, denoting $\mathrm{H}\equiv\dfrac{\Delta t}{\beta_{-1}}\mathrm{M}^{-1}$, we set $\mathrm{S}\equiv-\mathrm{B}\mathrm{H}\mathrm{B}^T$. Finally, matrix Q is associated with the so-called *pressure correction step* (see [6]). When solving system (1.1) replacing $\mathcal{A}$ with $\widehat{\mathcal{A}}$, we introduce a splitting error that is however confined to the continuity equation, as it is highlighted by the aggregate form of $\widehat{\mathcal{A}}$

$$\widehat{\mathcal{A}}=\begin{bmatrix}\mathrm{C}&0\\ \mathrm{B}&\mathrm{S}\end{bmatrix}\begin{bmatrix}\mathrm{I}&\mathrm{C}^{-1}\mathrm{B}^T\\ 0&\mathrm{Q}\end{bmatrix}=\begin{bmatrix}\mathrm{C}&\mathrm{B}^T\\ \mathrm{B}&\mathrm{S}\mathrm{Q}-\Sigma\end{bmatrix}. \tag{1.4}$$

An exact factorization would be yielded by the choice $\mathrm{Q}=\mathrm{S}^{-1}\Sigma\equiv\mathrm{Q}_{ex}$. This choice is however unfeasible, since it still requires the solution of the Schur complement Σ. Yosida pressure corrected schemes advocated in [6] and investigated in [2] and [1] stem from the selection of $\mathrm{Q}=\widehat{\mathrm{Q}}_p$ being p a parameter related to the accuracy of the method, such that $||\Sigma-\mathrm{S}\mathrm{Q}||=\mathcal{O}(\Delta t^{f(p)})$ where $f(p)$ is a proper function of p.

1.1.1. Selection of Q**.** As done in [1], we denote $\mathrm{D}_k\equiv\mathrm{B}\left(-\nu\mathrm{H}\mathrm{K}\right)^k\mathrm{H}\mathrm{B}^T$, $k\geq0$. It is possible to verify that $||\mathrm{D}_k||=\mathcal{O}(\Delta t^{k+1})$. In particular $\mathrm{S}=-\mathrm{D}_0$ and

$$\Sigma=-\sum_{k\geq0}\mathrm{D}_k\Rightarrow\Sigma^{-1}=\left(-\sum_{k\geq0}\mathrm{D}_k\right)^{-1}=\left(\mathrm{I}-\mathrm{S}^{-1}\sum_{k\geq1}\mathrm{D}_k\right)^{-1}\mathrm{S}^{-1}. \tag{1.5}$$

Still following notation introduced in [1], we set $\mathrm{R}=\mathrm{S}^{-1}\sum_{k\geq1}\mathrm{D}_k$. Observe that $||\mathrm{R}||=\mathcal{O}(\Delta t)$. By assuming that Δt is small enough so that $\rho(\mathrm{R})<1$, we have

$$\Sigma^{-1}=\left(\mathrm{I}-\mathrm{R}\right)^{-1}\mathrm{S}^{-1}=\sum_{k\geq0}\mathrm{R}^k\mathrm{S}^{-1}, \tag{1.6}$$

so that $\mathrm{Q}_{ex}^{-1}=\Sigma^{-1}\mathrm{S}=\sum_{k=0}^{p}\mathrm{R}^k+\sum_{k\geq p+1}\mathrm{R}^k=\mathrm{I}+\sum_{k=1}^{p}\mathrm{R}^k+\mathcal{O}(\Delta t^{p+1})$. A natural approximation of Q_{ex}^{-1} arises by neglecting the higher order term (h.o.t.) on the right hand side, by taking $\sum_{k=0}^{p}\mathrm{R}^k$. However, this is still unfeasible since R and its powers are series. It is also non optimal for computational purposes, since R^k even for $k\leq p$

still retains some terms of order higher than p. For the sake of clarity, we explicitly illustrate in the following table the case $p = 3$ (see [1] for more details).

	$\mathcal{O}(\Delta t)$		$\mathcal{O}(\Delta t^2)$		$\mathcal{O}(\Delta t^3)$		h.o.t.
R =	$\mathrm{S}^{-1}\mathrm{D}_1$	+	$\mathrm{S}^{-1}\mathrm{D}_2$	+	$\mathrm{S}^{-1}\mathrm{D}_3$	+	$\mathcal{O}(\Delta t^4)$
R^2 =			$(\mathrm{S}^{-1}\mathrm{D}_1)^2$	+	$\mathrm{S}^{-1}\mathrm{D}_1\mathrm{S}^{-1}\mathrm{D}_2 + \mathrm{S}^{-1}\mathrm{D}_2\mathrm{S}^{-1}\mathrm{D}_1$	+	$\mathcal{O}(\Delta t^4)$
R^3 =					$(\mathrm{S}^{-1}\mathrm{D}_1)^3$	+	$\mathcal{O}(\Delta t^4)$
	$\widetilde{\mathrm{R}}_1$		$\widetilde{\mathrm{R}}_2$		$\widetilde{\mathrm{R}}_3$		

Terms of R^k can be re-arranged on the basis of their dependence on Δt. Following the columnwise notation introduced in the previous table and setting $\widetilde{\mathrm{R}}_0 = \mathrm{R}^0 = \mathrm{I}$, we have $\sum_{k=0}^{p} \widetilde{\mathrm{R}}_k = \sum_{k=0}^{p} \mathrm{R}^k + \text{h.o.t.}$ Formally, a feasible approximation of Q_{ex} is therefore for any $p \in \mathbb{N}$

$$\mathrm{Q}_{ex}^{-1} \approx \widehat{\mathrm{Q}}_p^{-1} = \sum_{k=0}^{p} \widetilde{\mathrm{R}}_k. \tag{1.7}$$

2. Some properties of $\widehat{\mathrm{Q}}_p$ for $p \in \mathbb{N}$.

2.1. Nested pressure correction steps. The specific form of $\widehat{\mathrm{Q}}_p$ deserves to be carefully considered in view of practical implementation. A system in $\widehat{\mathrm{Q}}_p$ is solved as

$$\widehat{\mathrm{Q}}_p \mathbf{P}^{n+1} = \widetilde{\mathbf{P}}^{n+1} \Rightarrow \mathbf{P}^{n+1} = \sum_{k=0}^{p} \widetilde{\mathrm{R}}_k \widetilde{\mathbf{P}}^{n+1}.$$

Denoting $\mathbf{z}_k$ the vectors such that $\widetilde{\mathrm{R}}_k^{-1}\mathbf{z}_k = \widetilde{\mathbf{P}}_{n+1}$, we have $\mathbf{P}^{n+1} = \sum_{k=0}^{p} \mathbf{z}_k$. Solution to the sequence of systems in $\widetilde{\mathrm{R}}_k^{-1}$ can take advantage of the particular structure of the matrices at hand and can be rearranged in a hierarchical way. As a matter of fact, let us consider again the case $p = 3$:

$$\begin{aligned}
&\mathbf{z}_0 = \widetilde{\mathbf{P}}_{n+1} \\
&\mathbf{z}_1 = \mathrm{S}^{-1}\mathrm{D}_1\widetilde{\mathbf{P}}_{n+1} \qquad \Rightarrow \qquad \mathrm{S}\mathbf{z}_1 = \mathrm{D}_1\mathbf{z}_0 \\
&\mathbf{z}_2 = \left(\mathrm{S}^{-1}\mathrm{D}_2 + (\mathrm{S}^{-1}\mathrm{D}_1)^2\right)\widetilde{\mathbf{P}}_{n+1} \qquad \Rightarrow \qquad \mathrm{S}\mathbf{z}_2 = \mathrm{D}_2\mathbf{z}_0 + \mathrm{D}_1\mathbf{z}_1 \\
&\mathbf{z}_3 = \left(\mathrm{S}^{-1}\mathrm{D}_3 + \mathrm{S}^{-1}\mathrm{D}_2\mathrm{S}^{-1}\mathrm{D}_1 + \mathrm{S}^{-1}\mathrm{D}_1\mathrm{S}^{-1}\mathrm{D}_2 + (\mathrm{S}^{-1}\mathrm{D}_1)^3\right)\widetilde{\mathbf{P}}_{n+1} \quad \Rightarrow \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{S}\mathbf{z}_3 = \mathrm{D}_3\mathbf{z}_0 + \mathrm{D}_2\mathbf{z}_1 + \mathrm{D}_1\mathbf{z}_2.
\end{aligned} \tag{2.1}$$

In this framework it is clear that more terms in the approximation of Q_{ex} are retained by solving more systems that however entail the same matrix S. When possible direct methods for solving S should be recommended (for instance in 2D, by exploiting a QR factorization - see [7]). An algorithmic formulation of the method reads as follows (the time step index is understood for the easiness of notation).

```
Solve:  C U = F1; //intermediate velocity
Solve:  S P = F2 - B U; // intermediate pressure
z0 = P; // pressure correction - start
if p>0 then
   Solve S z1 = D z0;
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```
    if p>1 then
      Solve S z2 = D2 z0 + D1 z1;
      if p>2 then
        Solve S z3 = D3 z0 + D2 z1 + D1 z2;
        if p>3 then ...  end;
      end; //if p>2
    end; //if p>1
  end; //if p>0
  for (k=1; k<=p; k++) P=P+zk; end; //pressure correction - end
  Solve C U = F1 - BT P; // end-of-step velocity
```

Let us introduce the following notation.

$$\mathbf{P}_c \equiv \sum_{k=0}^{p-1} \mathbf{z}_k, \quad \mathbf{P}_f \equiv \mathbf{P}_c + \mathbf{z}_p, \quad \mathbf{U}_c \equiv \mathrm{C}^{-1}(\mathbf{F}_1 - \mathrm{B}^T\mathbf{P}_c), \quad \mathbf{y} \equiv -\mathrm{C}^{-1}\mathrm{B}^T\mathbf{z}_p$$

so that we define $\mathbf{U}_f = \mathbf{U}_c + \mathbf{y}$. In view of the convergence analysis carried out in [1], we have that $\mathbf{U}_c$ is a solution with accuracy $p + 1/2$ in time, while $\mathbf{U}_f$ features order $p + 3/2$ (w.r.t. the norms $l^\infty(L^2)$ and $l^2(H^1)$). Correspondingly, $\mathbf{P}_c$ and $\mathbf{P}_f$ have order p and $p + 1$ respectively (w.r.t. the norm $l^2(L^2)$). Vectors $\mathbf{z}_p$ and $\mathbf{y}$ are therefore *a-posteriori* error estimators that can be used for implementing time-adaptive strategies, the former being a by-product of the method, the latter requiring a solution of a system in C. Time adaptivity is an interesting strategy for pursing computational efficiency in specific classes of problems, such as blood flow, featuring the sequence of fast transients (when the aortic valve is open) and slow dynamics. This will be the subject of a forthcoming paper.

2.2. Non-singularity and consistency. Non-singularity of $\widehat{\mathrm{Q}}_p$ that is needed for the mathematical correctness of (1.7) has been investigated in [1] (Lemma 3.1) limitedly to the case $p = 0, 1, 2$. Consistency of the approximation has been investigated in [3], [6] and in Lemma 3.2 of [1]. In the latter reference, consistency analysis is limited to $p = 0, 1, 2$ in the statement of the Lemma, even if the source of limitation is not pointed out in the proof. However, hereafter we give a self-contained consistency analysis holding for any $p \in \mathbb{N}$.

PROPOSITION 2.1. *For any $p \in \mathbb{N}$*

1. *matrices $\widehat{\mathrm{Q}}_p$ are non-singular for Δt small enough;*
2. *consistency error is such that*

$$||\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_p|| = \mathcal{O}(\Delta t^{p+2}), \forall p. \tag{2.2}$$

Proof.

We prove that matrix $\sum_{k=0}^{p} \widetilde{\mathrm{R}}_k$ is non singular. Observe that

$$\widetilde{\mathrm{R}}_0 = \mathrm{I}, \text{ and } ||\widetilde{\mathrm{R}}^k|| = \mathcal{O}(\Delta t^k) \text{ for } k > 0,$$

so that $\sum_{k=0}^{p} \widetilde{\mathrm{R}}_k = \mathrm{I} + \mathcal{O}(\Delta t)$ for $p \geq 1$. This means that the eigenvalues of $\sum_{k=0}^{p} \widetilde{\mathrm{R}}_k$ are in the form $\lambda_i = 1 + c_i \Delta t$ and consequently the determinant shares a similar dependence on Δt. For Δt small enough $\sum_{k=0}^{p} \widetilde{\mathrm{R}}_k$ is therefore non singular. By construction, $\sum_{k=0}^{p} \widetilde{\mathrm{R}}_k = \widehat{\mathrm{Q}}_p^{-1}$ so that first part of the proposition is proved.

For the second part, we refer to the matrices $\mathrm{Q}_p \equiv \mathrm{S}^{-1}\left(-\sum_{k=0}^{p} \mathrm{D}_k\right)$ introduced in [1]. These are still approximations of Q_{ex}, and however their structure does not lend

itself to a nested reformulation as pointed out in the previous paragraph[1] for $\widehat{\mathrm{Q}}_p$. Let us prove that

$$\mathrm{I} - \widehat{\mathrm{Q}}_p^{-1}\mathrm{Q}_p = \mathcal{O}(\Delta t^{p+1}).$$

Observe that, since $\mathrm{D}_0 = -\mathrm{S}$ and $\mathrm{D}_k = \mathcal{O}(\Delta t^{k+1})$, by construction we have

$$\mathrm{R} = \mathrm{S}^{-1}\sum_{k\geq 1}\mathrm{D}_k = \mathrm{S}^{-1}\left(\sum_{k\geq 0}\mathrm{D}_k + \mathrm{S}\right) = \mathrm{S}^{-1}\left(\sum_{k=0}^{p}\mathrm{D}_k + \mathrm{S} + \mathcal{O}(\Delta t^{p+2})\right),$$

yielding $\mathrm{Q}_p = \mathrm{I} - \mathrm{R} + \mathcal{O}(\Delta t^{p+1})$. Moreover, by construction $\widehat{\mathrm{Q}}_p^{-1} = \sum_{k=0}^{p}\widetilde{\mathrm{R}}_k = \sum_{k\geq 0}\mathrm{R}^k + \mathcal{O}(\Delta t^{p+1}) = (\mathrm{I}-\mathrm{R})^{-1} + \mathcal{O}(\Delta t^{p+1})$. We have therefore $\widehat{\mathrm{Q}}_p^{-1}\mathrm{Q}_p = \mathrm{I} + \mathcal{O}(\Delta t^{p+1})$. Since $\widehat{\mathrm{Q}}_p = \mathcal{O}(1)$ and $\mathrm{S} = \mathcal{O}(\Delta t)$, it follows that

$$\mathrm{S}\widehat{\mathrm{Q}}_p - \mathrm{S}\mathrm{Q}_p = \mathrm{S}\widehat{\mathrm{Q}}_p\left(\mathrm{I} - \widehat{\mathrm{Q}}_p^{-1}\mathrm{Q}_p\right) = \mathcal{O}(\Delta t^{p+2}). \tag{2.3}$$

Now we have

$$\mathrm{S}\mathrm{Q}_p - \Sigma = \mathrm{S}\left(\mathrm{Q}_p - \mathrm{Q}_{ex}\right) = \sum_{k\geq p+1}\mathrm{D}_k = \mathcal{O}(\Delta t^{p+2}). \tag{2.4}$$

The thesis follows from (2.3), (2.4) and the triangular inequality $|||\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_p||| \leq |||\Sigma - \mathrm{S}\mathrm{Q}_p||| + |||\mathrm{S}\mathrm{Q}_p - \mathrm{S}\widehat{\mathrm{Q}}_p|||$. □

3. A note on the stability analysis of [1]. As pointed out in [6] and [1], time stability of the Yosida scheme can be affected by the splitting. In particular, is $\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_p$ is negative, then the absolute stability region of the adopted time advancing scheme is reduced. In [3], [6] and (with different arguments) [1] it is proved that $\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_0$ is positive and $\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_1$ is negative. In [1] it is proved that $\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_2$ is positive under the assumption that the matrix $\widetilde{\widetilde{\mathrm{B}}} \equiv -\mathrm{D}_3 - \mathrm{D}_1\mathrm{S}^{-1}\mathrm{D}_2 - \mathrm{D}_2\mathrm{S}^{-1}\mathrm{D}_1 - \mathrm{D}_1(\mathrm{S}^{-1}\mathrm{D}_1)^2$ is s.p.d. Quoting [1]: "...numerical computations have shown that $\widetilde{\widetilde{\mathrm{B}}}$ has real non-negative eigenvalues for any space discretization we have considered." Moreover, non-negativity of the eigenvalues is observed to be independent of Δt in [1]. We provide a formal explanation of this numerical evidence.

LEMMA 3.1. *Matrix $\widetilde{\widetilde{\mathrm{B}}}$ is symmetric positive semidefinite, independently of Δt.*

Proof.

Let us introduce the following matrices

$$\mathrm{Z} \equiv \mathrm{M}^{-1/2}\mathrm{K}\mathrm{M}^{-1/2},\ \mathrm{V} = \mathrm{M}^{-1/2}\mathrm{B}^T,\ \mathrm{G} = \mathrm{V}(\mathrm{V}^T\mathrm{V})^{-1}\mathrm{V}^T.$$

Matrix Z is readily recognized to be s.p.d. By exploiting the QR factorization of the rectangular matrix V, we have $\mathrm{G} = \mathrm{U}\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix}\mathrm{U}^T$ where U is orthogonal. Notice that $\mathrm{I} - \mathrm{G}$ can be written

$$\mathrm{I} - \mathrm{G} = \mathrm{U}\left(\begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} - \begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix}\right)\mathrm{U}^T = \mathrm{U}\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{bmatrix}\mathrm{U}^T,$$

[1] Another reason for preferring approximations given by $\widehat{\mathrm{Q}}_p$ refers to the constants driving the consistency error as pointed out in [1]

which is clearly positive semidefinite. With this notation we have $\mathrm{D}_k = \gamma_k \mathrm{V}^T \mathrm{Z}^k \mathrm{V}$ with $\gamma_k = (-\nu)^k (\Delta t/\beta_{-1})^{k+1}$. In particular, notice that $\gamma_3 = \gamma_2\gamma_1/\gamma_0 = \gamma_1^3/\gamma_0^2$ so that we can write

$$\begin{aligned}\widetilde{\widetilde{\mathrm{B}}} &= |\gamma_3| \left(\mathrm{V}^T\mathrm{Z}^3\mathrm{V} + \mathrm{V}^T\mathrm{ZGZGZV} - \mathrm{V}^T\mathrm{ZGZ}^2\mathrm{V} - \mathrm{V}^T\mathrm{Z}^2\mathrm{GZV}\right) = \\ &|\gamma_3| \mathrm{V}^T\mathrm{Z}\left(\mathrm{I}-\mathrm{G}\right)\mathrm{Z}\left(\mathrm{I}-\mathrm{G}\right)\mathrm{ZV}.\end{aligned} \tag{3.1}$$

Matrix $\mathrm{T} \equiv (\mathrm{I}-\mathrm{G})\mathrm{Z}(\mathrm{I}-\mathrm{G})$ is symmetric semidefinite positive and the thesis is therefore proved. □

Unfortunately, the previous result inhibits the use of Lemma 3.5 of [1], which is strongly based on the assumption of strict positivity of $\widetilde{\widetilde{\mathrm{B}}}$. However, conclusion about stability of the method can be still recovered with different arguments.

PROPOSITION 3.2. *If Δt is small enough, then $\mathrm{A} \equiv \Sigma - \mathrm{S}\widehat{\mathrm{Q}}_2$ is symmetric positive semidefinite.*

Proof. We rewrite $\Sigma - \mathrm{S}\widehat{\mathrm{Q}}_2 = -\mathrm{S}\widehat{\mathrm{Q}}_2 \left(\mathrm{I} - \widehat{\mathrm{Q}}_2^{-1}\mathrm{S}^{-1}\sum_{k\geq 0}\mathrm{D}_k\right)$. Matrix $-\mathrm{S}\widehat{\mathrm{Q}}_2$ is proved to be s.d.p. for Δt small enough in [1] (that implies that A is symmetric). The sign of A is therefore driven by the term into brackets. Bearing in mind that $\widehat{\mathrm{Q}}_2^{-1} = \mathrm{I} + \mathrm{S}^{-1}\mathrm{D}_1 + \mathrm{S}^{-1}\mathrm{D}_2 + \left(\mathrm{S}^{-1}\mathrm{D}_1\right)^2$, by a direct computation we get

$$\mathrm{I} - \widehat{\mathrm{Q}}_2^{-1}\mathrm{S}^{-1}\sum_{k\geq 0}\mathrm{D}_k = -\mathrm{S}^{-1}\left(\widetilde{\widetilde{\mathrm{B}}} + \widetilde{\widetilde{\mathrm{C}}}\right)$$

where $\widetilde{\widetilde{\mathrm{C}}} = -\mathrm{D}_4 - \mathrm{D}_1\mathrm{S}^{-1}\mathrm{D}_3 - \left(\mathrm{DS}_1^{-1}\right)^2\mathrm{D}_2 - \mathrm{D}_2\mathrm{S}^{-1}\mathrm{D}_2 + \text{h.o.t.}$. Again, since $-\mathrm{S}^{-1}$ is s.p.d. the sign of the matrix is driven by the matrix $\widetilde{\widetilde{\mathrm{B}}} + \widetilde{\widetilde{\mathrm{C}}}$. With computations similar to the ones carried out in the previous Lemma 3.1 we obtain

$$\widetilde{\widetilde{\mathrm{B}}} + \widetilde{\widetilde{\mathrm{C}}} = |\gamma_3|\mathrm{V}^T\mathrm{ZT}\left(\mathrm{I} - \frac{|\gamma_4|}{|\gamma_3|}\mathrm{Z} + \text{h.o.t.}\right)\mathrm{ZV}.$$

Matrix T is symmetric positive semidefinite, as proved in Lemma 3.1. Since $\gamma_4/\gamma_3 = \mathcal{O}(\Delta t)$, then for Δt small enough the bracketed term is positive and the thesis is proved. □

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