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A NEW APPROACH TO NUMERICAL SOLUTION OF DEFECTIVE BOUNDARY VALUE PROBLEMS IN INCOMPRESSIBLE FLUID DYNAMICS*

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Abstract. We consider the incompressible Navier-Stokes equations where on a part of the boundary flow rate and mean pressure boundary conditions are prescribed. There are basically two strategies for solving these defective boundary problems: the *variational approach* (see *J. Heywood, R. Rannacher, S. Turek, Int J Num Meth Fluids 22 (1996), pp. 325-352*) and the *augmented formulation* (see *L. Formaggia, J. F. Gerbeau, F. Nobile, A. Quarteroni, SIAM J Num Anal, 40-1 (2002), pp. 376–401,* and *A. Veneziani, C. Vergara, Int J Num Meth Fluids, 47 (2005), pp. 803–816*). However, both these approaches present some drawbacks. For the flow rate problem, the former resorts to non standard functional spaces, which are quite difficult to discretize. On the other hand, for the mean pressure problem, it yields exact solutions only in very special cases. The latter is applicable only to the flow rate problem, since for the mean pressure problem it provides unfeasible boundary conditions.

In this paper, we propose a new strategy, based on a reformulation of the problems at hand in terms of the minimization of an appropriate functional. This approach allows to treat the two kind of problems (flow rate and mean pressure) successfully within the same framework, that can be useful in view of mixed problem where the two conditions are simultaneously prescribed on different artificial boundaries. Moreover, it is more versatile, being prone to be extended to other kind of defective conditions. We analyze the problems obtained with this approach and propose some algorithms for their numerical solution. Several numerical results are presented supporting the effectiveness of our approach.

Key words. Navier-Stokes equations, flow rate boundary condition, mean pressure boundary condition, computational haemodynamics.

AMS subject classifications. 65N99

1. Introduction.

In computational fluid-dynamics the available boundary data are sometimes incomplete, in particular with reference to the *artificial boundaries*, i.e. the boundaries created just to limit the computational domain and not corresponding to a physical interface. On these boundaries, measurements or results coming from other computations (like in the *geometrical multiscale approach* - see [5, 14]) typically provide mean or, more in general, incomplete data. This kind of problems has been addressed in [10] and in [4, 16, 17, 18]. In particular, denoting with Γ an artificial portion of the boundary, two conditions have been considered on Γ, namely *the mean pressure problem*

$$\frac{1}{|\Gamma|}\int_{\Gamma} p \, d\gamma = P(t), \tag{1.1}$$

where $p = p(t, \boldsymbol{x})$ is the fluid pressure and $P = P(t)$ is a given function of time, and the *flow rate problem*, where we have

$$\rho \int_{\Gamma} \boldsymbol{u} \cdot \boldsymbol{n} \, d\gamma = Q(t), \tag{1.2}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ is the fluid velocity, $\boldsymbol{n}$ the outward unit vector, ρ the constant fluid density and $Q = Q(t)$ is a given function.

In [10] a variational approach is proposed for finding an appropriate numerical solution of these problems. A well posedness analysis of the resulting boundary value problems is given too. In [13] the same conditions on fluxes and pressures are considered as *asymptotic conditions* on infinite domains. In this approach, the artificial boundaries are replaced by far field conditions to

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be prescribed in an infinite domain. The analysis of this problem is by far not trivial, and however in [13] it is proved that flux and mean pressure conditions regarded as asymptotic conditions under appropriate assumptions yield well posed problems. In practical problems, however, the computational domain needs to be truncated, so an appropriate method for prescribing such conditions in finite domains is needed.

The variational approach advocated in [10] for the mean pressure problem gives reliable results only in the case of special geometries. On the other hand, in the case of flow rate problems, the proposed variational formulation of [10] resorts to non standard functional spaces. The construction of finite-dimensional subspaces is actually quite problematic. For this reason, a different approach is proposed in [4], based on an augmented formulation, where condition (1.2) is regarded as a constraint for the solution to be forced by means of a Lagrange multiplier (see also [16]). Unfortunately, the technique cannot treat the mean pressure problem, as it yields inaccurate solutions for the velocity (see [4, 18]).

In this paper, we propose a new general approach for the defective boundary data problems, which does not suffer from the limitations of the previous ones, so that both mean pressure and flow rates problems can be solved successfully within the same framework. This approach relies on the introduction of an appropriate functional which quantifies the difference between the boundary solution and the prescribed data (see, in a different context, [1]). Resorting to a technique similar to that of a control problem, we introduce control variables related to the fluid velocity and pressure on the boundary. Acting on these control variables, we seek the minimum of the functional at hand. This approach is general and can be applied in principle to all kind of defective boundary problems. The idea of using boundary data as control variables to force some properties of the solution has been already exploited: we quote for instance [8, 11, 2] and more recently [3]. In [7] it has been used for an effective splitting of velocity-pressure computation in standard unsteady Stokes problems. Boundary control variables were used here to force the incompressibility constraint. In general, in these papers the basic idea is to use point-wise velocity data on the boundary for driving the solution to some desired behavior. Here, we still use boundary data (not necessarily the velocity) for forcing the "optimal" fulfillment of the incomplete boundary conditions. We consider both the flow rate and the mean pressure problem. We discuss the effectiveness of this approach in comparison with the variational approach proposed in [10] and with the augmented formulation addressed in [4, 16]. Numerical results presented here confirm the flexibility of our method.

The outline of the paper is as follows. Section 2 is devoted to the flow rate problem. The approach based on using normal stresses as control variables is first presented for the generalized Stokes problem and then extended to the nonlinear case. For both cases a proof of well-posedness is given. In Section 3 we illustrate the approach for the mean pressure problem. We propose different formulations, featuring different sets of control variables. Well posedness analysis is in this case non trivial and will be carried out in future works, yet the effectiveness of the methods is demonstrated here by numerical tests. Section 4 is devoted to the presentation of some algorithms for the solution of the proposed problems and to their convergence analysis. Finally, in Section 5 we present and discuss some numerical results.

1.1. Basic notation. Let us denote by $\Omega \in \mathbb{R}^d$ a domain filled by an incompressible fluid, whose boundary can be split into two parts. The former is denoted by Γ_w and corresponds to a physical wall. The latter is given by the union of the artificial sections Γ_j $(j = 0, \dots, m)$ which limit the domain of interest, but do not have a direct physical significance. A possible domain of this sort is given in Figure 1.1. We assume that the walls are rigid, so that the velocity field is zero on Γ_w, and that the fluid is Newtonian. This domain is representative of incompressible flow problems in network of pipes, like in computational haemodynamics, that motivated the present study. The extension to the case of compliant vessels, which is of paramount interest for blood flow problems, requires a specific analysis still to be carried out (see [6]). The Navier-Stokes equations for the problem at hand are:

$$\begin{cases} \rho\dfrac{\partial \boldsymbol{u}}{\partial t} - \mu\triangle\boldsymbol{u} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } (0,T)\times\Omega, \end{cases} \tag{1.3}$$

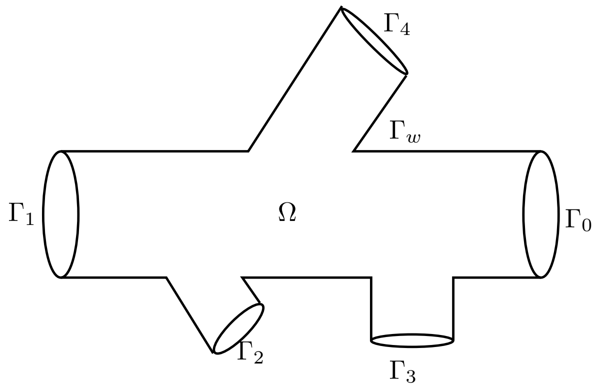

FIGURE 1.1. *Reference domain* Ω

where $\boldsymbol{f}(t,\boldsymbol{x})$ is a given forcing term and μ is the constant fluid viscosity. In the sequel, for the sake of simplicity and without loss of generality we will set $\rho = 1$. Moreover, we have *no-slip* boundary conditions on Γ_w, i.e.

$$\boldsymbol{u}|_{\Gamma_w} = \boldsymbol{0} \qquad \text{in } (0,T), \tag{1.4}$$

as well as the initial condition

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \qquad \text{in } \Omega, \tag{1.5}$$

where $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ is a given function, regular enough. Boundary conditions still have to specified on Γ_j. On those boundaries, we will consider the two possibilities addressed in the Introduction.

In the sequel we will refer to the functional spaces

$$L^2(\Omega) = \Big\{ v : \Omega \to \mathbb{R} : \int_\Omega v^2\, d\omega < \infty \Big\}, \qquad H^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \},$$
$$\boldsymbol{V} = \{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_{\Gamma_w} = \boldsymbol{0} \},$$

where $\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^d$ and use the following notation for scalar functions $s,\, q \in H^1(\Omega)$ and for vector functions $\boldsymbol{v}\,, \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$:

$$(s,q) = \int_\Omega s\, q\, d\omega, \quad (\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}\, d\omega = \sum_{i=1}^d \int_\Omega v_i w_i\, d\omega, \quad (s,q)_{H^1} = (s,q) + (\nabla s, \nabla q),$$

$$(\nabla\boldsymbol{v},\nabla\boldsymbol{w}) = \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w}\, d\omega = \sum_{i,j=1}^d \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\, d\omega, \quad (\boldsymbol{v},\boldsymbol{w})_{H^1} = (\boldsymbol{v},\boldsymbol{w}) + (\nabla\boldsymbol{v},\nabla\boldsymbol{w}),$$

$$a(\boldsymbol{v},\boldsymbol{w}) = \alpha(\boldsymbol{v},\boldsymbol{w}) + \mu(\nabla\boldsymbol{v},\nabla\boldsymbol{w}), \qquad c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}),$$

being $\alpha \geq 0$ a given parameter. We recall that $a(\cdot,\cdot)$ is coercive on the space $\boldsymbol{V}$, $\forall \alpha \geq 0$. Moreover, given a vector $\boldsymbol{q} \in \mathbb{R}^m$, we set

$$\|\boldsymbol{q}\|_p = \Big(\sum_{j=1}^m |q_j|^p \Big)^{1/p}.$$

Finally, given m Hilbert spaces $W_1, \ldots, W_m$, let $\boldsymbol{W} \equiv W_1 \times W_2 \times \ldots \times W_m$, $N : \boldsymbol{W} \to \mathbb{R}$, such that $(y_1,\ldots,y_m) \in \boldsymbol{W} \to N(y_1,\ldots,y_m) \in \mathbb{R}$ and $< \cdot,\cdot >$ the duality pairing between W' and $\boldsymbol{W}$. We denote

$$< dN_{y_j}[w_1,\ldots,w_m], z >= \lim_{\varepsilon\to 0} \Big(\frac{N(y_1,\ldots,y_j+\varepsilon z,\ldots,y_m) - N(y_1,\ldots,y_j,\ldots,y_m)}{\varepsilon} \Big)\Big|_{\boldsymbol{y}=\boldsymbol{w}}$$

the Gateaux differential of N, with the respect of y_j computed in the point $\boldsymbol{w} = (w_1,\ldots,w_m) \in \boldsymbol{W}$ along the direction $z \in W_j$.

2. Flow rate boundary conditions. We start by considering the following boundary conditions applied to the artificial boundaries Γ_i:

$$\begin{cases} (p\boldsymbol{n} - \mu\nabla\boldsymbol{u}\,\boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0}, & \text{in } (0,T), \\ \displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma = Q_i, \qquad i = 1,2,\ldots,m, & \text{in } (0,T), \end{cases} \tag{2.1}$$

where $Q_i = Q_i(t)$ are given functions of time. Since Γ_w is a rigid boundary, the incompressibility constraint implies that the flow rate on Γ_0 is $Q_0 = -\sum_{i=1}^{m} Q_i$. Conditions $(2.1)_2$ are not enough to make the problem at hand well posed in the classical setting, since d "point-wise" boundary data should be prescribed. The approach proposed in [10] to overcome this lack of data is based on the introduction of a variational formulation in the "null fluxes" subspace of $\boldsymbol{V}$ and by lifting the solution by means of an appropriate set of functions called *flux-carriers*. It is possible to prove (see e.g. [15]) that the boundary conditions implicitly prescribed on Γ_i by this formulation are constant-in space Neumann boundary conditions. From the numerical viewpoint this has the drawback of dealing with a non standard functional space, whose finite dimensional discretization is not easy to construct. For this reason, a different approach, based on Lagrange multipliers and on the reformulation of the problem in terms of a constraint minimization, has been investigated and numerically tested in [4] for the steady Stokes problem and then extended to the unsteady Navier-Stokes problem in [16, 17, 18]. The idea is to regard the flow rate boundary condition as a constraint to be forced by means of Lagrange multipliers. The *augmented variational formulation* associated with this approach leads to the following problem.

PROBLEM 1. *Given $\boldsymbol{u}_0 \in \boldsymbol{V}$, with $\nabla\cdot\boldsymbol{u}_0 = 0$, $\boldsymbol{f} \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ and a vector of functions $\boldsymbol{Q} \in (C^0([0,T]))^m$, satisfying the conditions $\int_{\Gamma_j} \boldsymbol{u}_0\cdot\boldsymbol{n}\,d\gamma = Q_j(0)$ for $j = 1,\ldots,m$, find $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, $p \in L^2(0,T;L^2(\Omega))$ and $\boldsymbol{\lambda} \in (L^2(0,T))^m$ such that at any $t \in (0,T)$*

$$\begin{cases} \left(\dfrac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + \mu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{j=1}^{m}\lambda_j\int_{\Gamma_j}\boldsymbol{v}\cdot\boldsymbol{n}\,d\gamma = (\boldsymbol{f},\boldsymbol{v}) & \text{in } \Omega, \\ (q,\nabla\cdot\boldsymbol{u}) = 0 & \text{in } \Omega, \\ \displaystyle\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma = Q_i, & i = 1,\ldots,m, \end{cases}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in L^2(\Omega)$ and with $\boldsymbol{u} = \boldsymbol{u}_0$ for $t = 0$. It has been proven in [4, 16] that this approach leads to the same natural boundary conditions on the artificial boundaries as the variational approach presented in [10]. However, in this case we deal with the standard functional space $\boldsymbol{V}$.

In the next section we show a different variational formulation, still prompt to numerical discretization. For the sake of simplicity we start considering the generalized Stokes problem. Then, in Section 2.2, we extend the approach to the non linear case.

2.1. The generalized Stokes problem. In order to address conditions $(2.1)_2$, let us consider the following generalized Stokes problem with complete (i.e. non defective) boundary conditions.

$$\begin{cases} \alpha\boldsymbol{u}(\boldsymbol{k}) - \mu\triangle\boldsymbol{u}(\boldsymbol{k}) + \nabla p(\boldsymbol{k}) = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}(\boldsymbol{k}) = 0 & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{k})|_{\Gamma_w} = \boldsymbol{0}, & \\ (-p(\boldsymbol{k})\boldsymbol{n} + \mu\nabla\boldsymbol{u}(\boldsymbol{k})\,\boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0}, & \\ (-p(\boldsymbol{k})\boldsymbol{n} + \mu\nabla\boldsymbol{u}(\boldsymbol{k})\,\boldsymbol{n})|_{\Gamma_i} = -k_i\boldsymbol{n}, & i = 1,\ldots,m, \end{cases} \tag{2.2}$$

where $\boldsymbol{k} = (k_1,\ldots,k_m) \in \mathbb{R}^m$ denotes the vector of the constant values of the normal stresses, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is given and $\alpha \geq 0$ is a given parameter. For the solution of unsteady problems, α is related to the time step and to the time advancing scheme (see Sect. 2.1.1). As our notation suggests, we regard velocity and pressure fields as a function of the vector $\boldsymbol{k}$. More precisely, for any given $\boldsymbol{k} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{k})$ and $p(\boldsymbol{k})$ denote the velocity and the pressure field obtained by solving (2.2). In this context, we consider $\boldsymbol{k}$ as *control variable*, to be set such that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{k})$ fulfills the constraint $(2.1)_2$ in some sense to be specified. To this aim we introduce the functional $J_Q : \boldsymbol{V} \to \mathbb{R}^+$

$$J_Q(\boldsymbol{w}) = \frac{1}{2}\sum_{i=1}^{m}\Big(\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)^2, \tag{2.3}$$

which is clearly minimal and equal to zero when $(2.1)_2$ is fulfilled. For each $\boldsymbol{w} \in \boldsymbol{V}$, $s \in L^2(\Omega)$ and $\boldsymbol{\eta} \in \mathbb{R}^m$, we build the following Lagrange functional, where equations (2.2) play the role of *constraints* for the solution:

$$\mathcal{L}(\boldsymbol{w}, s; \boldsymbol{\lambda}_w, \lambda_s; \boldsymbol{\eta}) = J_Q(\boldsymbol{w}) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) - (s, \nabla\cdot\boldsymbol{\lambda}_w) + \sum_{i=1}^{m}\int_{\Gamma_i} \eta_i \boldsymbol{\lambda}_w\cdot\boldsymbol{n}\,d\gamma - (\boldsymbol{f}, \boldsymbol{\lambda}_w) - (\lambda_s, \nabla\cdot\boldsymbol{w}). \tag{2.4}$$

Here, $\boldsymbol{\lambda}_w \in \boldsymbol{V}$ and $\lambda_s \in L^2(\Omega)$ are the *adjoint* variables associated to $\boldsymbol{w}$ and s respectively. In order to find the corresponding Euler equations, we force the Gateaux differentials of $\mathcal{L}$ evaluated for any test function to vanish in correspondence of the solution $\boldsymbol{\chi} = [\boldsymbol{u}, p; \boldsymbol{\lambda}_u, \lambda_p; \boldsymbol{k}]$. That is, we will consider the following problem, where for the sake of notation we omit to specify that the differentials are computed in $\boldsymbol{\chi}$:

Problem 2. *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{Q} \in \mathbb{R}^m$, find $\boldsymbol{k} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{k}) \in \boldsymbol{V}, p(\boldsymbol{k}) \in L^2(\Omega), \boldsymbol{\lambda}_u \in \boldsymbol{V}$ and $\lambda_p \in L^2(\Omega)$, such that, for all $\boldsymbol{v} \in \boldsymbol{V}$, $q \in L^2(\Omega)$ and $\nu \in \mathbb{R}$:*

$$\begin{cases} (P)\begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_w}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{i=1}^{m}\int_{\Gamma_i} k_i \boldsymbol{v}\cdot\boldsymbol{n}\,d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_s}, q >= -(q, \nabla\cdot\boldsymbol{u}) = 0, \end{cases} \\ (A)\begin{cases} < d\mathcal{L}_{\boldsymbol{w}}, \boldsymbol{v} >= a(\boldsymbol{v}, \boldsymbol{\lambda}_u) + \displaystyle\sum_{i=1}^{m}\Big(\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\,d\gamma - (\lambda_p, \nabla\cdot\boldsymbol{v}) = 0, \\ < d\mathcal{L}_s, q >= -(q, \nabla\cdot\boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_j) \quad < d\mathcal{L}_{\eta_j}, \nu >= \nu\displaystyle\int_{\Gamma_j} \boldsymbol{\lambda}_u\cdot\boldsymbol{n}\,d\gamma = 0, \quad j = 1,\dots,m. \end{cases} \tag{2.5}$$

This system couples a generalized Stokes problem (P) with its adjoint (A) and m scalar equations (*optimality conditions*), denoted by (C_j). Observe that the latter force the adjoint variable $\boldsymbol{\lambda}_u$ to have null flux on the artificial boundaries.

By exploiting the symmetry of the bilinear form $a(\cdot,\cdot)$, the strong formulation of the adjoint problem (A) can be readily deduced, yielding

$$\begin{cases} \alpha\boldsymbol{\lambda}_u - \mu\triangle\boldsymbol{\lambda}_u + \nabla\lambda_p = \boldsymbol{0} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{\lambda}_u = 0 & \text{in } \Omega, \\ \boldsymbol{\lambda}_u|_{\Gamma_w} = \boldsymbol{0}, \\ (-\lambda_p\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda}_u\ \boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0}, \\ (-\lambda_p\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda}_u\ \boldsymbol{n})|_{\Gamma_i} = -\Big(\displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)\boldsymbol{n}, & i = 1,\dots,m. \end{cases} \tag{2.6}$$

Numerical solution of Problem 2 can be pursued by means of iterative techniques, as we will see in Section 4. It is worth noting that, if the iterative process converges, at the limit (i.e. when $J_Q(\boldsymbol{u}) = 0$), the fulfillment of (A) and (C_j) implies $\boldsymbol{\lambda}_u = \boldsymbol{0}$ and $\lambda_p = 0$. This is promptly verified by bearing in mind conditions (C_j) and by selecting $\boldsymbol{v} = \boldsymbol{\lambda}_u$ in (A). The adjoint variables are however needed to drive the iterative scheme to the optimal solution, as it will be illustrated in Section 4.

A general theory of well posedness of problems arising in flow control is presented in [8], Chap. 6. Here, we prove a well posedness results strongly based on the specific feature of the problems at hand.

PROPOSITION 2.1. *Problem 2 admits a unique solution* $[\boldsymbol{u}(\boldsymbol{k}), p(\boldsymbol{k}); \boldsymbol{\lambda}_u, \lambda_p; \boldsymbol{k}]$.
Proof. We introduce the following operators:

1. $\mathcal{P}_{\boldsymbol{f}} : \mathbb{R}^m \to \boldsymbol{V}$: given a vector $\boldsymbol{h} \in \mathbb{R}^m$, it associates it with the function $\boldsymbol{w}$, where $[\boldsymbol{w}, s]$ is the solution of
$$\begin{cases} a(\boldsymbol{w}, \boldsymbol{v}) - (s, \nabla \cdot \boldsymbol{v}) = -\sum_{j=1}^{m} h_j \int_{\Gamma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\gamma + (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ (q, \nabla \cdot \boldsymbol{w}) = 0, & \forall q \in L^2(\Omega). \end{cases} \tag{2.7}$$
In particular, if $\boldsymbol{f} = \boldsymbol{0}$, we will write $\mathcal{P}_{\boldsymbol{0}}$. We point out that operator $\mathcal{P}_{\boldsymbol{f}}$ is well defined since, for a given $\boldsymbol{h}$ (2.7) admits an unique solution, being a generalized Stokes system with constant mixed Neumann and Dirichlet boundary condition (see e.g. [10]). By exploiting the linearity of the problem we have that $\boldsymbol{w} = \mathcal{P}_{\boldsymbol{f}}(\boldsymbol{h}) = \boldsymbol{w}_1 + \boldsymbol{w}_2$, where
$$\boldsymbol{w}_1 = \mathcal{P}_{\boldsymbol{f}}(\boldsymbol{0}), \quad \boldsymbol{w}_2 = \mathcal{P}_{\boldsymbol{0}}(\boldsymbol{h}).$$
2. $\mathcal{A} : \boldsymbol{V} \to \mathbb{R}^m$, such that
$$\mathcal{A}\boldsymbol{w} = \Big(\int_{\Gamma_1} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma, \int_{\Gamma_2} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma, \ldots, \int_{\Gamma_m} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma \Big). \tag{2.8}$$
3. We define $\mathcal{B}_{\boldsymbol{f}} = \mathcal{A} \circ \mathcal{P}_{\boldsymbol{f}}$ and $\mathcal{B}_{\boldsymbol{0}} = \mathcal{A} \circ \mathcal{P}_{\boldsymbol{0}}$, respectively. Operator $\mathcal{B}_{\boldsymbol{f}}$ is an affine operator from $\mathbb{R}^m$ to $\mathbb{R}^m$, while $\mathcal{B}_{\boldsymbol{0}}$ is linear. Observe that
$$\mathcal{B}_{\boldsymbol{f}}(\boldsymbol{h}) = \mathcal{A}\,(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathcal{B}_{\boldsymbol{0}}\boldsymbol{h} + \mathcal{B}_{\boldsymbol{f}}(\boldsymbol{0}). \tag{2.9}$$

We point out that Problem 2 is equivalent to solving the following linear system for the unknown $\boldsymbol{k}$,
$$\mathcal{B}_{\boldsymbol{0}}[\mathcal{B}_{\boldsymbol{f}}(\boldsymbol{k}) - \boldsymbol{Q}] = \boldsymbol{0}. \tag{2.10}$$
Let us prove that $\mathcal{B}_{\boldsymbol{0}}$ is an invertible operator, by providing its explicit representation. This is a linear operator from $\mathbb{R}^m$ to $\mathbb{R}^m$, so it can be represented by a matrix. Let us denote by $\boldsymbol{w}_j$ the solution of (2.7) for $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{e}_j$, being $\boldsymbol{e}_j$ the $j-$th unit vector. For a generic vector $\boldsymbol{h} = \sum_{j=1}^{m} h_j \boldsymbol{e}_j$, we have $\mathcal{P}_{\boldsymbol{0}}(\boldsymbol{h}) = \sum_{j=1}^{m} h_j \boldsymbol{w}_j$.

Let us now exploit $(2.7)_2$ in $(2.7)_1$, by taking $\boldsymbol{v} = \boldsymbol{w}_j$ and $\boldsymbol{h} = \boldsymbol{e}_i$. We obtain
$$a(\boldsymbol{w}_i, \boldsymbol{w}_j) = -\int_{\Gamma_i} \boldsymbol{w}_j \cdot \boldsymbol{n} d\gamma.$$
This implies that
$$[\mathcal{B}_{\boldsymbol{0}}\boldsymbol{h}]_i = -\sum_{j=1}^{m} a(\boldsymbol{w}_i, \boldsymbol{w}_j) h_j$$
so we finally have:
$$[\mathcal{B}_{\boldsymbol{0}}]_{ij} = -a(\boldsymbol{w}_i, \boldsymbol{w}_j).$$
Since $a(\cdot, \cdot)$ is coercive, symmetric bilinear form, matrix $\mathcal{B}_{\boldsymbol{0}}$ is symmetric and negative definite. Therefore, it follows that (2.10) reduces to
$$\mathcal{B}_{\boldsymbol{f}}(\boldsymbol{k}) = \boldsymbol{Q}.$$
Thanks to the linearity, this in turn can be rewritten
$$\mathcal{B}_{\boldsymbol{0}}\boldsymbol{k} = \boldsymbol{Q} - \mathcal{B}_{\boldsymbol{f}}(\boldsymbol{0}).$$
Again, thanks to the non singularity of $\mathcal{B}_{\boldsymbol{0}}$, we conclude that solution $\boldsymbol{k}$ exists and is unique. The corresponding $\boldsymbol{u}$ and p, are obtained by solving (2.7) with $\boldsymbol{h} = \boldsymbol{k}$ and satisfy the constraints by construction. The associated adjoint variables $\boldsymbol{\lambda}_u$ and λ_p are zero. □

2.1.1. The unsteady case. When one wants to solve the unsteady Stokes problem

$$\begin{cases} \dfrac{\partial \boldsymbol{u}(\boldsymbol{k})}{\partial t} - \mu \triangle \boldsymbol{u}(\boldsymbol{k}) + \nabla p(\boldsymbol{k}) = \boldsymbol{f} & \text{in } (0,T)\times\Omega, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{k}) = 0 & \text{in } (0,T)\times\Omega, \\ \boldsymbol{u}(\boldsymbol{k})|_{t=0} = \boldsymbol{u}_0 & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{k})|_{\Gamma_w} = \boldsymbol{0} & \text{in } (0,T), \\ (-p(\boldsymbol{k})\boldsymbol{n} + \mu\nabla\boldsymbol{u}(\boldsymbol{k})\,\boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0} & \text{in } (0,T), \\ (-p(\boldsymbol{k})\boldsymbol{n} + \mu\nabla\boldsymbol{u}(\boldsymbol{k})\,\boldsymbol{n})|_{\Gamma_i} = -k_i(t)\boldsymbol{n}, \quad i=1,\dots,m, & \text{in } (0,T), \end{cases} \tag{2.11}$$

with $\boldsymbol{f} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$, $\nabla \cdot \boldsymbol{u}_0 = 0$ and $k_i(t)$, $i = 1,\dots,m$, chosen such that (2.4) is minimal at each time step, a possible approach would rely on a Lagrangian functional for the time dependent problem. However, this implies a time dependent adjoint equation with a final time condition. The solution of the associated control problem in the time interval $(0,T)$ would lead to iteratively solve (and store) the solution of the primal problem in order to find the time-reversed adjoint solution. A different, *discretize then optimize* approach consists of applying our control technique on the time discretization of (2.11). See in different contexts [2, 12]. In this way, we avoid the solution of the time adjoint problem, since we already know its exact solution, which is zero. In fact, here the adjoint problem is just a tool for enforcing the mean flux boundary conditions. In particular, denoting $t^n = n\Delta t$, with Δt the time discretization step and referring to a BDF time advancing scheme, we obtain, for each n, problem (2.2) and the Lagrangian (2.4) with $\alpha = \beta_0/\Delta t$ and forcing term equal to $\boldsymbol{f}^n + \sum_{i=1}^r \beta_i/\Delta t\, \boldsymbol{u}^{n-i}$, where $\boldsymbol{k}^n = \boldsymbol{k}(t^n)$, $\boldsymbol{u}^n = \boldsymbol{u}^n(\boldsymbol{k}^n) = \boldsymbol{u}(t^n,\boldsymbol{x})$, $p^n = p^n(\boldsymbol{k}^n) = p(t^n,\boldsymbol{x})$, $\boldsymbol{f}^n = \boldsymbol{f}(t^n,\boldsymbol{x})$, $\boldsymbol{u}^{n-i}$ is the approximation of $\boldsymbol{u}(t^{n-i},\boldsymbol{x})$ and $\beta_i\,(i = 0,1,\dots,r \le n)$ are the coefficients of the time discretization. Therefore, with this notation, we can interpret Problem 2 as a technique for the solution of a *discrete-in-time* unsteady flow rate Stokes problem. In this case the strong formulation of the adjoint problem (A) is given by (2.6) with $\alpha = \beta_0/\Delta t$. It may be noted that in this case system (2.6) is exactly the BDF discretization at time $t = t^n$ of the backward-in-time adjoint problem when the exact solution $(\boldsymbol{\lambda}_u^{n+k} = \boldsymbol{0}, \lambda_p^{n+k} = 0)$ for $k = 1,2,\dots,r$ is used.

2.2. The non linear case. We focus now on the system arising when the non-linear convective term is present in the fluid equations. More precisely, we consider the following Navier-Stokes problem:

$$\alpha\boldsymbol{u}(\boldsymbol{k}) - \mu\triangle\boldsymbol{u}(\boldsymbol{k}) + (\boldsymbol{u}(\boldsymbol{k})\cdot\nabla)\boldsymbol{u}(\boldsymbol{k}) + \nabla p(\boldsymbol{k}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.12}$$

together with $(2.2)_{2-5}$. To minimize (2.3) with the constraint given by (2.12) together with $(2.2)_{2-5}$, we have to find the stationary point of the following Lagrangian functional:

$$\mathcal{L}(\boldsymbol{w}, s; \boldsymbol{\lambda}_w, \lambda_s; \boldsymbol{\eta}) = J_Q(\boldsymbol{w}) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) + c(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{\lambda}_w) - (s, \nabla\cdot\boldsymbol{\lambda}_w) +$$

$$+ \sum_{i=1}^m \int_{\Gamma_i} \eta_i \boldsymbol{\lambda}_w \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{\lambda}_w) - (\lambda_p, \nabla\cdot\boldsymbol{w}).$$

This leads to the following

PROBLEM 3. *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{Q} \in \mathbb{R}^m$, find $\boldsymbol{k} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{k}) \in \boldsymbol{V}, p(\boldsymbol{k}) \in L^2(\Omega), \boldsymbol{\lambda}_u \in \boldsymbol{V}$ and $\lambda_p \in L^2(\Omega)$, such that, for all $\boldsymbol{v} \in \boldsymbol{V}$, $q \in L^2(\Omega)$ and $\nu \in \mathbb{R}$:*

$$\begin{cases} (P) \begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{i=i}^m \int_{\Gamma_i} k_i \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma - (\boldsymbol{f},\boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= -(q, \nabla\cdot\boldsymbol{u}) = 0, \end{cases} \\ (A) \begin{cases} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} > \; = a(\boldsymbol{v}, \boldsymbol{\lambda}_u) + c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\lambda}_u) + c(\boldsymbol{v},\boldsymbol{u},\boldsymbol{\lambda}_u) + \\ \qquad -(\lambda_p, \nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{i=1}^m \Big(\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i \Big) \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma = 0, \\ < d\mathcal{L}_p, q > \; = -(q, \nabla\cdot\boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_j) \quad < d\mathcal{L}_{k_j}, \nu >= \displaystyle\int_{\Gamma_j} \nu \boldsymbol{\lambda}_u \cdot \boldsymbol{n}\, d\gamma = 0, \quad j = 1,\dots,m. \end{cases}$$

In order to extend Proposition 2.1 to the non linear case, we introduce the following operators.

1. $\mathcal{F}_{\boldsymbol{f}} : \mathbb{R}^m \to \boldsymbol{V}$ associates to a vector $\boldsymbol{h} \in \mathbb{R}^m$ the function $\boldsymbol{w}$, where $[\boldsymbol{w}, s]$ is the solution of the Navier-Stokes problem:

$$\begin{cases} a(\boldsymbol{w},\boldsymbol{v}) + c(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}) - (s, \nabla\cdot\boldsymbol{v}) = -\sum_{j=1}^m h_j \int_{\Gamma_j} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma + (\boldsymbol{f},\boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ (q, \nabla\cdot\boldsymbol{w}) = 0, & \forall q \in L^2(\Omega). \end{cases} \tag{2.13}$$

2. Given $\boldsymbol{z} \in \boldsymbol{V}$, $\mathcal{M}[\boldsymbol{z}] : \mathbb{R}^m \to \boldsymbol{V}$ associates to a vector $\boldsymbol{h} \in \mathbb{R}^m$ the function $\boldsymbol{w}$, where $[\boldsymbol{w}, s]$ is the solution of the problem:

$$\begin{cases} a(\boldsymbol{w},\boldsymbol{v}) + c(\boldsymbol{z},\boldsymbol{v},\boldsymbol{w}) + c(\boldsymbol{v},\boldsymbol{z},\boldsymbol{w}) - (s, \nabla\cdot\boldsymbol{v}) = -\sum_{j=1}^m h_j \int_{\Gamma_j} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ (q, \nabla\cdot\boldsymbol{w}) = 0, & \forall q \in L^2(\Omega). \end{cases} \tag{2.14}$$

Let us denote with $\boldsymbol{Q}^0 = \mathcal{A} \circ \mathcal{F}_{\boldsymbol{f}}(\boldsymbol{0})$ the vector of flow rates (an entry for each artificial boundary) of the solution of the problem (2.13) with $h_j = 0$ for any j. $\mathcal{A}$ has been defined in (2.8).

Proposition 2.2. *If the viscosity μ is large enough, and $\sum_i |Q_i - Q_i^0|$ and $\|\boldsymbol{f}\|$ are small enough, then Problem 3 admits a solution $[\boldsymbol{u}(\boldsymbol{k}), p(\boldsymbol{k}); \boldsymbol{\lambda}_u, \lambda_p; \boldsymbol{k}]$.*

Proof. Assume that the bilinear form $b_{\boldsymbol{z}}(\boldsymbol{w},\boldsymbol{v}) := a(\boldsymbol{w},\boldsymbol{v}) + c(\boldsymbol{z},\boldsymbol{v},\boldsymbol{w}) + c(\boldsymbol{v},\boldsymbol{z},\boldsymbol{w})$ is continuous and coercive for any $\boldsymbol{z} \in \boldsymbol{V}$. This is actually possible if the first term dominates the other two, namely if the viscosity μ is large enough. In this case, problem $\boldsymbol{w} = \mathcal{M}[\boldsymbol{z}](\boldsymbol{h})$ admits an unique solution for any $\boldsymbol{z} \in \boldsymbol{V}$. Let us introduce the following operators from $\mathbb{R}^m$ to $\mathbb{R}^m$: $\mathcal{C}[\boldsymbol{z}] = \mathcal{A}\circ\mathcal{M}[\boldsymbol{z}]$, and $\mathcal{D}_{\boldsymbol{f}} = \mathcal{A} \circ \mathcal{F}_{\boldsymbol{f}}$. Problem 3 is then equivalent to solve

$$\mathcal{C}[\mathcal{F}_{\boldsymbol{f}}(\boldsymbol{k})](\mathcal{D}_{\boldsymbol{f}}(\boldsymbol{k}) - \boldsymbol{Q}) = \boldsymbol{0}. \tag{2.15}$$

Now, for any $\boldsymbol{z} \in \boldsymbol{V}$, $\mathrm{Ker}(\mathcal{C}[\boldsymbol{z}]) = \{\boldsymbol{0}\}$. Indeed, $\mathcal{C}[\boldsymbol{z}](\boldsymbol{h})$ corresponds to compute the volume fluxes on Γ_i, $i = 1,\ldots,m$, of the velocity field solution of (2.14). Being $\mathcal{C}[\boldsymbol{z}]$ a linear operator, it is in fact represented by matrix whose elements are $[\mathcal{C}[\boldsymbol{z}]]_{ij} = -b_{\boldsymbol{z}}(\boldsymbol{w}_j, \boldsymbol{w}_i)$, and where the velocity fields $\boldsymbol{w}_s$ are defined as in Proposition 2.1. Since $b_z(\cdot,\cdot)$ is coercive, we conclude that $\mathcal{C}[\boldsymbol{z}]$ is invertible. Therefore, system (2.15) is equivalent to the non-linear problem

$$\mathcal{D}_{\boldsymbol{f}}(\boldsymbol{k}) = \boldsymbol{Q}, \tag{2.16}$$

that of finding suitably the k_j's such that the velocity field solution of system (P) in Problem 3 satisfies the flow rate condition $(2.1)_2$. This problem coincides with the augmented formulation proposed in [4, 16], with the difference that there the k_j were treated as Lagrange multipliers while here act as control variables. In [16] it is shown that if $\sum_i |Q_i|$ is small enough then an unique solution $[\boldsymbol{u}, p, \boldsymbol{k}]$ of the augmented problem exists in $[0, T]$, for a suitable $T > 0$, in the case $\boldsymbol{f} = \boldsymbol{0}$. Using similar arguments we can extend this result to the case $\boldsymbol{f} \neq \boldsymbol{0}$ by asking that $\sum_i |Q_i - Q_i^0|$ and $\|\boldsymbol{f}\|$ are small enough. Under these assumptions, problem (2.16) admits (at least) a solution $\boldsymbol{k}$ in $[0, T]$.

The corresponding $\boldsymbol{u}$ and p, are obtained by solving system (P) in Problem 3 and satisfy the constraints by construction. The associated adjoint variables $\boldsymbol{\lambda}_u$ and λ_p are zero. □

The hypothesis on the smallness of $\sum_i |Q_i - Q_i^0|$ requires in fact that the prescribed fluxes are sufficiently close the ones featured by the solution of the Navier-Stokes problem fulfilling homogeneous Neumann conditions in correspondence of the artificial boundaries.

3. Mean pressure boundary conditions. We consider now the mean pressure problem. Still referring to Figure 1.1, we consider (1.3) together with the boundary condition (1.4), initial condition (1.5) and the following defective condition on the artificial sections Γ_j at any $t \in (0, T)$

$$\frac{1}{|\Gamma_j|} \int_{\Gamma_j} p\, d\gamma = P_j, \quad j = 0, 1, \ldots, m, \tag{3.1}$$

where the $P_j = P_j(t)$ are given functions of time. We will specify later on the regularity of the data we address. Conditions (3.1) are still not sufficient to make the problem at hand well posed too.

The variational approach for this type of problem advocated in [10] is based on the forcing of implicit natural homogeneous boundary conditions leading in this case to constant (in space) normal stresses and to zero tangential stresses on each Γ_i. Thus, this formulation is in fact just an approximation of the mean pressure problem and gives the expected results only for specific cases (see Section 3.2). On the other hand, the Lagrange multiplier approach is not suited in this case. Its direct application indeed implies a constant normal velocity on each artificial section, which is general not compatible with the conditions on Γ_w. Indeed, referring, for the sake of simplicity, to the linear case, let us write the Lagrangian functional

$$\mathcal{M}(\boldsymbol{w}, s, \eta_0, \ldots, \eta_m) = \frac{1}{2}a(\boldsymbol{w}, \boldsymbol{w}) + \sum_{j=0}^{m} \eta_j \int_{\Gamma_j} (s - P_j)\, d\gamma + (\nabla s, \boldsymbol{w}) - (\boldsymbol{f}, \boldsymbol{w}),$$

where $\boldsymbol{\eta} \in \mathbb{R}^{m+1}$ is the vector of the Lagrange multipliers related to the boundary conditions (3.1). Introducing the function

$$\Psi(\varepsilon, \varepsilon, \alpha_0, \ldots, \alpha_m) = \mathcal{M}(\boldsymbol{w} + \varepsilon \boldsymbol{v}, s + \varepsilon q, \eta_0 + \alpha_0 \chi_0, \ldots, \eta_m + \alpha_m \chi_m)$$

which takes into account variations of the velocity, of the pressure and of the Lagrange multipliers along the directions $\boldsymbol{v}$, q and χ_j, respectively, we impose $\frac{d\psi}{d\varepsilon}(0, 0, 0, \ldots, 0) = 0$ and $\frac{d\psi}{d\alpha_j}(0, 0, 0, \ldots, 0) = 0$, $j = 0, \ldots, m$, obtaining the following augmented problem:
For all $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in H^1(\Omega)$, find $\boldsymbol{u} \in V$, $p \in H^1(\Omega)$ and $\boldsymbol{\lambda} \in \mathbb{R}^{m+1}$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \\ (\nabla q, \boldsymbol{u}) + \displaystyle\sum_{j=0}^{m} \lambda_j \int_{\Gamma_j} q\, d\gamma = 0, \\ \dfrac{1}{|\Gamma_j|} \displaystyle\int_{\Gamma_j} p\, d\gamma = P_j, \quad j = 0, 1, \ldots, m. \end{cases}$$

The choice of the space $H^1(\Omega)$ for the pressures is done to guarantee the existence of their trace (see Sect. 3.1). The second equation of the previous system gives

$$-(q, \nabla \cdot \boldsymbol{u}) + \int_{\partial\Omega} q\, \boldsymbol{u} \cdot \boldsymbol{n}\, d\gamma + \sum_{j=0}^{m} \lambda_j \int_{\Gamma_j} q\, d\gamma = 0,$$

that is $(\boldsymbol{u} \cdot \boldsymbol{n})|_{\Gamma_j} = \lambda_j$, $\forall j = 0, \ldots, m$, meaning that the normal component of the velocity is constant on each Γ_j. These conditions are not in general compatible with the no-slip boundary condition (1.4) on Γ_w.

Here we propose as a possible alternative an extension of the approach introduced in the previous section. To this aim, it is crucial to select an appropriate control variable. For the moment, the (constant) normal stresses $\boldsymbol{k}$ will be retained as control variables.

3.1. Basic approach. Let us consider the generalized Stokes problem (2.2) where also at Γ_0 we prescribe a condition like $(2.2)_5$. The control variables $\boldsymbol{k}$ are determined so that the following functional is minimized:

$$J_P(s) = \frac{1}{2}\Big(\sum_{i=0}^{m} \frac{1}{|\Gamma_i|} \int_{\Gamma_i} s\, d\gamma - P_i \Big)^2. \tag{3.2}$$

We refer to the following Lagrangian functional:

$$\mathcal{L}(\boldsymbol{w}, s; \boldsymbol{\lambda}_w, \lambda_s; \boldsymbol{\eta}) = J_P(s) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) - (s, \nabla \cdot \boldsymbol{\lambda}_w) + \sum_{i=0}^{m} \int_{\Gamma_i} \eta_i \boldsymbol{\lambda}_w \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{\lambda}_w) - (\lambda_s, \nabla \cdot \boldsymbol{w}), \tag{3.3}$$

for all $\boldsymbol{w} \in \boldsymbol{V}$, $s \in H^1(\Omega)$, $\boldsymbol{\lambda}_w \in \boldsymbol{V}$, $\lambda_s \in H^1(\Omega)$ and $\boldsymbol{\eta} \in \mathbb{R}^{m+1}$. We point out that we take $s \in H^1(\Omega)$ so that its trace on Γ_i, $i = 0, \dots, m$, is meaningful. Of course, we assume that Ω is sufficient regular. The stationary point of $\mathcal{L}$ fulfills the following

Problem 4. *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{P} \in \mathbb{R}^{m+1}$, *find* $\boldsymbol{k} \in \mathbb{R}^{m+1}$, $\boldsymbol{u}(\boldsymbol{k}) \in \boldsymbol{V}, p(\boldsymbol{k}) \in H^1(\Omega), \boldsymbol{\lambda}_u \in \boldsymbol{V}$ *and* $\lambda_p \in H^1(\Omega)$, *such that, for all* $\boldsymbol{v} \in \boldsymbol{V}$, $q \in H^1(\Omega)$ *and* $\nu \in \mathbb{R}$,

$$\begin{cases} (P) \begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + \sum_{i=0}^{m} \int_{\Gamma_i} k_i \boldsymbol{v} \cdot \boldsymbol{n} \, d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= -(q, \nabla \cdot \boldsymbol{u}) = 0, \end{cases} \\ (A) \begin{cases} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= a(\boldsymbol{v}, \boldsymbol{\lambda}_u) - (\lambda_p, \nabla \cdot \boldsymbol{v}) = 0, \\ < d\mathcal{L}_p, q >= \sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|} \int_{\Gamma_i} p \, d\gamma - P_i \Big) \frac{1}{|\Gamma_i|} \int_{\Gamma_i} q \, d\gamma - (q, \nabla \cdot \boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_j) \quad < d\mathcal{L}_{k_i}, \nu >= \int_{\Gamma_i} \nu \boldsymbol{\lambda}_u \cdot \boldsymbol{n} \, d\gamma = 0, \quad i = 0, \dots, m. \end{cases}$$

In this case, we obtain a system coupling a generalized Stokes (P) and a fluid problem featuring a non zero divergence velocity (A). These problems are well posed in the spaces $[\boldsymbol{V}, H^1(\Omega)]$ when considered separately. Indeed, the satisfaction of the inf-sup condition for the spaces $[\boldsymbol{V}, L^2(\Omega)]$ guarantees that it holds in particular for the functions $q \in H^1(\Omega)$. Moreover, thanks to the density of $H^1(\Omega)$ in $L^2(\Omega)$, for a given $g \in L^2(\Omega)$, the equality $(q, \nabla \cdot \boldsymbol{v}) = (q, g)$, $\forall q \in H^1(\Omega)$, implies $\nabla \cdot \boldsymbol{v} = g$ in $L^2(\Omega)$, that is in the weak sense.

The analysis of this control problem seems however more complex than that of Problem 2. The explicit definition of the operator mapping the given data to the control variables resorts to pseudo-differential operators involving the pressure. On the other hand, general theoretical results on the convergence of Lagrange multipliers, such as the ones reported in [8, Chap. 6], cannot be applied, since the functional at hand does not contain the control variables explicitly. The well posedness of this formulation is therefore still an open problem. Numerical results suggest however that the problem can in fact be solved (see Sect. 4.2).

3.2. More complex functionals. In solving defective boundary problems, it is sometimes worth to include into the formulation of the problem specific features of the solution that are not represented in terms of boundary conditions. However, the knowledge of these features can be useful in completing a defective boundary data set. The approach presented here is quite versatile and prone to include these features. In fact, the functional to be minimized can be properly designed for including these data. Consequently, the set of control variables for the optimization process can be driven by the definition of a modified functional. For the sake of clarity, let us consider a domain given by an almost rectilinear pipe, like the one depicted in Fig. 3.1, where we prescribe a mean value for the pressure on an artificial boundary not normal to the pipe axis. Geometrical considerations suggest that the flow is mainly directed along the axial direction **a**. On the contrary, solution to Problem 4 would in general provide a significant

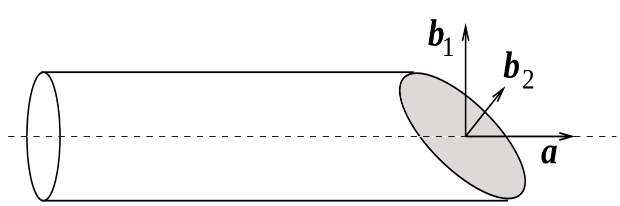

Figure 3.1. *Artificial section* Γ_i, *axial direction* $\boldsymbol{a}$ *and orthogonal directions* $\boldsymbol{b}_1$ *and* $\boldsymbol{b}_2$.

transverse velocity component on the artificial boundary. Problem 4 indeed forces zero tangential stresses on Γ_i as natural boundary condition. We can, however, modify the functional in such a way the minimization process could include the transverse velocity component. This can be done for instance by assuming

$$J_P(\boldsymbol{w}, s) = \frac{1}{2}\sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} s\,d\gamma - P_i\Big)^2 + \mathcal{S}(\boldsymbol{w}, s, \Omega), \tag{3.4}$$

where $\mathcal{S}$ depends also on the velocity field and on the domain. For example, for the case at hand a possible expression for $\mathcal{S}$ is

$$\mathcal{S} = \mathcal{S}_1(\boldsymbol{w}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{d-1}) = \frac{1}{2}\sum_{l=1}^{d-1}\sum_{i=0}^{m}\int_{\Gamma_i} |\boldsymbol{w}\cdot\boldsymbol{b}_l|^2\,d\gamma, \tag{3.5}$$

where $\boldsymbol{b}_l$, $l = 1, \dots, d-1$, are, for each section Γ_i, the orthogonal unit vectors to the axial direction $\boldsymbol{a}$ (see Fig. 3.1). In order to improve the effectiveness of the control procedure on the boundary velocity, we augment the set of control variables. For instance, we do not fix a specific direction for the normal stresses and set

$$(-p\boldsymbol{n} + \mu\nabla\boldsymbol{u}\,\boldsymbol{n})|_{\Gamma_j} = -\boldsymbol{K}_j(\boldsymbol{x}), \quad j = 0, \dots, m, \tag{3.6}$$

where, for instance, the $\boldsymbol{K}_j \in (L^2(\Gamma_j))^d$, $j = 0, \dots, m$, are the control variables. ¿From the Lagrangian functional associated with (3.4), (3.5), with the constraints $(2.2)_1$, $(2.2)_2$, $(2.2)_3$ and (3.6), we obtain the following

Problem 5. *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{P} \in \mathbb{R}^{m+1}$, *find* $\boldsymbol{K}_j \in (L^2(\Gamma_j))^d$, $j = 0, \dots, m$, $\boldsymbol{u}(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in \boldsymbol{V}$, $p(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in H^1(\Omega)$, $\boldsymbol{\lambda}_u \in \boldsymbol{V}$ *and* $\lambda_p \in H^1(\Omega)$, *such that,* $\forall \boldsymbol{v} \in \boldsymbol{V}$, $q \in H^1(\Omega)$ *and* $\boldsymbol{\nu} \in (L^2(\Gamma_i))^d$, $i = 0, \dots, m$*:*

$$\begin{cases} (P)\begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{i=0}^{m}\int_{\Gamma_i} \boldsymbol{K}_i\cdot\boldsymbol{v}\,d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= -(q, \nabla\cdot\boldsymbol{u}) = 0, \end{cases} \\ (A)\begin{cases} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= \displaystyle\sum_{l=1}^{d-1}\sum_{i=0}^{m}\int_{\Gamma_i} (\boldsymbol{u}\cdot\boldsymbol{b}_l)(\boldsymbol{v}\cdot\boldsymbol{b}_l)\,d\gamma + a(\boldsymbol{v}, \boldsymbol{\lambda}_u) - (\lambda_p, \nabla\cdot\boldsymbol{v}) = 0, \\ < d\mathcal{L}_p, q >= \displaystyle\sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p\,d\gamma - P_i\Big)\frac{1}{|\Gamma_i|}\int_{\Gamma_i} q\,d\gamma - (q, \nabla\cdot\boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_i) \quad < d\mathcal{L}_{\boldsymbol{k}_i}, \boldsymbol{\nu} >= \displaystyle\int_{\Gamma_i} \boldsymbol{\nu}\cdot\boldsymbol{\lambda}_u\,d\gamma = 0, \quad i = 0, \dots, m. \end{cases}$$

Observe that in this case the optimality conditions (C_i) imply that $\boldsymbol{\lambda}_u|_{\Gamma_i} = \boldsymbol{0}$, $i = 0, \dots, m$. The conditions of fulfillment of the optimal state are more restrictive of the ones in Problem 4 having a larger control variables set.

Other forms for the perturbing term $\mathcal{S}$ can be considered as well. For instance, we could minimize the functional

$$\mathcal{S}_2(\boldsymbol{w}, \boldsymbol{a}) = \frac{1}{2}\sum_{i=0}^{m}\int_{\Gamma_i} \|\nabla\boldsymbol{w}\,\boldsymbol{a}\|_2^2\,d\gamma. \tag{3.7}$$

Here $\|\cdot\|_2$ is the Euclidean norm made on the components of the vector $\nabla\boldsymbol{w}\,\boldsymbol{a}$. In this way the variation of the velocity along $\boldsymbol{a}$ on the artificial boundary is minimized. Due to the incompressibility constraint, a variation of the velocity along $\boldsymbol{a}$ will be compensated by variations of the components orthogonal to $\boldsymbol{a}$. For this reason, this form of $\mathcal{S}$ can be considered as an indirect way of forcing null tangential velocity. This leads to the following

Problem 6. *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{P} \in \mathbb{R}^{m+1}$, *find* $\boldsymbol{K}_j \in (L^2(\Gamma_j))^d$, $j = 0, \dots, m$, $\boldsymbol{u}(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in \boldsymbol{V}$, $p(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in H^1(\Omega)$, $\boldsymbol{\lambda}_u \in \boldsymbol{V}$ *and* $\lambda_p \in H^1(\Omega)$, *such that, for all*

$\boldsymbol{v} \in \boldsymbol{V}$, $q \in H^1(\Omega)$ *and* $\boldsymbol{\nu} \in (L^2(\Gamma_i))^d$, $i = 0, \dots, m$*:*

$$\begin{cases} (P) \begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + \sum_{i=0}^{m} \int_{\Gamma_i} \boldsymbol{K}_i \cdot \boldsymbol{v} \, d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= -(q, \nabla \cdot \boldsymbol{u}) = 0, \end{cases} \\ (A) \begin{cases} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= \sum_{i=0}^{m} \int_{\Gamma_i} (\nabla \boldsymbol{u} \, \boldsymbol{a}) \cdot (\nabla \boldsymbol{v} \ \boldsymbol{a}) \, d\gamma + a(\boldsymbol{v}, \boldsymbol{\lambda}_u) - (\lambda_p, \nabla \cdot \boldsymbol{v}) = 0, \\ < d\mathcal{L}_p, q >= \sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|} \int_{\Gamma_i} p \, d\gamma - P_i \Big) \frac{1}{|\Gamma_i|} \int_{\Gamma_i} q \, d\gamma - (q, \nabla \cdot \boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_i) \quad < d\mathcal{L}_{\boldsymbol{K}_i}, \boldsymbol{\nu} >= \int_{\Gamma_i} \boldsymbol{\nu} \cdot \boldsymbol{\lambda}_u \, d\gamma = 0, \quad i = 0, \dots, m \end{cases}$$

We point out that it is possible to extend the previous control problems to the non linear case.

REMARK 1. *It is well known that the correct formulation of the diffusive term in the weak form of the momentum equation is* $\int_\Omega \mu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)\nabla \boldsymbol{v}$*. This is crucial for instance in fluid structure interaction problems in correct computing the stresses exchanged by fluid and structure. On the other hand, in [10] it is observed that the complete diffusive term can have some drawbacks in applying the variational formulation for solving pressure drop problems, due to boundary tangential velocity components induced by this approach. These drawbacks in the present framework can be dominated by resorting to functionals (3.4), (3.5) or (3.4), (3.7). This aspect will be investigated in [6], in the case of problems in deformable domains.*

3.3. Using the flow rates as control variables. For solving the mean pressure problem we can also pursue a sort of "dual" approach to the one proposed in Section 2 for the flow rate problem. Here, the control variables are given by the flow rates Q_j on Γ_j, $j = 1, \dots, m$. More precisely, we consider the generalized Stokes problem

$$\begin{cases} \alpha \boldsymbol{u}(\boldsymbol{Q}) - \mu \triangle \boldsymbol{u}(\boldsymbol{Q}) + \nabla p(\boldsymbol{Q}) = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{Q}) = 0 & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{Q})|_{\Gamma_w} = \boldsymbol{0}, \\ (-p(\boldsymbol{Q})\boldsymbol{n} + \mu \nabla \boldsymbol{u}(\boldsymbol{Q})\,\boldsymbol{n})|_{\Gamma_0} = -P_0 \boldsymbol{n}, \\ \int_{\Gamma_j} \boldsymbol{u}(\boldsymbol{Q}) \cdot \boldsymbol{n} \, d\gamma = Q_j, & j = 1, \dots, m, \end{cases} \tag{3.8}$$

where P_0 is a given constant. Thanks to the Neumann condition on Γ_0, there is no need of compatibility conditions on the flow rate data. When we formulate the corresponding control problem where (3.2) (or its possible modifications) is the functional to be minimized, while (3.8) are the state equations, the latter can be solved according to the augmented Lagrangian approach (Problem 1). Therefore, we need to introduce the following Lagrangian functional

$$\begin{aligned} \mathcal{L}(\boldsymbol{w}, s, \boldsymbol{\xi}; \boldsymbol{\lambda}_w, \lambda_s, \boldsymbol{\lambda}_\xi; \boldsymbol{\eta}) = \quad & J_P(s) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) - (s, \nabla \cdot \boldsymbol{\lambda}_w) + \sum_{i=1}^{m} \xi_i \int_{\Gamma_i} \boldsymbol{\lambda}_w \cdot \boldsymbol{n} \, d\gamma + \\ & -(\boldsymbol{f}, \boldsymbol{\lambda}_w) - (\lambda_s, \nabla \cdot \boldsymbol{w}) + \sum_{i=1}^{m} \lambda_{\xi_i} \Big(\int_{\Gamma_i} \boldsymbol{w} \cdot \boldsymbol{n} \, d\gamma - \eta_i \Big), \end{aligned}$$

where the $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ are the Lagrange multipliers of the augmented formulation (see [4, 16]). The quantities $\boldsymbol{\lambda}_w, \lambda_s$ and $\lambda_{\xi_1}, \dots, \lambda_{\xi_m}$ are the adjoint variables related to $\boldsymbol{w}, s$ and $\xi_1, \dots, \xi_m$ respectively. By forcing the derivatives of $\mathcal{L}$ with respect all the variables in correspondence of the solution $[\boldsymbol{u}, p, \boldsymbol{\zeta}; \boldsymbol{\lambda}_u, \lambda_p, \boldsymbol{\lambda}_\zeta; \boldsymbol{Q}]$ to vanish, we obtain

PROBLEM 7. *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{P} \in \mathbb{R}^m$*, find* $\boldsymbol{Q} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{Q}) \in \boldsymbol{V}, p(\boldsymbol{Q}) \in H^1(\Omega), \zeta_j(\boldsymbol{Q}) \in \mathbb{R}$, $j = 1, \dots, m$, $\boldsymbol{\lambda}_u \in \boldsymbol{V}, \lambda_p \in H^1(\Omega)$ *and* $\lambda_{\zeta_j} \in \mathbb{R}$, $j = 1, \dots, m$*, such that, for all* $\boldsymbol{v} \in \boldsymbol{V}$, $q \in$

$H^1(\Omega)$ *and* $\nu \in \mathbb{R}$*:*

$$
\begin{cases}
(P) \begin{cases}
< d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u},\boldsymbol{v}) - (p, \nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{i=1}^{m} \zeta_i \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma - (\boldsymbol{f},\boldsymbol{v}) = 0,\\
< d\mathcal{L}_{\lambda_p}, q >= -(q, \nabla\cdot\boldsymbol{u}) = 0,\\
< d\mathcal{L}_{\lambda_{\zeta_i}}, \nu >= \Big(\displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i\Big)\nu = 0,
\end{cases}\\
(A) \begin{cases}
< d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= a(\boldsymbol{v},\boldsymbol{\lambda}_u) - (\lambda_p, \nabla\cdot\boldsymbol{v}) + \displaystyle\sum_{i=1}^{m} \lambda_{\zeta_i} \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma = 0,\\
< d\mathcal{L}_p, q >= \displaystyle\sum_{i=1}^{m} \Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p\, d\gamma - P_i\Big)\frac{1}{|\Gamma_i|}\int_{\Gamma_i} q\, d\gamma - (q, \nabla\cdot\boldsymbol{\lambda}_u) = 0,\\
< d\mathcal{L}_{\zeta_i}, \nu >= \Big(\displaystyle\int_{\Gamma_i} \boldsymbol{\lambda}_u\cdot\boldsymbol{n}\, d\gamma\Big)\nu = 0,
\end{cases}\\
(C_i) \quad < d\mathcal{L}_{Q_i}, \nu >= -\lambda_{\zeta_i}\nu = 0, \quad i = 1,\dots,m.
\end{cases}
$$

The compatibility conditions in this case yield $\lambda_{\zeta_i} = 0$, $i = 1,\dots,m$, that amounts to force normal stress in the adjoint problem is zero on each artificial section (see [4]). This is the dual situation to what we found for the flow rate problems of Sect. 2, in which fluxes of the adjoint problem were zero. Also in this case, in correspondence of the solution of Problem 7 the adjoint problem features the (unique) zero solution.

The previous approach can be extended to the non-linear case, in analogy with Section 2.

Also for solving Problem 7 we resort to an iterative scheme, where the optimality conditions (C_i), $\lambda_{\zeta_i} = 0$ drive the convergence (see [8]).

4. Numerical algorithms.

We present now some numerical procedures for the solution of the problems introduced in the previous sections. In particular, in Section 4.1 we investigate the flow rate problem. We first introduce a general strategy applied to Problem 2; then we present a special algorithm for the prescription of just one flow rate condition. In Section 4.2 we illustrate numerical algorithms for solving Problem 4 as well, as an example of mean pressure problem. Methods presented can be extended to different defective boundary problems.

4.1. Flow rate problems.

Let us consider Problem 2. For its numerical solution, we can resort to an iterative method such that at each iteration we solve separately problems (P) and (A) and check condition (C_j) until convergence.

Let us notice that the adjoint problem (A) in Problem 2 depends linearly on the values of the natural boundary condition on Γ_j. Therefore, let $(\boldsymbol{\lambda}_{u,i}, \boldsymbol{\lambda}_{p,i})$ be the *reference solutions* of the problems featuring unit normal stresses:

$$
\begin{cases}
a(\boldsymbol{v},\boldsymbol{\lambda}_{u,i}) - (\lambda_{p,i}, \nabla\cdot\boldsymbol{v}) + \displaystyle\int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma = 0, & \forall \boldsymbol{v} \in \boldsymbol{V},\\
(q, \nabla\cdot\boldsymbol{\lambda}_{u,i}) = 0, & \forall q \in L^2(\Omega),
\end{cases}
\tag{4.1}
$$

for all $i = 1,\dots,m$, then the solution of (A) in Problem 2 can be written as a linear combination of such reference solutions

$$
\boldsymbol{\lambda}_u = \sum_{i=1}^{m} \Big(\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i\Big)\boldsymbol{\lambda}_{u,i}, \qquad \lambda_p = \sum_{i=1}^{m} \Big(\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i\Big)\lambda_{p,i}.
$$

Hence, the computational cost of the adjoint problem reduces to the m problems (4.1), which can be computed preliminarily. Obviously, this is not true anymore for the non-linear case.

Observe first of all that if we introduce the affine map $\mathcal{T}_\tau : \mathbb{R}^m \to \mathbb{R}^m$:

$$
\mathcal{T}_\tau(\boldsymbol{k}) := \boldsymbol{k} + \tau\mathcal{B}_{\boldsymbol{0}}(\mathcal{B}_{\boldsymbol{f}}(\boldsymbol{k}) - \boldsymbol{Q}),
$$

where $\mathcal{B}_{\boldsymbol{f}}$ and $\mathcal{B}_{\mathbf{0}}$ have been introduced in Sect. 2.1 and τ is a given real coefficient, Problem 2 can be reformulated as the computation of a fixed point solution:

$$\hat{\boldsymbol{k}} = \mathcal{T}_\tau(\hat{\boldsymbol{k}}). \tag{4.2}$$

This naturally leads to the numerical method based on the fixed point iteration

$$\boldsymbol{k}^{l+1} = \mathcal{T}_\tau(\boldsymbol{k}^l),$$

where l denotes the iteration index. The following algorithm reflects this scheme.

ALGORITHM 1.

1. `Solve for` $j = 1, \dots, m$ `problems (4.1) discretized in space, giving solutions` $\boldsymbol{\lambda}_{u,j,h}$ `and` $\lambda_{p,j,h}$`, for` $j = 1, \dots, m$.
2. `Loop: given` $k^1_{j,h}$, $j = 1, \dots, m$, `and` ε, `set` $l = 1$ `and do until convergence`
   - `Solve`

$$\begin{cases} a(\boldsymbol{u}_h^l, \boldsymbol{v}_h) - (p_h^l, \nabla \cdot \boldsymbol{v}_h) + \displaystyle\sum_{i=1}^m \int_{\Gamma_i} k^l_{i,h} \boldsymbol{v}_h \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{v}_h) = 0, & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \\ (q_h, \nabla \cdot \boldsymbol{u}_h^l) = 0, & \forall q_h \in Q_h \end{cases}$$

   - `Compute the adjoint solutions`

$$\boldsymbol{\lambda}^l_{u,h} = \sum_{i=1}^m \Big(\int_{\Gamma_i} \boldsymbol{u}_h^l \cdot \boldsymbol{n}\, d\gamma - Q_i \Big) \boldsymbol{\lambda}_{u,i,h}, \qquad \lambda^l_{p,h} = \sum_{i=1}^m \Big(\int_{\Gamma_i} \boldsymbol{u}_h^l \cdot \boldsymbol{n}\, d\gamma - Q_i \Big) \lambda_{p,i,h}.$$

   - `Convergence test: if` $\Big| \int_{\Gamma_j} \boldsymbol{\lambda}^l_{u,h} \cdot \boldsymbol{n}\, d\gamma \Big| < \varepsilon, \forall j = 1, \dots, m$ `then break`

     `else` $k^{l+1}_{j,h} = k^l_{j,h} + \tau^l \int_{\Gamma_j} \boldsymbol{\lambda}^l_{u,h} \cdot \boldsymbol{n}\, d\gamma, \quad \forall j = 1, \dots, m,$ `and set` $l = l + 1$.

   `end`

We have the following Proposition.

PROPOSITION 4.1. *For an appropriate choice of τ, Algorithm 1 converges to the solution of Problem 2.*

Proof. Let us prove that $\mathcal{T}_\tau$ is a contraction for suitable values of τ. For a given increment $\delta \boldsymbol{k}$, thanks to the linearity of the involved operators, we obtain

$$\mathcal{T}_\tau(\boldsymbol{k} + \delta \boldsymbol{k}) - \mathcal{T}_\tau(\boldsymbol{k}) = (\mathcal{I} + \tau \mathcal{B}_{\mathbf{0}}^2) \delta \boldsymbol{k}, \tag{4.3}$$

where $\mathcal{I} + \tau \mathcal{B}_{\mathbf{0}}^2$ is a square matrix, being $\mathcal{I}$ the identity. Being $\mathcal{B}_{\mathbf{0}}$ symmetric and negative definite (see Prop. 2.1), T_τ is a contraction for

$$-\frac{2}{\rho(\mathcal{B}_{\mathbf{0}}^2)} < \tau < 0,$$

where $\rho(\mathcal{B}_{\mathbf{0}})$ is the spectral radius of $\mathcal{B}_{\mathbf{0}}$. For any τ in this range, Algorithm 1 converges to the solution of Problem 2. The optimal value for the coefficient τ, which guarantees the highest convergence rate, is given by

$$\tau_{opt} = -\frac{2}{\rho(\mathcal{B}_0^2) + \mu_{min}},$$

where μ_{min} is the minimum eigenvalue of $\mathcal{B}_0^2$. □

The knowledge of the optimal value of τ is not exploitable in practice, as it is based on the knowledge of the spectrum of $\mathcal{B}_{\mathbf{0}}$, which is not in general available. Dynamic strategies for the selection of the parameter τ, regarded as a function of the iteration l, can be considered. In particular, denoting by $\boldsymbol{\chi}_h^l$ the numerical solution obtained at iteration l, we can resort to the choice

$$\tau^l = \tau_N^l = -\frac{J_Q(\boldsymbol{u}_h^l)}{\|\mathcal{L}_{\boldsymbol{k}}(\boldsymbol{\chi}_h^l)\|_2^2}, \tag{4.4}$$

that amounts to update the k_j as follows:

$$k_{j,h}^{l+1} = k_{j,h}^l - \frac{\frac{1}{2}\sum_{j=1}^m \left(\int_{\Gamma_j} \boldsymbol{u}_h^l \cdot \boldsymbol{n}\, d\gamma - Q_j \right)^2 \int_{\Gamma_j} \boldsymbol{\lambda}_{u,h}^l \cdot \boldsymbol{n}\, d\gamma}{\sum_{j=1}^m |\int_{\Gamma_j} \boldsymbol{\lambda}_{u,h}^l \cdot \boldsymbol{n}\, d\gamma|^2}.$$

This choice stems from the application of the classical Newton method for the equation $J_Q(\boldsymbol{k}) = 0$. Indeed, it can be shown that $< J'_Q(\boldsymbol{k}), \boldsymbol{\nu} > = < \mathcal{L}_{\boldsymbol{k}}(\boldsymbol{s}), \boldsymbol{\nu} >$, $\forall \boldsymbol{\nu} \in \mathbb{R}^m$. A further improvement can be yielded by observing that J_Q is a quadratic functional and the associate solution is supposed to be of multiplicity 2, so that we select $\tau^l = 2\tau_N^l$. As we will see in Sect. 5 this is an affective choice.

Algorithm 1 can be suitably extended to non-linear Problem 3.

4.1.1. Prescription of a single flow rate. In the particular case of only one prescribed flux, we can resort to a different algorithm, based on the variational formulation of (2.2) (or (2.12) for the non linear case). For instance, considering equation (2.12) together with $(2.2)_{2-5}$ and $(2.1)_2$ and with $m = 1$, setting $\boldsymbol{v} = \boldsymbol{u}$ we obtain:

$$a(\boldsymbol{u}, \boldsymbol{u}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{u}) + \int_\Gamma k\boldsymbol{u} \cdot \boldsymbol{n}\, d\gamma = (\boldsymbol{f}, \boldsymbol{u}),$$

and then, exploiting the fact that k is constant, if $Q \neq 0$ we can calculate directly

$$k = \frac{(\boldsymbol{f}, \boldsymbol{u}(k)) - \|\boldsymbol{u}(k)\|_a^2 - ((\boldsymbol{u}(k) \cdot \nabla)\boldsymbol{u}(k), \boldsymbol{u}(k))}{Q} = \phi(k), \tag{4.5}$$

where we have set $\|\boldsymbol{v}\|_a^2 := a(\boldsymbol{v}, \boldsymbol{v})$. A fixed point iterations may then be set up as follows:

ALGORITHM 2. `Given` k_h^1, $\boldsymbol{u}_h^0$ `and` ε, `set` $l = 1$ `and do until convergence`

1. `Solve`

$$\begin{cases} a(\boldsymbol{u}_h^l, \boldsymbol{v}_h) + ((\boldsymbol{u}_h^{l-1} \cdot \nabla)\boldsymbol{u}_h^l, \boldsymbol{v}_h) - (p_h^l, \nabla \cdot \boldsymbol{v}_h) + \int_\Gamma k_h^l \boldsymbol{v}_h \cdot \boldsymbol{n}\, d\gamma = (\boldsymbol{f}, \boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (q_h, \nabla \cdot \boldsymbol{u}_h^l) = 0, & \forall q_h \in Q_h. \end{cases}$$

2. `Compute` $k_h^{l+1} = \dfrac{(\boldsymbol{f}, \boldsymbol{u}_h^l) - \|\boldsymbol{u}_h^l\|_a^2 - ((\boldsymbol{u}_h^{l-1} \cdot \nabla)\boldsymbol{u}_h^l, \boldsymbol{u}_h^l)}{Q}.$
3. `Convergence test: if` $|k_h^{l+1} - k_h^l| < \varepsilon$ `then exit`
4. `else` $l = l + 1$

`end`

REMARK 2. *In the case of unsteady problem, Algorithm 2 fails if there exists a* $\widetilde{t}$ *such that* $Q(\widetilde{t})$=*0.*

In the particular case of a linear problem with $\boldsymbol{f} = \boldsymbol{0}$, we do not even need to resort to a fixed point iteration. Indeed, in this case the velocity $\boldsymbol{u}$ depends linearly on k. Therefore, denoting with $\widetilde{\boldsymbol{u}}$ the solution obtained for $k = 1$, $\boldsymbol{u}$ is computed thanks to $\boldsymbol{u} = k\widetilde{\boldsymbol{u}}$. Therefore, from (4.5), we obtain $k = -k^2\|\widetilde{\boldsymbol{u}}\|_a^2/Q$, yielding the following explicit expression for k:

$$k = -\frac{Q}{\|\widetilde{\boldsymbol{u}}\|_a^2}.$$

4.2. Mean pressure problem. Algorithm 1 can be properly extended to the mean pressure problem presented in Section 3. If the control variable is a vector of constants and the problem at hand is linear (like in Problem 4 and 7), the adjoint problem can be still solved effectively by solving a suitable problem out of the iterative loop. For instance, if we refer to Problem 4, and we denote by $(\widetilde{\boldsymbol{\lambda}}_{u,i}, \widetilde{\lambda}_{p,i})$ the solutions of

$$\begin{cases} a(\widetilde{\boldsymbol{\lambda}}_{u,i}, \boldsymbol{v}) - (\widetilde{\lambda}_{p,i}, \nabla\cdot\boldsymbol{v}) = 0, & \forall \boldsymbol{v}\in\boldsymbol{V}, \\ -(q, \nabla\cdot\widetilde{\boldsymbol{\lambda}}_{u,i}) = -\displaystyle\int_{\Gamma_i} q\,d\gamma, & \forall q\in H^1(\Omega), \end{cases} \tag{4.6}$$

then we can obtain the solution to the adjoint problem (A) by setting

$$\boldsymbol{\lambda}_u = \sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(\boldsymbol{k})\,d\gamma - P_i\Big)\widetilde{\boldsymbol{\lambda}}_{u,i}, \qquad \lambda_p = \sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(\boldsymbol{k})\,d\gamma - P_i\Big)\widetilde{\lambda}_{p,i}.$$

Let $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $Q_h^1 \subset H^1(\Omega)$ respectively be a couple of inf-sup compatible finite dimensional spaces. We denote by $\{\boldsymbol{\varphi}_i\}$ $(i = 1,\dots,n_{vel})$ and $\{\psi_i\}$ $(i = 1,\dots,n_p)$ the basis function set of $\boldsymbol{V}_h$ and Q_h^1 respectively, being n_{vel} and n_p their dimensions.

For solving Problem 4, we propose the following algorithm.

ALGORITHM 3.

1. `Solve, for i=0,...,m, problems (4.6) discretized in space, giving the solutions` $\widetilde{\boldsymbol{\lambda}}_{u,i,h}$ `and` $\widetilde{\lambda}_{p,i,h}$.
2. `Loop: given` $k_{j,h}^1$, $j = 0,\dots,m$, `and` ε, `set` $l = 1$ `and do until convergence`
   - `Solve`

$$\begin{cases} a(\boldsymbol{u}_h^l, \boldsymbol{v}_h) - (p_h^l, \nabla\cdot\boldsymbol{v}_h) + \displaystyle\sum_{i=0}^{m}\int_{\Gamma_i} k_{i,h}^l \boldsymbol{v}_h\cdot\boldsymbol{n}\,d\gamma - (\boldsymbol{f}, \boldsymbol{v}_h) = 0, & \forall \boldsymbol{v}_h\in\boldsymbol{V}_h \\ (q_h, \nabla\cdot\boldsymbol{u}_h^l) = 0, & \forall q_h \in Q_h^1. \end{cases}$$

   - `Compute the adjoint solutions`

$$\boldsymbol{\lambda}_{u,h}^l = \sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p_h^l\,d\gamma - P_i\Big)\widetilde{\boldsymbol{\lambda}}_{u,i,h}, \qquad \lambda_{p,h}^l = \sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p_h^l\,d\gamma - P_i\Big)\widetilde{\lambda}_{p,i,h}$$

   - `Convergence test: if` $\Big|\displaystyle\int_{\Gamma_j}\boldsymbol{\lambda}_{u,h}^l\cdot\boldsymbol{n}\,d\gamma\Big| < \varepsilon, \forall j = 0,\dots,m$ `then exit`
   - `else` $k_{j,h}^{l+1} = k_{j,h}^l + \tau^l\displaystyle\int_{\Gamma_j}\boldsymbol{\lambda}_{u,h}^l\cdot\boldsymbol{n}\,d\gamma, \quad \forall j = 0,\dots,m$ `and set` $l = l+1$
3. `end`

In order to investigate the properties of this algorithm, we introduce the following operators.

1. $\mathcal{N}_{\boldsymbol{f},h} : \mathbb{R}^{m+1} \to Q_h^1$. This operator maps a vector $\boldsymbol{h} \in \mathbb{R}^{m+1}$ to the function s_h, where $[\boldsymbol{w}_h, s_h]$ is the unique solution of

$$\begin{cases} a(\boldsymbol{w}_h, \boldsymbol{v}_h) - (s_h, \nabla\cdot\boldsymbol{v}_h) = -\displaystyle\sum_{j=0}^{m} h_j\int_{\Gamma_j}\boldsymbol{v}_h\cdot\boldsymbol{n}\,d\gamma + (\boldsymbol{f}, \boldsymbol{v}_h), & \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \\ (q_h, \nabla\cdot\boldsymbol{w}_h) = 0, & \forall q_h\in Q_h^1. \end{cases}$$

Let A denote the $n_{vel}\times n_{vel}$ matrix with entries $a_{ij} = a(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i)$, B the $n_{vel}\times n_p$ matrix $b_{ij} = -(\psi_j, \boldsymbol{\varphi}_i)$, G the matrix with entries $g_{ij} = \int_{\Gamma_i}\boldsymbol{\varphi}_j\cdot\boldsymbol{n}\,d\gamma$ and $\widetilde{\mathbf{f}}$ the vector whose components are $(\boldsymbol{f}, \boldsymbol{\varphi}_i)$. Then, the algebraic counterpart of $\mathcal{N}_{\boldsymbol{f},h}$ reads:

$$\mathbf{s} = \left(BA^{-1}B^T\right)^{-1}\left(-BA^{-1}G^T\mathbf{h} + BA^{-1}\widetilde{\mathbf{f}}\right).$$

We observe, by the way, that G is the algebraic representation of the finite dimensional operator $\mathcal{A}_h$ obtained by taking the analogous of operator $\mathcal{A}$ defined in (2.8) from the subspace $\mathbf{V}_h$ to $\mathbb{R}^{m+1}$.

2. $\widetilde{\mathcal{G}}_h : Q_h^1 \to \mathbb{R}^{m+1}$, such that $\widetilde{\mathcal{G}}_h(s_h) = \Big(\frac{1}{|\Gamma_0|}\int_{\Gamma_0} s_h\, d\gamma, \frac{1}{|\Gamma_1|}\int_{\Gamma_1} s_h\, d\gamma, \ldots, \frac{1}{|\Gamma_m|}\int_{\Gamma_m} s_h\, d\gamma\Big)$.
   The algebraic counterpart of this operator is given by the matrix denoted by $\widetilde{G}$ with entries $\widetilde{g}_{ij} = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \psi_j\, d\gamma$.
3. $\mathcal{S}_h : \mathbb{R}^{m+1} \to \boldsymbol{V}_h$ maps a vector $\mathbf{r}$ in $\boldsymbol{w}_h$, where $[\boldsymbol{w}_h, s_h]$ is the unique solution of the generalized Stokes problem

$$\begin{cases} a(\boldsymbol{w}_h, \boldsymbol{v}_h) - (s_h, \nabla\cdot\boldsymbol{v}_h) = 0 & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ -(q_h, \nabla\cdot\boldsymbol{w}_h) = -\sum_{i=0}^{m} r_i \frac{1}{|\Gamma_i|}\int_{\Gamma_i} q_h\, d\gamma, & \forall q_h \in Q_h^1. \end{cases} \tag{4.7}$$

   The algebraic formulation of this operator reads:

$$\mathbf{w} = A^{-1}B^T\left(BA^{-1}B^T\right)^{-1}\widetilde{G}^T\mathbf{r}.$$

Now, set $\mathcal{H}_h = \mathcal{A}_h \circ \mathcal{S}_h$ and $\mathcal{E}_{\boldsymbol{f},h} = \widetilde{\mathcal{G}}_h \circ \mathcal{N}_{\boldsymbol{f},h}$. Then, it is promptly verified that the discrete control problem at hand reads:

$$\mathcal{H} \circ (\mathcal{E}_{\boldsymbol{f},h}(\mathbf{k}) - \mathbf{P}) = 0. \tag{4.8}$$

Correspondingly, observe that the algorithm proposed above can be regarded as the Richardson iteration: given $\mathbf{k}^l$, find $\mathbf{k}^{l+1}$ such that

$$\mathbf{k}^{l+1} = \mathbf{k}^l + \tau^l\mathcal{H}_h \circ \left(\mathbf{P} - \mathcal{E}_{\boldsymbol{f},h}(\mathbf{k}^l)\right).$$

Unfortunately, an analysis proving the definiteness of the residual operator on the right hand side in this case, that would imply the convergence of the iterative scheme, is still missing.

We limit ourselves to point out that in algebraic terms, if $\Sigma = BA^{-1}B^T$ is the Schur complement associated to the Stokes problem, (4.8) can be rewritten as:

$$GA^{-1}B^T\Sigma^{-1}\widetilde{G}^T\left(\widetilde{G}\Sigma^{-1}BA^{-1}\left(-G^T\mathbf{k} + \mathbf{f}\right) - \mathbf{P}\right) = 0$$

We set $Z = \widetilde{G}\Sigma^{-1}BA^{-1}G^T$ so that, thanks to the symmetry of Σ and A, the matrix problem can be rewritten:

$$Z^TZ\mathbf{k} = Z^T\left(\widetilde{G}\Sigma^{-1}BA^{-1}\mathbf{f} - \mathbf{P}\right).$$

This formulation highlights that the matrix at hand is semidefinite positive. This excludes that the algorithm could yield divergent sequences, even if the convergence is still not ensured. This issue will be investigated in a future work. Numerical evidence suggests however that with a proper choice of the parameter τ the numerical solution converges.

Remark 3. *A similar algorithm can be devised for Problem 7. In this case both the adjoint problem out of the iterative cycle and the problem in the iterative cycle are solved with the augmented formulation, in particular resorting to the GMRes+Schur complement scheme introduced in [16].*

5. Numerical results. In this Section we present several numerical results to validate the algorithms introduced in Section 4. In particular, in Section 5.1 we focus on the flow rate problem, while in Section 5.2 on the mean pressure problem.

5.1. Flow rate problems. All simulations of this Section and of Section 5.2 have been implemented using the 2D finite element library *Freefem++* (see [9]), with a discretization time step $\Delta t = 0.01\,s$ and using conforming $\mathbb{P}_2 - \mathbb{P}_1$ elements. In the first simulations we present, the computational domain is a rectangle R of $6 \times 1\,cm$ (see Figure 5.1, left), while the fluid kinematic viscosity is $\mu = 0.035\,cm^2/s$. This is the usual value adopted for blood viscosity in computational haemodynamics. We prescribed both a steady ($Q = 0.1\,cm^2/s$) and a pulsatile

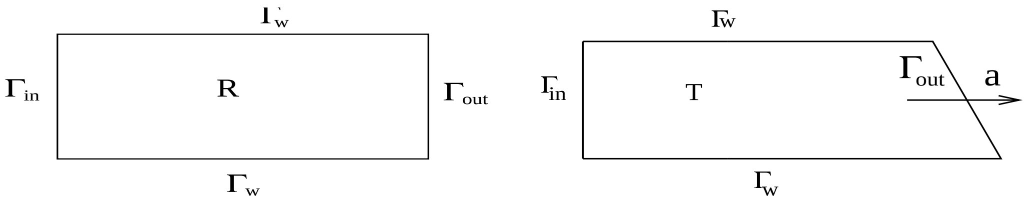

FIGURE 5.1. *Computational domains R and T.*

($Q = 0.15 + 0.1\cos(2\pi t)\, cm^2/s$) flow rate at the inlet Γ_{in} of R. We have used an almost uniform triangular grid with spacing $h = 0.05\, cm$. We have considered in this first test case the Stokes problem.

In order to verify the accuracy of the solutions obtained with the two algorithms, we provide comparisons with the analytical Poiseuille solution in the steady case (see Tab. 5.1, being $D_I = \|\boldsymbol{u}_{pois} - \boldsymbol{u}_I\|_{L^2(\Gamma_{in})}/\|\boldsymbol{u}_{pois}\|_{L^2(\Gamma_{in})}$ and $D_{II} = \|\boldsymbol{u}_{pois} - \boldsymbol{u}_{II}\|_{L^2(\Gamma_{in})}/\|\boldsymbol{u}_{pois}\|_{L^2(\Gamma_{in})}$, $\boldsymbol{u}_{pois}, \boldsymbol{u}_I$ and $\boldsymbol{u}_{II}$ are the Poiseuille solution, the numerical solutions obtained with Algorithm 1 and Algorithm 2, respectively).

In the unsteady case we compare the results with the solution obtained by the augmented formulation (see [16, 18]). In particular, this formulation has been faced with the *GMRes+Schur complement* (GS) algorithm. In Table 5.1 we report $D_I = \|\boldsymbol{u}_{augm} - \boldsymbol{u}_I\|_{L^2(\Gamma_{in})}/\|\boldsymbol{u}_{augm}\|_{L^2(\Gamma_{in})}$ and $D_{II} = \|\boldsymbol{u}_{augm} - \boldsymbol{u}_{II}\|_{L^2(\Gamma_{in})}/\|\boldsymbol{u}_{augm}\|_{L^2(\Gamma_{in})}$, where $\boldsymbol{u}_{augm}$ is the numerical solutions obtained with the augmented formulation. In Figure 5.2, the axial velocity at the inlet Γ_{in}, computed with

	Time	D_I	D_{II}
steady case		0.0000003	0.0000050
unsteady case	$t = 1.25\, s$	0.000230	0.000160
	$t = 1.30\, s$	0.000154	0.000156
	$t = 1.60\, s$	0.000348	0.000348

TABLE 5.1

Relative errors of the numerical solutions obtained with Algorithm 1 and Algorithm 2, compared to the reference solution.

the two algorithms in the steady (top, left) and in the unsteady case, is compared with the reference solution. Differences among the different solutions are clearly small.

The number of iterations for the three algorithms is shown in Table 5.2. In the unsteady case, we give the mean values over the entire computation in time. For Algorithm 1 we compare different possible choices of τ. An Aitken acceleration procedure has been successfully implemented to speed up convergence. Algorithm 2 has been considered only in the case of Aitken procedure implementation.

In the steady case, as expected from Section 4.1.1, Algorithm 2 is very fast. In the unsteady case, Algorithm 1 with $\tau = 2\tau_N^l$ gives the best performances. Number of iterations is in fact comparable with the one of the GMRes algorithm for the augmented problem. The latter is supposed (in exact arithmetic) to converge in $m+1$ iterations per time step, where m is the number of prescribed flow rates (see [16]). In these two cases, we also compute the (mean) CPU time used to perform an iteration. These values are shown in Tab. 5.2 in the brackets and they highlights that also the computational costs of Algorithm 1 with $\tau = 2\tau_N^l$ and of the GMRes algorithm for the augmented problem are comparable.

In the second test case, we consider the computational domain depicted in Figure 5.3 representing a 2D simplified model of a by-pass anastomosis (see e.g. [18]). Space discretization step is of $h = 0.1\, cm$. We solve the Navier-Stokes equations by extending Algorithm 1 to the non-linear case. In this case the adjoint problem needs to be solved in the time advancing loop. We prescribe the flow rates $Q_1 = \cos(2\pi t)\, cm^2/s$ on the upper inlet and $Q_2 = 0.5\cos(2\pi t)\, cm^2/s$ on the lower inlet. We show in Figure 5.3 the axial velocity on the left and the difference with the

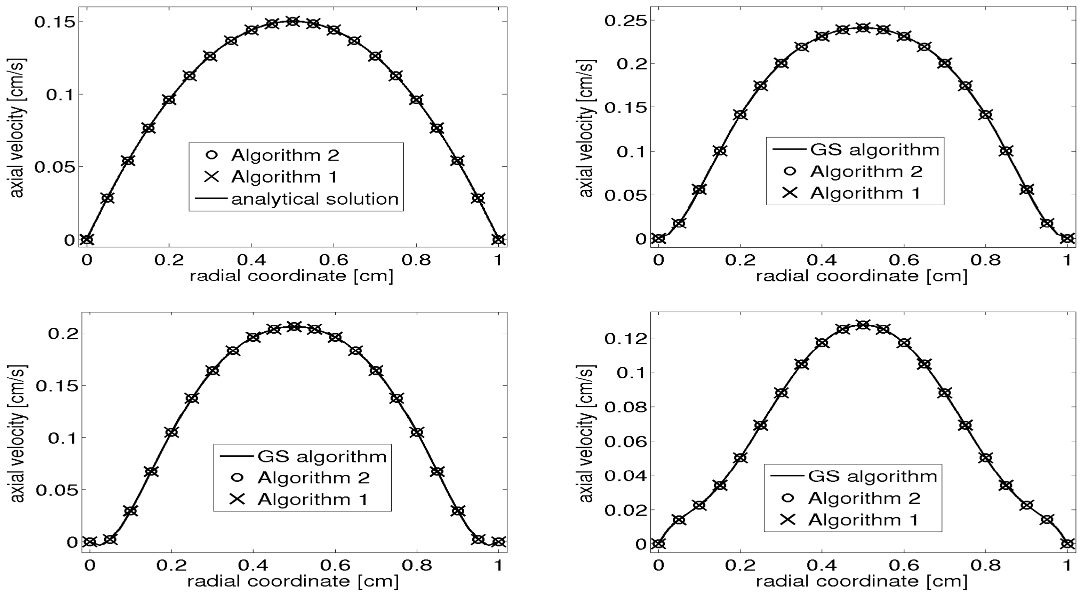

FIGURE 5.2. *Axial velocity computed with Algorithm 1 and Algorithm 2 and reference solution - steady simulation (top, left) and unsteady simulation, $t = 1.25\,s$ (top, right), $t = 1.30\,s$ (bottom, left) and $t = 1.60\,s$ (bottom, right) - toll $= 10^{-7}$.*

	Alg. 1 $\tau = -1$	Alg. 1 $\tau = \tau_N^l$	Alg. 1 $\tau = \tau_N^l$+Aitken	Alg. 1 $\tau = 2\tau_N^l$	Alg.2 +Aitken	GS
Steady case	87	22	4	2	1	2
Unsteady case	-	5.08	3.81	1.98 ($6.37\,s$)	4.50	2.00 ($6.41\,s$)

TABLE 5.2
Number of iterations for the convergence of Algorithm 1, Algorithm 2 and GS algorithm (in average for the unsteady case) and, in brackets, CPU times in seconds for Algorithm 1 with $\tau = 2\tau_N^l$ and for GS algorithm.

solution obtained using the augmented formulation on the right. The two numerical results are in excellent agreement being the difference bounded by 10^{-5}. Moreover, we point out that this error in independent on the mesh size. The mean number of iterations per time step required in this case by Algorithm 1 with the choice $\tau = 2\tau_N^l$ is 18.22, versus the $m + 1 = 3$ iterations needed by the GS algorithm. The (mean) CPU times per iteration are $169.2\,s$ and $16.1\,s$, respectively. The slow convergence of Algorithm 1 is due to the ill-conditioning of the problem when prescribing 2 flow rates. In this case it is well known that the convergence of the steepest descent method can be quite slow. One possibility, that will be investigated in a future work, is to search for conjugate directions.

5.2. Mean pressure problems. In this Section, we illustrate the performances of Algorithm 3 and its extensions to solve Problems 5, 6 and 7. In the first test case, we prescribe the mean pressure $P = \sin(2\pi t)\, g/(s^2\, cm)$ at the outlet Γ_{out} of the domain R (with $h = 0.1\, cm$) of Fig. 5.1 left. We compare the two strategies of using normal component of the normal stress as control variable (strategy PS, see Problem 4) and the one using flow rates (strategy PF, see Problem 7). Computations are performed for the Stokes problem.

Figure 5.4 and Table 5.3 show that the two strategies give close results. However, the PF strategy is more problematic in terms of convergence of the numerical scheme. As a matter of fact, we observed that it diverges for the non stationary choices of τ we tested. Finally, we resorted to set a static value $\tau = -10^{-6}$. The average number of iterations in this case has been 31.31. It is worth to observe that this strategy is more expensive also in terms of CPU times, since for

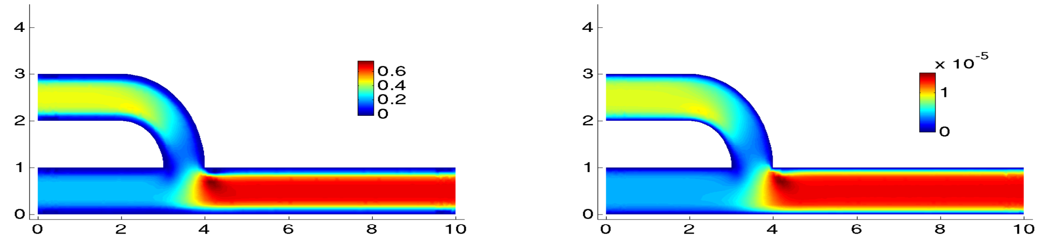

FIGURE 5.3. *Anastomosis simulation - Algorithm 1 -* $t = 1.2s$ *- on the left the axial velocity, on the right the difference with the GS algorithm - toll* $= 10^{-9}$.

each iteration it requires the solution of $m + 1 = 2$ generalized Stokes problems for solving the augmented formulation.

On the contrary, convergence with the PS strategy is obtained with an average of 2.00 iterations (with $\tau = 2\tau_N^l$). Each iteration require the solution of only one generalized Stokes problem.

We conclude that if the control variables can be chosen, strategy PS is definitely the best one.

Time	PF	PS
$t = 1.10\,s$	$9.327 \cdot 10^{-5}$	$9.329 \cdot 10^{-5}$
$t = 1.40\,s$	$3.569 \cdot 10^{-5}$	$3.578 \cdot 10^{-5}$

TABLE 5.3

Relative errors in the $L^2(\Gamma)$ *norm of the solutions obtained using the flow rate (*PF*) and the normal stress (*PS*), respectively, as control variable.*

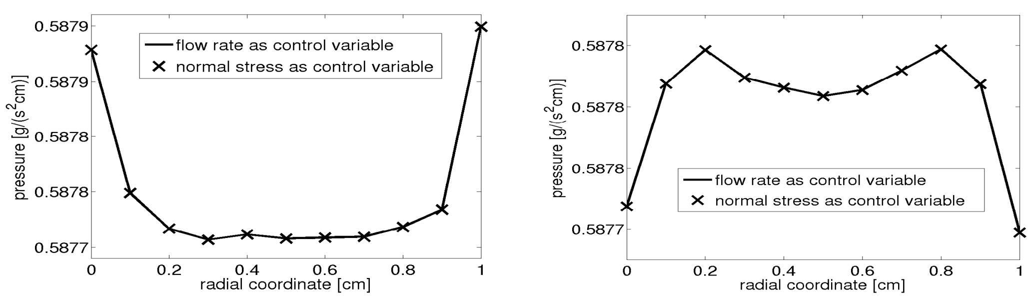

FIGURE 5.4. *Mean pressure prescribed at the outlet at t=1.1 s,* $P = 0.5878\,g/(s^2\,cm)$ *(left) and at t=1.4 s,* $P = 0.5878\,g/(s^2\,cm)$ *(right) - toll* $= 10^{-7}$.

In the second set of simulations, we prescribe the mean pressure $P = 1\,g/(s^2\,cm)$ at the outlet Γ_{out} of the domain T (see Figure 5.1, right). We indicate with $\boldsymbol{a}$ the axial direction. By minimizing functional (3.2), an undesirable transverse velocity and, consequently, an incorrect axial velocity at the outlet occur (Figures 5.5, top). To avoid these effects, we solve Problems 5 and 6, i.e. we minimize functional (3.4) with (3.5) and (3.7), respectively. We consider the same iterative scheme of Algorithm 3. Figures 5.5 illustrate the effectiveness of these strategies, yielding errors of about 3×10^{-4} in the former case and 5×10^{-5} in the latter one.

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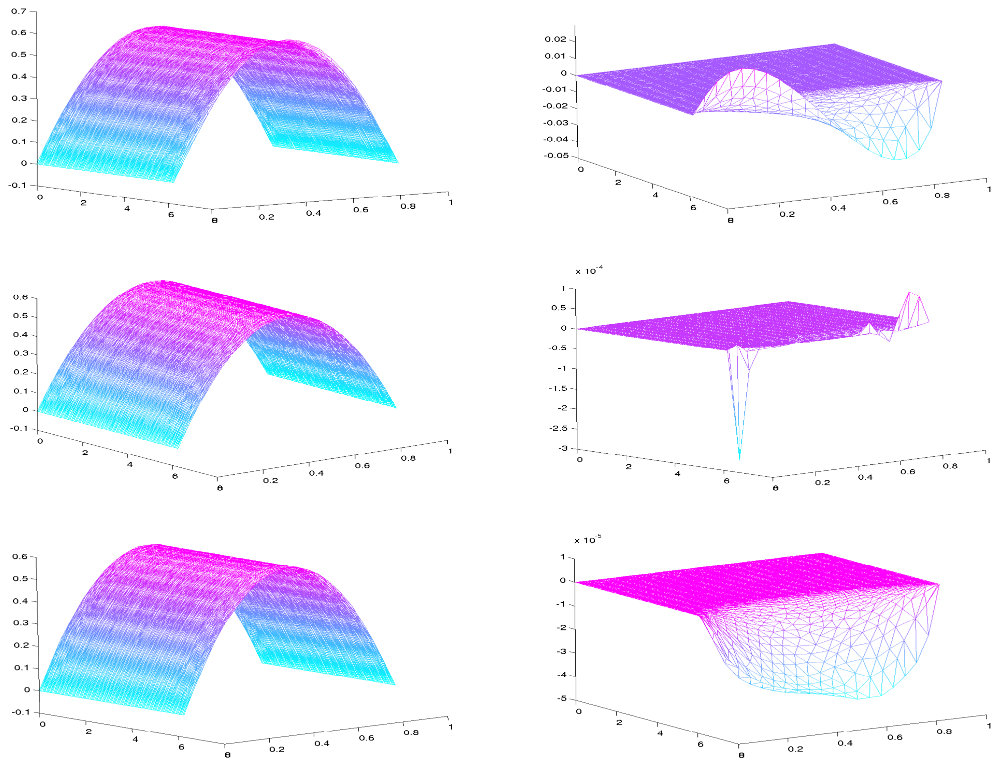

FIGURE 5.5. *Axial (left) and tangential (right) velocity in* cm^2/s *obtained prescribing* $P = 1\,g/(s^2\,cm)$ *at the outlet of* T *minimizing (3.2) (top), (3.4) with (3.5) (middle) and (3.4) with (3.7) (bottom) - toll* $= 10^{-7}$.

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