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Special Kaehler-Ricci potentials and Ricci solitons

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SPECIAL KÄHLER-RICCI POTENTIALS AND RICCI SOLITONS

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Abstract. On a manifold of dimension at least six, let (g, τ) be a pair consisting of a Kähler metric g which is locally Kähler irreducible, and a nonconstant smooth function τ. Off the zero set of τ, if the metric $\widehat{g} = g/\tau^2$ is a gradient Ricci soliton which has soliton function $1/\tau$, we show that $\widehat{g}$ is Kähler with respect to another complex structure, and locally of a type first described by Koiso. Moreover, τ is a special Kähler-Ricci potential, a notion defined in earlier works of Derdzinski and Maschler. The result extends to dimension four with additional assumptions. We also discuss a *Ricci-Hessian equation*, which is a generalization of the soliton equation, and observe that the set of pairs (g, τ) satisfying a Ricci-Hessian equation is invariant, in a suitable sense, under the map $(g, \tau) \to (\widehat{g}, 1/\tau)$.

1. Introduction

In this paper we study pairs (g, τ) on a manifold M of dimension larger than two, where g is a Riemannian metric and τ is a smooth nonconstant function. In this context, an important role will be played by the map $(g, \tau) \to (\widehat{g}, \widehat{\tau}) = (g/\tau^2, 1/\tau)$ on the set of all such pairs. On $M \smallsetminus \tau^{-1}(0)$, this map is a well-defined involution. We will call $(\widehat{g}, \widehat{\tau})$ the *associated pair* of (g, τ).

We say that a pair satisfies a *Ricci-Hessian equation* if

$$\alpha \nabla d\tau + \mathrm{r} = \gamma\, g \tag{1.1}$$

holds for the Hessian of τ, the Ricci tensor r of g, and some C^∞ coefficient functions α and γ. If α and γ are constant, the pair, or sometimes just the metric, is called a gradient Ricci soliton.

Before stating our main result, we note a closely related fact: the set of pairs satisfying a Ricci-Hessian equation is invariant under our involution. The latter is well-defined once the domain of each pair is further restricted (see §2.2). In this setting we call it the duality map.

A part of our main result may be stated informally as follows. Consider two subsets of the set of all pairs on M: those for which the metric is Kähler (and locally irreducible in a suitable sense, see below), and pairs which are gradient Ricci solitons. Assuming a restriction on the dimension of the manifold, if the involution maps an element of the first subset to an element of the second one, the latter element lies in the intersection of the two subsets. More precisely,

Theorem A. *Let M be a manifold of dimension at least six, and (g, τ) a pair as above, with g a Kähler metric. Suppose g is not a local product of Kähler metrics in any neighborhood of some point of M. If the associated pair $(\widehat{g}, \widehat{\tau})$ is a gradient Ricci soliton, then, on $M \smallsetminus \tau^{-1}(0)$, the metric $\widehat{g}$ is Kähler.*

The complex structures giving the Kähler structures of g and $\widehat{g}$ are oppositely oriented. Also, with an extra assumption, the result extends to real dimension four.

The proof, in fact, yields much more information on both pairs. First, τ is a special Kähler-Ricci potential. This notion (Definition 3.4) was first defined in [6], and implies that τ is a Killing potential, and that (1.1) holds in some (generically nonempty) open set of the manifold. Second, the Kähler-Ricci soliton $(\widehat{g}, \widehat{\tau})$ is locally of a type first described by Koiso in [10] (see also [4]).

Although our result is of a local character, one should note that there exist compact manifolds, specifically toric Fano ones, which admit Kähler-Ricci solitons [12], most of which are not of the form found by Koiso.

In the following we describe a few related problems of a broader scope. The involution above is defined in part via a conformal change, and one can ask whether, starting with a Kähler metric g, one can find a metric g/τ^2, for some function τ as above, which is a Ricci soliton. The case where g/τ^2 is Einstein, was the subject of the study of [6, 7, 8], where local and global classifications were given, and, in all even dimensions larger than four, τ turns out to be, in fact, a special Kähler-Ricci potential. In dimension four this need not be the case, and different compact examples where recently shown to exist in [5].

For the case of Ricci solitons, no such general results are known, even if one *assumes* that τ is a special Kähler-Ricci potential. Our result can be considered a first step in an attempt to answer this question. For more on this topic, see the material surrounding Proposition 3.1.

In various talks, G. Tian has asked whether there exist compact non-Kähler Ricci solitons in dimension four. Extending the question to all dimensions, one may answer it affirmatively via constructions involving products. Ignoring these fairly simple examples, one can try to produce such a Ricci soliton in the conformal class of a Kähler metric (at least in dimension four, it is not too difficult to see that there can be no more than two Kähler metrics in a given conformal class). Our result can be regarded as implying that, in a special case, such an attempt will fail.

Finally, recall the result of Schur, stating that if $\mathrm{r} = \phi\, g$ for some function ϕ, then, except in dimension two, ϕ must be constant. A similar principle holds for Kähler-Ricci solitons, for the coefficient of g, and one may ask whether it holds for any Ricci soliton, or, equivalently, whether there exist pairs for which (1.1) holds with the coefficient of $\nabla d\tau$ constant, but not the coefficient of g. In unpublished work, A. Derdzinski has shown that such pairs do exist on compact manifolds. In fact, on these manifolds there are families of pairs (g, τ), with g Kähler, for which the associated pairs $(\widehat{g}, \widehat{\tau})$ are each of this type, and in fact are obtained by deforming one of the Einstein metrics in [8].

This paper is arranged as follows. Riemannian preliminaries on duality and Ricci solitons appear in §2. Consequences of the Kähler condition for (1.1), along with a review of the basic properties of metrics with a special Kähler-Ricci potential, are given in §3. Ordinary differential equations associated with a large class of such metrics are studied in §4, especially in relation to the assumption that the associated pair forms a Ricci soliton, and we give an analysis of their solution set. After recalling the geometric structure of a Kähler metric admitting a special Kähler-Ricci potential, and presenting a duality result in this context in §5, we summarize our results in §6 by proving Theorem 6.1, from which Theorem A easily follows. Our conventions throughout closely follow [6].

2. Ricci-Hessian equations, duality and Ricci solitons

2.1. **conformal changes.** Let (M, g) be a Riemannian manifold of dimension n, and $\tau : M \to \mathbb{R}$ a nonconstant C^∞ function. We write metrics conformally related to g in the form $\widehat{g} = g/\tau^2$, and we set $Q = g(\nabla\tau, \nabla\tau)$. We always consider the metric $\widehat{g}$ on its domain of definition, i.e. the set $M \smallsetminus \tau^{-1}(0)$. With respect to $\widehat{g}$, the Hessian of a given C^2 function f on $M \smallsetminus \tau^{-1}(0)$ is given by

$$\widehat{\nabla} df \;=\; \nabla df \;+\; \tau^{-1}[2\, d\tau \odot df - \; g(\nabla\tau, \nabla f) g], \tag{2.1}$$

where $d\tau \odot df = (d\tau \otimes df + df \otimes d\tau)/2$. We will be concerned primarily with the case where $df \wedge d\tau = 0$, i.e., at points where $d\tau \neq 0$, f is given locally as a composition $f = H \circ \tau$. In this case, (2.1) becomes $\widehat{\nabla} df \;= f' \,\nabla d\tau + (f'' + 2\tau^{-1} f')\; d\tau \otimes d\tau - f' \,\tau^{-1}\, Q\, g$, with $'$ denoting differentiation with respect to τ. For the particular choice of $f = \tau^{-1}$, this expression simplifies:

$$\widehat{\nabla} d\tau^{-1} \;= -\tau^{-2}(\,\nabla d\tau \;-\; \tau^{-1}\, Q\, g), \quad \text{ if } \widehat{g} = g/\tau^2 \tag{2.2}$$

We conclude by recording the conformal change expression relating the Ricci tensors of g and $\widehat{g}$, with Δ denoting the Laplace operator:

$$\widehat{\mathrm{r}} \;=\; \mathrm{r} \;+\; (n-2)\,\tau^{-1}\nabla d\tau \;+\; \left[\tau^{-1}\Delta\tau \;-\; (n-1)\,\tau^{-2} Q\right] g. \tag{2.3}$$

2.2. **Ricci-Hessian equations and duality.** With M, g, τ and other notations as above, we say that the pair (g, τ) satisfies a *Ricci-Hessian equation* on M (or often just on an open set of M), if (1.1) holds there. We record this equation more explicitly as

$$\alpha \nabla d\tau + \mathrm{r} = \gamma\, g, \quad \text{with } \tau \text{ nonconstant}, \tag{2.4}$$

where $\nabla d\tau$ and r are as above, and α, γ are C^∞ coefficient functions. What we will call duality may be regarded informally as an involution on the space of pairs satisfying (2.4):

Proposition 2.1. *Let M have dimension $n > 3$, and suppose a pair (g, τ) as above satisfies a Ricci-Hessian equation (2.4) on M. Then the pair $(\widehat{g}, \widehat{\tau}) = (g/\tau^2, 1/\tau)$ also satisfies a Ricci-Hessian equation $\widehat{\alpha}\widehat{\nabla} d\tau + \widehat{\mathrm{r}} = \widehat{\gamma}\,\widehat{g}$, on $M \smallsetminus \tau^{-1}(0)$, with coefficients*

$$\widehat{\alpha} = (n-2)\tau - \tau^2\alpha, \quad \widehat{\gamma} = \gamma\tau^2 - (1 + \alpha\tau)Q + \tau\Delta\tau. \tag{2.5}$$

In fact, letting β denote the coefficient of g in (2.3), one has, by (2.2) and (2.3),

$$\begin{aligned} \widehat{\alpha}\, \widehat{\nabla} d\widehat{\tau} + \widehat{\mathrm{r}} \;&=\; ((n-2)\tau - \tau^2\alpha)(-\tau^{-2}\, \nabla d\tau \, + \tau^{-3} Q g) + \mathrm{r} \, + (n-2)\tau^{-1}\nabla d\tau + \beta g \\ &=\; \alpha\, \nabla d\tau \, + \mathrm{r} \, + (\widehat{\alpha}\tau^{-3} Q + \beta) g = (\gamma + \widehat{\alpha}\tau^{-3} Q + \beta)\tau^2 \widehat{g}, \end{aligned}$$

and one easily checks that the last expression is $\widehat{\gamma}\,\widehat{g}$.

Remark 2.2. As mentioned in the introduction, the pair $(\widehat{g}, \widehat{\tau})$ will be called the associated pair. It is not necessarily defined on all of M, and it also does not determine the coefficients α, γ uniquely at every point of M. Hence, to consider $(g, \tau) \to (\widehat{g}, \widehat{\tau})$ as an involution on the set of metrics satisfying (2.4) for some smooth coefficients α and γ, one has to restrict consideration to the complement in M of $\tau^{-1}(0)$ and the closed set of points where $\nabla d\tau$ is a multiple of g. In some cases one may also consider coefficients with isolated singularities, in which case those singularities also must be excluded

To verify the involutive property, one easily checks that $\widehat{\widehat{\alpha}} = \alpha$, while $\widehat{\widehat{\gamma}} = \gamma$ follows from this, as g, τ and α determine γ uniquely. One can also check the last relation directly, using the following formulas for the two functions $\widehat{Q} = \widehat{g}(\widehat{\nabla}\widehat{\tau}, \widehat{\nabla}\widehat{\tau})$ and $\widehat{\Delta}\widehat{\tau}$:

$$\widehat{Q} = \tau^{-2}Q, \qquad \widehat{\Delta}\widehat{\tau} = n\tau^{-1}Q - \Delta\tau.$$

Remark 2.3. For any pair (g, τ) satisfying (2.4), one can produce another such pair by an affine change in τ (a nonconstant one). If this affine change involves only a change by an additive constant, it leaves equation (2.4) invariant. This freedom induces, of course, a freedom in the choice of $\widehat{\tau}$, which will be exploited in Proposition 5.1.

2.3. **Ricci solitons.** A *Ricci soliton* [9] is a Riemannian manifold $(M, \widehat{g})$ such that $\pounds_v\widehat{g} + \widehat{\mathrm{r}} = e\,\widehat{g}$ for some constant e and C^∞ vector field v on M. Here $\pounds_v$ is the Lie derivative and $\widehat{\mathrm{r}}$ denotes the Ricci tensor of $\widehat{g}$. We will only be interested in *gradient* Ricci solitons, in which M admits a C^∞ function $f : M \to \mathbb{R}$ with

$$\widehat{\nabla} df + \widehat{\mathrm{r}} = e\,\widehat{g} \quad \text{for a constant } e. \tag{2.6}$$

We will call f the *soliton function*. By a result of Perelman [11, Remark 3.2], every compact Ricci soliton $(M, \widehat{g})$ is a gradient Ricci soliton. Recall also that a metric is Einstein if its Ricci tensor is a multiple of it.

Thus a gradient Ricci soliton is nothing but a pair $(\widehat{g}, f)$ satisfying a Ricci-Hessian equation with constant coefficients. Using (2.2) and (2.3), or, more naturally, the duality of Proposition 2.1 (slightly modified to allow $\widehat{\tau}$ to be multiplied by a constant), we have

Proposition 2.4. *Let (M, g) be a Riemannian manifold of dimension $n > 2$ and τ a nonconstant C^∞ function. The Ricci soliton equation $\widehat{\nabla} d\,(b\tau^{-1}) + \widehat{\mathrm{r}} = e\,\widehat{g}$, with b a constant, holds for $\widehat{g} = \tau^{-2}g$ on $M \setminus \tau^{-1}(0)$, if and only if g satisfies a Ricci-Hessian equation (2.4) with coefficients*

$$\alpha = (n-2)\,\tau^{-1} - b\,\tau^{-2}, \quad \gamma = e\,\tau^{-2} - \tau^{-1}\Delta\tau + \big((n-1)\,\tau^{-2} - b\,\tau^{-3}\big)\,Q. \tag{2.7}$$

Remark 2.5. The introduction of the constant b is not, strictly speaking, necessary for later development, but serves to compare with the conformally Einstein case, which occurs when $b = 0$: relations (2.7) with $b = 0$ are implied by [6, (6.1) and (6.2)], which hold in that case. It follows from this that an Einstein metric cannot also satisfy a Ricci soliton equation (2.6) with the soliton function a *nonzero* multiple of τ^{-1}. Note that for other nonconstant soliton functions, this is possible.

Remark 2.6. Here and in Proposition 3.1 we briefly consider the conformal change equation for g, which yields a gradient Ricci soliton $\widehat{g}$ with an arbitrary soliton function f. An analogous calculation using equation (2.3) yields

$$\begin{aligned} \mathrm{r} \;+\; & (n-2)\tau^{-1}\nabla d\tau + \nabla df + 2\tau^{-1}\,d\tau \odot df \\ =\; & \big[e\tau^{-2} + (n-1)\tau^{-2}Q - \tau^{-1}\Delta\tau + \tau^{-1}g(\nabla\tau, \nabla f)\big]\,g. \end{aligned} \tag{2.8}$$

If $df \wedge d\tau = 0$, this gives

$$\begin{aligned} \mathrm{r} \;+\; & \big(f' + (n-2)\,\tau^{-1}\big)\,\nabla d\tau + \big(f'' + 2\tau^{-1}f'\big)\,d\tau \otimes d\tau \\ =\; & \big[e\,\tau^{-2} - \tau^{-1}\Delta\tau + \big((n-1)\tau^{-2} + \tau^{-1}f'\big)\,Q\big]\,g. \end{aligned} \tag{2.9}$$

We can conclude from this that a particular choice for f will eliminate the Hessian term. Namely, setting $f = -(n-2)\log|\tau|$, the metric $\widehat{g}$ is a Ricci soliton metric precisely when

$$\mathrm{r} - (n-2)\,\tau^{-2}\, d\tau \otimes d\tau = \left[\tau^{-2}\,(e+Q) - \tau^{-1}\Delta\tau\right] g\,.$$

However, while this equation looks quite simple, it implies that g cannot be Kähler (unless $n = 2$ or τ is constant). This is another reason, apart from duality considerations, why we will focus on the case of a soliton function proportional to τ^{-1}.

3. Ricci-Hessian equations and special Kähler-Ricci potentials

3.1. **The Ricci-Hessian equation and Ricci solitons.** Let (M, J) be a complex manifold, with J the associated almost complex structure. Suppose g is a Kähler metric on M, i.e a Riemannian metric for which J is parallel. Let (g, τ) be a pair satisfying the Ricci-Hessian equation (2.4) on M. The Kähler property implies that both g and r are Hermitian, hence so is $\nabla d\tau$ on the support of α. In many of our applications, this support will be a dense set in M. This property of $\nabla d\tau$ is equivalent to the statement that τ is a Killing potential, i.e. a C^∞ function for which $J\nabla\tau$ is a Killing vector field (cf. [6, Lemma 5.2]).

In the Kähler case, if g/τ^2 is a Ricci soliton, certain restrictions on the soliton function force it to be proportional to, or at least affine in τ^{-1}.

Proposition 3.1. *Let (M, g) be a Kähler manifold with a Killing potential τ, and $\widehat{g} = g/\tau^2$ a Ricci soliton with a τ-dependent soliton function f. Then f is an affine function in τ^{-1}.*

Proof. As (2.9) holds under our assumptions, and $d\tau \otimes d\tau$ is the only term in it that is not Hermitian, its coefficient $f'' + 2\tau^{-1} f'$ must vanish, implying the conclusion. □

Remark 3.2. If the Killing assumption above is replaced by (2.4) for some τ, α not identically zero and γ, the conclusion still follows on the support of α. If one then drops the τ-dependency assumption on the soliton function f, all that (2.4) implies, in combination with (2.8), is that $\nabla df + 2\tau^{-1} d\tau \odot df = (2\tau^2)^{-1} \pounds_{(\tau^2\nabla f)}\, g$ is Hermitian on the support of α.

We will be especially interested in the case where α and γ in (2.4) are functions of τ. We note that, this always holds for α in (2.7), while it holds for γ there if both $d\tau \wedge d\Delta\tau = 0$ and $d\tau \wedge dQ = 0$. One may attempt to weaken these assumptions using methods akin to those of [6, (6.5) and Proposition 6.4]. We choose to follow here the quicker approach of [1, §1.4], which, however, works only for $m > 2$.

Proposition 3.3. *If (2.4) holds for a Kähler metric of complex dimension $m > 2$, and $d\alpha \wedge d\tau = 0$, then $d\gamma \wedge d\tau = 0$.*

Proof. Composing (2.4) with J and applying d to the result gives $d\alpha \wedge d(\imath_{\nabla\tau}\omega/2) = (d\alpha/d\tau)\, d\tau \wedge d(\imath_{\nabla\tau}\omega/2) = d\gamma \wedge \omega$, using [6, (5.3)] (here ω is the Kähler form of g). Exterior multiplication with $d\tau$ gives $d\tau \wedge d\gamma \wedge \omega = 0$, and the result follows because the operation $\wedge\omega$ is injective on 2-forms for $m > 2$. □

Thus the coefficients of the Ricci-Hessian equation (2.7) will be functions of τ, provided (M, g, τ) is Kähler of dimension $m > 2$, and g/τ^2 is a Ricci soliton with soliton function proportional to τ^{-1}.

3.2. **Special Kähler-Ricci potentials.** Below, we denote by M_τ the complement, in a manifold M, of the critical set of a smooth function τ. For a Killing potential on a Kähler manifold, M_τ is open and dense in M.

Definition 3.4. [6] A nonconstant Killing potential τ on a Kähler manifold (M, J, g) is called a *special Kähler-Ricci potential* if, on the set M_τ, all non-zero tangent vectors orthogonal to $\nabla\tau$ and $J\nabla\tau$ are eigenvectors of both $\nabla d\tau$ and r.

We will call a metric admitting a special Kähler-Ricci potential a *s.k.r.p. metric*, and occasionally will declare (g, τ) to be a *s.k.r.p. pair*. Among the more important characteristics of such a metric is the existence of a Ricci-Hessian equation. More precisely

Proposition 3.5. [6, Corollary 9.2, Remarks 7.1 and 7.4] *Let (M, g) be a Kähler manifold of complex dimension $m \geq 2$. If (2.4) holds for some C^∞ functions α, γ and (nonconstant) τ, with $d\alpha \wedge d\tau = 0$, $d\gamma \wedge d\tau = 0$ and $\alpha\, d\alpha \neq 0$ everywhere in M_τ, then τ is a special Kähler-Ricci potential. Conversely, if (M, g) admits a Kähler-Ricci potential τ, then (2.4) holds on an open subset of M_τ, namely away from points where $\nabla d\tau$ is a multiple of g.*

Remark 3.6. In [6], we have actually written the Ricci-Hessian equation in the form $\nabla d\tau + \chi \mathrm{r} = \sigma\, g$. Note that the domains of the coefficient functions may vary as one switches between these two forms. In general, any statement involving the Ricci-Hessian equation of a s.k.r.p. metric refers to the largest domain on which (2.4) holds. Moreover, this change results in a slightly different statement of the first part of Proposition 3.5, while to get the second part (and its proof), one need only to switch r with $\nabla d\tau$ in [6, second paragraph of Remark 7.4].

Corollary 3.7. If (M, g) is Kähler, of complex dimension $m > 2$, and $\widehat{g} = g/\tau^2$ is a Ricci soliton, with soliton function $b\tau^{-1}$, where b is a constant, then τ is a special Kähler-Ricci potential.

Proof. We combine Propositions 2.4, 3.3 and 3.5, except that for α in (2.7), $\alpha\, d\alpha = 0$ on the set where $\tau = (n-2)/b$ and on the set where $\tau = 2b/(n-2)$, so that τ is a special Kähler-Ricci potential, and hence a Killing potential away from these sets. By [6, Lemma 5.2], $\nabla d\tau$ is Hermitian away from these sets, yet it is also clearly Hermitian in the interior of each of these sets, so that by continuity, it is Hermitian on all of M_τ. Again using [6, Lemma 5.2], this means that $\nabla\tau$ is holomorphic on M_τ, which implies that the interiors of the above mentioned two sets are empty. As the s.k.r.p. condition is defined by equalities, continuity now implies that τ satisfies it on all of M_τ. □

By [6, Definition 7.2, Remark 7.3], the s.k.r.p. condition on (g, τ) is equivalent to the existence, on M_τ, of an orthogonal decomposition $TM = \mathcal{V} \oplus \mathcal{H}$, with $\mathcal{V} = \mathrm{span}(\nabla\tau, J\nabla\tau)$, along with four smooth functions ϕ, ψ, λ, μ which are pointwise eigenvalues for either $\nabla d\tau$ or r, i.e., they satisfy

$$\begin{aligned} \nabla d\tau|_{\mathcal{H}} &= \phi\, g|_{\mathcal{H}}, & \nabla d\tau|_{\mathcal{V}} &= \psi\, g|_{\mathcal{V}}, \\ \mathrm{r}|_{\mathcal{H}} &= \lambda\, g|_{\mathcal{H}}, & \mathrm{r}|_{\mathcal{V}} &= \mu\, g|_{\mathcal{V}}. \end{aligned} \tag{3.1}$$

This decomposition is also r- and $\nabla d\tau$-orthogonal.

Remark 3.8. By [6, Lemma 12.5], ϕ either vanishes identically on M_τ, or never vanishes there. In the former case, g is reducible to a local product of Kähler metrics near any point (see [6, Corollary 13.2] and [7, Remark 16.4]). In the latter case, we call g a *nontrivial* s.k.r.p. metric.

Remark 3.9. For a nontrivial s.k.r.p. metric, consider $c = \tau - Q/(2\phi)$, with $Q = g(\nabla d\tau, \nabla d\tau)$, and $\kappa = \mathrm{sgn}(\phi)(\Delta\tau + \lambda\, Q/\phi)$, regarded as functions $M_\tau \to \mathbb{R}$. By [6, Lemma 12.5], c is constant on M_τ, and will be called the *s.k.r.p. constant.* In any complex dimension $m \geq 2$, we will call a nontrivial s.k.r.p. metric *standard* if κ is constant (and also use "standard s.k.r.p. pair" as a designation for (g, τ)). According to [6, §27, using (10.1) and Lemma 11.1], constancy of κ holds if $m > 2$, so that the designation "standard" involves an extra assumption as compared with "nontrivial" only when $m = 2$. The geometric meaning of κ will be recalled in §5.1.

Remark 3.10. Using Proposition 3.5, for any s.k.r.p. metric, (2.4) holds on points of M_τ for which $\phi \neq \psi$. On this subset, we regard (2.4) as an equation of operators, and equate eigenvalues to obtain $\alpha\phi + \lambda = \gamma = \alpha\psi + \mu$, so that

$$\lambda - \mu = (\psi - \phi)\alpha. \tag{3.2}$$

According to [6, Lemma 11.1a], Q, $\Delta\tau$, ϕ, ψ and μ are locally C^∞ functions of τ on M_τ. If g is a standard s.k.r.p. metric, λ is also such a function, as one concludes from the equation defining κ. Hence, by (3.2), the same holds for α on its domain of definition.

4. Associated differential equations

4.1. **The s.k.r.p. differential equation.** A number of ordinary differential equations are associated with nontrivial s.k.r.p. metrics. Special cases of these were given in [6]. They are derived below from the Ricci-Hessian equation (2.4), i.e.

$$\alpha\nabla d\tau + \mathrm{r} = \gamma\, g.$$

In the next proposition, α will be as in (2.4), ϕ as in (3.1), c and κ as in Remark 3.9 and a prime denotes the derivative operator $d/d\tau$.

Proposition 4.1. *Let (g, τ) be a s.k.r.p. pair with g nontrivial, on a manifold M of complex dimension m. Then, the equation*

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = -\mathrm{sgn}(\phi)\kappa/2. \tag{4.1}$$

holds at points of M_τ for which $\phi'(\tau)$ is nonzero. If g is standard, (4.1) is an ordinary differential equation, which, upon differentiation and division by $\tau - c$, takes the homogeneous form

$$(\tau - c)\phi''' = [(\tau - c)\alpha - m - 2]\,\phi'' + [(\tau - c)\alpha' + 2\alpha]\,\phi'. \tag{4.2}$$

A special case of equation (4.2) was important in [8], but is given here mainly for the sake of completeness. We will only be using equation (4.1).

Proof. By Remark 3.10, on M_τ, each of Q, ϕ, ψ, $\Delta\tau$ and μ is locally a function of τ. In fact, we have

$$\begin{array}{llll} \text{a)} & \psi = \phi + (\tau - c)\phi', & \text{b)} & \psi' = 2\phi' + (\tau - c)\phi'', \\ \text{c)} & \Delta\tau = 2m\phi + 2(\tau - c)\phi', & \text{d)} & \mu = -(m+1)\phi' - (\tau - c)\phi''. \end{array} \tag{4.3}$$

Namely, [6, Lemma 11.1(b)] gives $2\psi = Q'$, which yields (4.3.a) (and hence (4.3.b)), since $Q = 2(\tau - c)\phi$ due to the definition of c. Next, (4.3.c) is immediate from (4.3.a), as $\Delta\tau = \mathrm{tr}_g \nabla d\tau = 2\psi + 2(m-1)\phi$. Finally, $2\mu = -(\Delta\tau)'$ by [6, Lemma 11.1(b)], and so, differentiating (4.3.c), we obtain (4.3.d).

Next, by Remark 3.10, equation (2.4) holds on points of M_τ for which $\phi \neq \psi$. Since, using (4.3.a), the latter inequality holds when $(\tau - c)\phi' \neq 0$, and $\tau \neq c$ on M_τ (as $Q = 2(\tau - c)\phi$ and $Q > 0$ on M_τ), we see that this set consists exactly of the points of M_τ for which $\phi'(\tau)$ is nonzero.

As on this subset of M_τ, (2.4) holds, so does (3.2), which along with $Q = 2(\tau - c)\phi$ and the definitions of κ and c easily yields $\mathrm{sgn}(\phi)\kappa/2 = \Delta\tau/2 + (\tau - c)\lambda = \Delta\tau/2 + (\tau - c)[\mu + (\psi - \phi)\alpha]$. Replacing μ, ψ and $\Delta\tau$ with the expressions provided by (4.3), we get (4.1). If g is standard, κ is constant, so α is a function of τ by Remark 3.10. Hence equation (4.1) is an ordinary differential equation, and (4.2) then follows as described in the body of the proposition. $\square$

Remark 4.2. A converse statement to this result can be made, where (4.1) implies (2.4) for a standard s.k.r.p. metric, under the following extra assumptions.

Let ϕ be *globally* a function of τ, in the sense that it is the composite of τ with some C^∞ function $I' \to \mathbb{R}$ on the image interval $I' = \tau(M_\tau)$. (That I' is indeed an interval is known, see [7, §10 and §11].) Assuming ϕ', as a function of τ, is nonzero at all points of a dense subset of I', and (4.1) holds on I' for a C^∞ function $\alpha : I' \to \mathbb{R}$, it follows that (2.4) is satisfied on M_τ by $\alpha = \alpha(\tau)$ and some γ.

In fact, the assumption involving ϕ' means, as we have seen in the proof above, that (2.4) holds on a dense subset of M_τ, with some α that must coincide with the one above: they both satisfy (4.1) with the same ϕ on a dense subset of I', and hence everywhere in I'.

4.2. **The differential equations in relation to Ricci solitons.** Let (g, τ) be a standard s.k.r.p. pair on a manifold M of complex dimension m. Suppose g/τ^2 is a Ricci soliton with soliton function $b\tau^{-1}$, where b is a constant. By Proposition 2.4, equation (2.4) holds on $M \smallsetminus \tau^{-1}(0)$, with

$$\alpha = (2(m-1)\,\tau - b)/\tau^2. \tag{4.4}$$

Hence, in this case, the differential equations appearing in Proposition 4.1 take the form

$$\tau^2(\tau - c)^2\phi'' + (\tau - c)\left[m\tau^2 - (\tau - c)(2(m-1)\tau - b)\right]\phi' - m\tau^2\phi = -\mathrm{sgn}\,(\phi)\kappa\,\tau^2/2 \tag{4.5}$$

and

$$\tau^3(\tau - c)\phi''' = [(m-4)\tau^3 - (2(m-1)c + b)\tau^2 + bc\tau]\phi'' + [2(m-1)\tau(\tau + c) - 2bc]\phi'.$$

These equations hold for τ values corresponding to points of M_τ on which $\phi'(\tau)$ is nonzero.

Another ordinary differential equation is obtained on the same set as follows. The term γ in (2.4) is given, by Remark 3.10, as $\gamma = \alpha\psi + \mu$, and we substitute for ψ and μ their respective expressions (4.3.a) and (4.3.d), to obtain

$$\gamma = \alpha\phi + (\alpha(\tau - c) - (m+1))\,\phi' - (\tau - c)\phi''.$$

In the case at hand, γ also has an expression derived from the last term of (2.7), in which we replace Q by $2(\tau-c)\phi$, and $\Delta\tau$ by (4.3.c). Equating the two expressions, and replacing α by (4.4), we get after rearranging terms and multiplying by τ that

$$\begin{aligned} -\tau^3(\tau-c)\phi'' &+ [(2m\tau-b)\tau(\tau-c)-\tau^3(m+1)]\phi' \\ &+ [2(2m-1)\tau^2-b\tau+2(\tau-c)(b-(2m-1)\tau)]\phi = e\tau. \end{aligned} \tag{4.6}$$

The fact shown shortly, that (4.6) is not, in general, a consequence of (4.5), is the main local difference between the case where the s.k.r.p. metric is conformal to a non-Einstein Ricci soliton of the type we are considering, with $b \neq 0$, and the one where it is conformal to an Einstein metric ($b=0$). The latter was the object of study of [6, 7, 8].

4.3. **Solutions of the system $\{(4.5), (4.6)\}$.** To examine the solutions of the system $\{(4.5), (4.6)\}$, we note the following

Lemma 4.3. *Let $\{\phi'+p\phi=q,\ A\phi''+B\phi'+C\phi=D\}$ be a system of ordinary differential equations in the variable τ, with coefficients p, q, A, B, C and D that are rational functions. Then, on any nonempty interval admitting a solution ϕ, either*

$$A(p^2-p')-Bp+C=0 \tag{4.7}$$

holds identically, or

$$\phi = (D-A(q'-pq)-Bq)\,/\,\big(A(p^2-p')-Bp+C\big)\,. \tag{4.8}$$

holds away from the (isolated) singularities of the right hand side.

Proof. Let ϕ be a solution on an interval as above. We have $\phi'=q-p\phi$, so that $\phi''=q'-p'\phi-p\phi'=q'-p'\phi-p(q-p\phi)=(p^2-p')\phi+q'-pq$. Substituting this in the second equation, while collecting terms involving ϕ, gives

$$\big(A(p^2-p')-Bp+C\big)\,\phi+A(q'-pq)+Bq=D,$$

from which the result follows at once. ☐

To apply this lemma to $\{(4.5), (4.6)\}$, we multiply (4.6) by $(\tau-c)/\tau$ and add it to (4.5), giving a first order equation which, after multiplying by τ, simplifies to

$$\begin{aligned} \tau^2(\tau-c)(\tau-2c)\phi' &+ \big[-m\tau^3+(2(2m-1)c+b)\tau^2-(2(2m-1)c^2+3bc)\tau+2bc^2\big]\phi \\ &= -\mathrm{sgn}(\phi)\kappa\tau^3/2+e\tau(\tau-c). \end{aligned} \tag{4.9}$$

We will be applying Lemma 4.3 to the system $\{(4.9), (4.5)\}$ (modifying (4.9) appropriately). This system has, a solution set identical to that of $\{(4.5), (4.6)\}$ (certainly on intervals not containing 0, c and $2c$, and by a continuity argument, on any interval). To compute (4.7) in this case, note that p, given as a ratio of the coefficient of ϕ to that of ϕ' in (4.9), has a partial fraction decomposition of the form $p=m/(\tau-c)-1/(\tau-2c)+b/\tau^2-(2m-1)/\tau$. Similarly, $q=\mathrm{sgn}(\phi)\kappa/(2(\tau-c))+(e-2\mathrm{sgn}(\phi)\kappa c)/(2c(\tau-2c))-e/(2\tau c)$. Using $A=\tau^2(\tau-c)^2$, $B=(\tau-c)[m\tau^2-(\tau-c)(2(m-1)\tau-b)]$, $C=-m\tau^2$ and $D=-\mathrm{sgn}(\phi)k\tau^2/2$, two long but direct computations gives

$$\begin{aligned} D-A(q'-pq)-Bq &= 0, \\ A(p^2-p')-Bp+C &= -2bc(\tau-c)^2/(\tau(\tau-2c)). \end{aligned} \tag{4.10}$$

This immediately gives

Proposition 4.4. *Suppose $bc \neq 0$. Then the system $\{(4.5), (4.6)\}$ has no nonzero solutions on any nonempty open interval.*

Proof. Assume $bc \neq 0$. Then the second of equations (4.10) does not vanish identically on the given interval. Hence Lemma 4.3 implies that any solution is the ratio of the two equations in (4.10), away from the point c. This ratio is the zero function. By continuity, $\{(4.9), (4.5)\}$, as well as $\{(4.5), (4.6)\}$, admit no nonzero solutions on the given interval. □

4.4. **Solutions for the case** $c = 0$**.** If $c = 0$, equations $\{(4.5), (4.6)\}$ take the form,

$$\begin{array}{rcl} \tau^4\phi'' + [(2-m)\tau^3 + b\tau^2]\,\phi' - m\tau^2\phi &=& -\mathrm{sgn}\,(\phi)\kappa\,\tau^2/2, \\ -\tau^4\phi'' + [((m-1)\tau^3 - b\tau^2]\phi' + b\tau\phi &=& e\tau, \end{array} \tag{4.11}$$

with m a positive integer, and $b \neq 0$. As special solutions, one can take $\mathrm{sgn}(\kappa)\kappa/(2m)$ for the first, and e/b for the second. A basis of solutions to each associated homogeneous equation is given by $\{\tau^m \exp(b/\tau), \sum b^{m-l}\tau^l/(m-l)!\}$, where the sum ranges over $l = 0 \ldots m-1$ for the first, and $l = 1 \ldots m-1$ for the second. Thus, the general solution to the system has the form

$$\phi = A + B\sum_{l=1}^{m-1} \frac{b^{m-l}}{(m-l)!}\tau^l + C\tau^m \exp(\,b/\tau) \tag{4.12}$$

for arbitrary constants A, B and C (where A represents the sum of an arbitrary multiple of $b^m/m!$ with $\mathrm{sgn}(\kappa)\kappa/(2m) + e/b$).

5. Geometry and duality for s.k.r.p. metrics

5.1. **Local Geometry of s.k.r.p. metrics.** We recall here the main case in the geometric classification of s.k.r.p. metrics. Let $\pi : (L, \langle\cdot,\cdot\rangle) \to (N, h)$ be a hermitian holomorphic line bundle over a Kähler-Einstein manifold of complex dimension $m-1$. Assume that the curvature of $\langle\cdot,\cdot\rangle$ is a multiple of the Kähler form of h. Note that, if N is compact and h is not Ricci flat, this implies that L is smoothly isomorphic to a rational power of the anti-canonical bundle of N.

Consider, on $L \smallsetminus N$ (the total space of L excluding the zero section), the metric g given by

$$g|_{\mathcal{H}} = 2|\tau - c|\,\pi^* h, \quad g|_{\mathcal{V}} = \frac{Q(\tau)}{(ar)^2}\,\mathrm{Re}\,\langle\cdot,\cdot\rangle, \tag{5.1}$$

where

– $\mathcal{V}, \mathcal{H}$ are the vertical/horizontal distributions of L, respectively, the latter determined via the Chern connection of $\langle\cdot,\cdot\rangle$,

– $c, a \neq 0$ are constants,

– r is the norm induced by $\langle\cdot,\cdot\rangle$,

– τ is a function on $L \smallsetminus N$, obtained by composing on r another function, denoted via abuse of notation by $\tau(r)$, and obtained as follows: one fixes an open interval I and a positive C^∞ function $Q(\tau)$ on I, solves the differential equation $(a/Q)\,d\tau = d(\log r)$ to obtain a diffeomorphism $r(\tau) : I \to (0, \infty)$, and defines $\tau(r)$ as the inverse of this diffeomorphism.

The pair (g, τ), with $\tau = \tau(r)$, is a s.k.r.p. pair (see [6, §8 and §16], as well as [7, §4]), and $|\nabla\tau|_g^2 = Q(\tau(r))$. If g is nontrivial, the connection on L will not be flat. The constant κ of Remark 3.9 is the Einstein constant of h, so that if g is nontrivial, it is in fact standard (for an arbitrary s.k.r.p. metric, h need not be Einstein if $m = 2$). For g standard, or merely nontrivial, the s.k.r.p. constant c (see again Remark 3.9) coincides with c of (5.1).

Conversely, for any standard nontrivial s.k.r.p. metric (M, J, g, τ), any point in M_τ has a neighborhood biholomorphically isometric to an open set in some triple $(L \smallsetminus N, g, \tau(r))$ as above (this is a special case of [6, Theorem 18.1]). This biholomorphic isometry identifies span $(\nabla\tau, J\nabla\tau)$ and its orthogonal complement, with $\mathcal{V}$ and, respectively, $\mathcal{H}$. Moreover, one can extend $(g, \tau(r))$ to all of L, and then the biholomorphic isometry can also be defined on neighborhoods of points in $M \smallsetminus M_\tau$ [7, Remark 16.4].

5.2. **Duality for s.k.r.p. metrics.** By Proposition 3.5, a s.k.r.p. pair (g, τ) satisfies a Ricci-Hessian equation (2.4), on points of the noncritical set M_τ in which $\nabla d\tau$ is not a mutliple of g. On this set (with $\tau^{-1}(0)$ excluded), the involution of §2.2 yields a new pair $(\widehat{g}, \widehat{\tau})$, which also satisfies a Ricci-Hessian equation. In general, not much can be said about $\widehat{g}$. However, a special case of the affine change mentioned in Remark 2.3 involves changing τ by an additive constant. This produces a new Killing potential t, with (g, t) a s.k.r.p. pair very closely related to (g, τ). If the additive constant is chosen to be minus the s.k.r.p. constant c, applying the involution to (g, t) yields metrics which are Kähler with respect to an oppositely oriented complex structure. In fact, they are even s.k.r.p. metrics. We provide a proof of this in the following proposition, for the sake of completeness. Similar less detailed statements appear in [8, Remark 28.4] and [1, end of §5.5 and §5.6].

Proposition 5.1. *Let g be a standard s.k.r.p. metric on (M, J), with Killing potential τ and corresponding s.k.r.p. constant c. If $t = \tau - c$, then the associated pair $(\widehat{g}, \widehat{t}) = (g/t^2, 1/t)$ is a standard s.k.r.p. pair on $M \smallsetminus \tau^{-1}(c)$.*

In fact, the proof will imply that the metric $\widehat{g}$ is Kähler with respect to the complex structure $\bar{J}$ given by $\bar{J}|_{\mathcal{H}} = J|_{\mathcal{H}}$, $\bar{J}|_{\mathcal{V}} = -J|_{\mathcal{V}}$, where $\mathcal{H}$ is the orthogonal complement of $\mathcal{V} = \mathrm{span}\,(\nabla\tau, J\nabla\tau)$. This structure, defined on M_τ, extends uniquely to M (see [7, Remark 16.4]), and the corresponding extension of $\widehat{g}$ (see end of §5.1) to $M \smallsetminus \tau^{-1}(0)$ is still Kähler with respect to it.

Proof. By the classification of s.k.r.p. metrics, it is enough to consider M as a subset of the model line bundle L of §5.1. For simplicity, we take $M = L \smallsetminus N$. On L, the complex structure $\bar{J}$ defines the complex conjugate bundle structure, which we denote $\bar{L}$. We will show that the metric $\widehat{g}$ is a s.k.r.p. metric, by constructing it explicitly as in §5.1, but on the line bundle $\bar{L}$. This line bundle is smoothly isomorphic to the dual bundle L^*, and hence the construction will transfer to a holomorphic line bundle, which is one of the requirements for the data used in §5.1. The proof that such structures are Kähler is indicated in [6, §16] (or, quite efficiently, via the methods in [1]).

The metric $\widehat{g}$ is obtained from the model metric g as follows: first replace $\langle\cdot,\cdot\rangle$, a, τ and I, respectively, with the complex conjugate fiber metric $\overline{\langle\cdot,\cdot\rangle}$, the constant $\widehat{a} = -a$, the function $\widehat{t} = 1/(\tau - c)$ and the open interval $\widehat{I}$ which is the image of the decreasing diffeomorphism $I \ni \tau \to \widehat{t} \in \widehat{I}$. We then replace c by $\widehat{c} = 0$, and have

Q replaced with a function $\widehat{Q}$ which is a solution to the equation $a\,\widehat{Q}/Q = \widehat{a}\,d\widehat{t}/d\tau$. Finally, using these new data, along with h, r and $\mathcal{H}$, one defines a new standard s.k.r.p. metric exactly as in (5.1). Note that the definition of $\widehat{Q}$ guarantees that the required relation $(\widehat{a}/\widehat{Q})\,d\,\widehat{t} = d(\log r)$ holds, and positivity of $\widehat{Q}$ follows from its defining equation together with the fact that $\widehat{t}(\tau)$ is decreasing. To conclude that this standard s.k.r.p. metric is indeed $\widehat{g} = g/(\tau - c)^2$, one computes its two factors to be $2|\widehat{t}-\widehat{c}| = 2/|\tau - c|$ and $\widehat{Q}(\widehat{t})/(\widehat{a}\,r)^2 = -[Q(\tau)/(a\,r)^2]\,d\widehat{t}/d\tau = Q(\tau)/[(a\,r)^2(\tau-c)^2]$. □

Remark 5.2. In the case $c = \widehat{c}$, i.e. $c = 0$ we have $t = 1/\tau$, so that $(\widehat{g}, t) = (\widehat{g}, \widehat{\tau})$. Then, by Remark 3.9, one has $Q = 2\tau\phi$, and similarly for $\widehat{Q}$. Hence $\widehat{\phi}/\phi = \tau\widehat{Q}/(tQ) = -(\tau/t)\,dt/d\tau = -(\tau/(1/\tau))\cdot(-1/\tau^2) = 1$, i.e. $\widehat{\phi}(t) = \phi(\tau)$. The same conclusion can be reached without the use of the geometric description of s.k.r.p. metrics in §5.1, by restricting (2.2) to $\mathcal{H}$ and using (3.1) and Remark 3.9.

Remark 5.3. Still assuming $c = 0$, and using all the above conventions, suppose one fixes all the data defining g in (5.1), except for $Q = 2\tau\phi$, which varies only by changing $\phi(\tau)$ in the solution space of equation (4.1). If, in these circumstances, for each such solution, ϕ' satisfies the requirement in Remark 4.2, it follows that (2.4), and in particular, $\alpha = \alpha(\tau)$ does not vary for all these metrics. As they all share the same associated equation (4.1), the corresponding dual metrics $\widehat{g}$ also share their own associated equation (4.1), written with $t = \widehat{\tau}$ and $\widehat{\alpha}$, the latter determined as in (2.5). Since the solution space determines the coefficients of a linear differential equation, the result $\widehat{\phi}(t) = \phi(\tau)$ now implies that (4.1) for $(\widehat{g}, \widehat{\tau})$ is obtained from (4.1) of (g, τ) simply by the change of variable $\tau \to \widehat{\tau} = 1/\tau$.

6. Proof of the Theorem A

Theorem 6.1. *Given a standard s.k.r.p. pair* (g, τ)*, if* $\widehat{g} = g/\tau^2$ *is a non-Einstein Ricci soliton with soliton function a multiple of* τ^{-1}*, then* $\widehat{g}$ *is Kähler, and locally of the type given by Koiso in* [10] *(or Cao in* [4]*).*

Proof. In fact, as the pair (g, τ) is standard, the associated function ϕ cannot be identically zero (see remark 3.8). But then, as a function on the image of τ, the function ϕ solves the system $\{(4.5), (4.6)\}$ only if $bc = 0$ (here $b\tau^{-1}$ denotes, as in §4.2, the soliton function). As $\widehat{g}$ is a non-Einstein Ricci soliton, $b \neq 0$ (Remark 2.5). Hence the s.k.r.p. constant c is zero. This implies, by Proposition 5.1 and the paragraph past it, that the soliton $\widehat{g}$ is Kähler on $M \smallsetminus \tau^{-1}(0)$, with respect to a complex structure oppositely oriented to that with respect to which g is Kähler. By Remark 5.2, $\widehat{\phi}(\widehat{\tau}) = \phi(\tau)$, so that, using (4.12) and the definition of c in Remark 3.9,

$$\widehat{Q} = 2\widehat{\tau}\widehat{\phi} = \frac{2}{\tau}\left[A + B\sum_{l=1}^{m-1}\frac{b^{m-l}}{(m-l)!}\frac{1}{\tau^l} + C\frac{1}{\tau^m}\exp(\,b\tau)\right],$$

for some constants A, B, C. It is known (cf. [3, §2]) that a metric $\widehat{g}$ with the characteristics given in §5.1, and such an expression for $\widehat{Q}$, is (locally) of the form found by Koiso. □

We end with the

Proof of Theorem A. Let M be of dimension at least six, with (g, τ) a pair for which g is Kähler. If the associated pair $(\widehat{g}, \widehat{\tau})$ is a Ricci soliton, then, by Corollary 3.7, (g, τ) is a s.k.r.p. pair. (This will also hold in dimension four if Q and $\Delta\tau$ are τ-dependent, see the paragraph before Proposition 3.3.) The non-reducibilty assumption on g implies, in these dimensions, that it is a standard s.k.r.p. metric. As the metric $\widehat{g}$ cannot be Einstein by Remark 2.5, Theorem 6.1 implies that $\widehat{g}$ is Kähler (and locally of the type given by Koiso).

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