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## An Augmented Lagrangian Approach to Linearized Problems in Hydrodynamic Stability

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# AN AUGMENTED LAGRANGIAN APPROACH TO LINEARIZED PROBLEMS IN HYDRODYNAMIC STABILITY

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**Abstract.** The solution of linear systems arising from the linear stability analysis of solutions of the Navier–Stokes equations is considered. Due to indefiniteness of the submatrix corresponding to the velocities, these systems pose a serious challenge for iterative solution methods. In this paper, the augmented Lagrangian-based block triangular preconditioner introduced by the authors in [2] is extended to this class of problems. We prove eigenvalue estimates for the velocity submatrix and deduce several representations of the Schur complement operator which are relevant to numerical properties of the augmented system. Numerical experiments on several model problems demonstrate the effectiveness and robustness of the preconditioner over a wide range of problem parameters.

**Key words.** Navier–Stokes equations, incompressible flow, linear stability analysis, eigenvalues, finite elements, preconditioning, iterative methods, multigrid

**AMS subject classifications.** 65F10, 65N22, 65F50.

## 1. Introduction.

In this paper we consider the numerical solution of the following problem: Given a mean velocity field $U$, a forcing term $\mathbf{f}$, a scalar $\alpha \geq 0$ and a viscosity coefficient $\nu$, find a velocity-pressure pair $\{\mathbf{u}, p\}$ which solves

$$-\nu\Delta\mathbf{u} - \alpha\mathbf{u} + (U\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)U + \nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{1.1}$$

$$-\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \tag{1.3}$$

on a given domain $\Omega \subset \mathbb{R}^d$ (with $d = 2$ or 3). We assume $\Omega$ to be bounded and with a sufficiently smooth boundary $\partial\Omega$, except in section 4 where we briefly consider the case $\Omega = \mathbb{R}^d$. Imposing on the pressure the additional condition $\int_\Omega p\,\mathrm{d}\mathbf{x} = 0$, we assume the system to have exactly one solution.

This problem typically arises in the linear stability analysis of solutions of the Navier–Stokes equations; see, e.g., [8, Section 7.2.1]. Such analysis leads to the solution of an eigenvalue problem, in particular, to the determination of eigenvalues close to the imaginary axis. Indeed, a necessary condition for the (original) flow solution to be linearly stable is that the real parts of all the eigenvalues are negative. This type of analysis is especially useful in the determination of values of the Reynolds number above which a steady state flow becomes unstable. Shift-and-invert type methods are often used for the solution of the eigenvalue problem, leading (on the continuous level) to systems of the form (1.1)–(1.3); see, e.g., [5].

As a prototypical problem (with $U = \mathbf{0}$) for the linearized Navier–Stokes equations we also consider the following indefinite Stokes-type problem:

$$-\nu\Delta\mathbf{u} - \alpha\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{1.4}$$

$$-\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{1.5}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\,. \tag{1.6}$$

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This problem also arises in the stability analysis of the Ladyzhenskaya–Babuška–Brezzi (LBB) condition for incompressible finite elements for linear elasticity or Stokes flow; see [15]. It is worth noting that similar problems arise in other contexts as well, e.g., electromagnetism. As we shall see, the development of solvers for (1.4)–(1.6) and a good understanding of their capabilities and limitations are crucial steps towards efficient numerical solution methods for (1.1)–(1.3); these, in turn, are necessary for analyzing the spectra (or pseudo-spectra, see [22, 23]) of operators arising in fluid mechanics.

Discretization of (1.4)–(1.6) using LBB-stable finite elements (see, e.g., [8]) results in a saddle point system of the form

$$\begin{bmatrix} A-\alpha M_u & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{1.7}$$

where $A$ is the discretization of the vector Laplacian, $M_u$ the velocity mass matrix, and $B^T$ the discrete gradient. Note that $A$ is symmetric positive definite, whereas $A-\alpha M_u$ is indefinite for $\alpha > 0$ sufficiently large, making the system (1.7) difficult to solve. In the case of the full system (1.1)–(1.3), the matrix $A$ also contains the discretization of the first-order terms in (1.1), and is nonsymmetric. Again, the matrix $A-\alpha M_u$ will generally have eigenvalues on both sides of the imaginary axis, making the solution of system (1.7) by iterative methods a challenge. The present paper is devoted to the development of such methods, building on the work described in [2] for the case $\alpha = 0$.

The remainder of the paper is organized as follows. In section 2 we briefly recall the augmented Lagrangian-based block preconditioner from [2]. Section 3 is devoted to an analysis of the spectrum of the $(1,1)$ block in the augmented system corresponding to the discrete Stokes-like problem (1.7) with $A$ symmetric and positive definite. Analysis of the preconditioner also requires knowledge of the eigenvalue distribution of the Schur complement of the augmented system; some analysis of the spectrum of this operator is presented, for a few different model problems, in section 4. Numerical experiments illustrating the performance of the preconditioner are discussed in section 5. Some conclusive remarks are given in section 6.

**2. Augmented Lagrangian approach.** Here we briefly recall the augmented Lagrangian (AL) approach used in [2] for the case with $\alpha = 0$. For convenience, define $A_\alpha := A - \alpha M_u$. The original system (1.7) is replaced with the equivalent one

$$\begin{bmatrix} A_\alpha + \gamma B^T M_p^{-1} B & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{2.1}$$

where $\gamma > 0$ is a parameter and $M_p$ is a diagonal approximation to the pressure mass matrix. In our case $M_p$ is a scaled identity. We consider a block triangular preconditioner of the form

$$\mathcal{P} = \begin{bmatrix} \hat{A}_\alpha & B^T \\ 0 & \hat{S} \end{bmatrix}. \tag{2.2}$$

Here the matrix $\hat{A}_\alpha$ is not given explicitly; rather, $\hat{A}_\alpha^{-1}$ represents an inexact solver for linear systems involving the matrix $A_\alpha + \gamma B^T W^{-1} B$. For the case $\alpha = 0$, excellent results were obtained in [2] with a multigrid iteration based on a method due to Schöberl [21]. We discuss the approximate multigrid solver $\hat{A}_\alpha^{-1}$ in the section on numerical experiments. For the choice of $\hat{S}$ we consider two possibilities: a simple scaled mass matrix preconditioner

$$\hat{S}^{-1} := -(\nu+\gamma)\hat{M}_p^{-1}\,, \tag{2.3}$$

and one which takes into account the presence of the $\alpha$-term, namely

$$\hat{S}^{-1} := -(\nu+\gamma)\hat{M}_p^{-1} + \alpha\big(B\hat{M}_u^{-1}B^T\big)^{-1}, \tag{2.4}$$

where $\hat{M}_u$ is a diagonal approximation to the velocity mass matrix. Note that $B\hat{M}_u^{-1}B^T$ can be seen as a mixed approximation to the pressure Poisson problem with Neumann boundary conditions and that (2.4) resembles the Cahouet–Chabard Schur complement preconditioner [4] initially proposed for the instationary Stokes problem. However, since the reactive term in (1.4) is now *negative*, the $\alpha$-term enters (2.4) with the opposite sign compared to the Cahouet–Chabard preconditioner.

The block triangular preconditioner (2.2) can be used with any Krylov subspace method for nonsymmetric linear systems, such as GMRES [20] or BiCGStab [25]; if, however, the action of $\hat{A}_\alpha^{-1}$ or of $\hat{S}^{-1}$ is computed via a nonstationary inner iteration, a flexible variant (such as FGMRES [19]) must be used.

We note that some preliminary experiments with a block triangular preconditioner (2.2) for systems of the form (1.7) arising from marker-and-cell (MAC) discretizations of flow problems can be found in [1]. The results in [1] show the good performance of the AL-based approach, especially in terms of robustness with respect to problem and algorithmic parameters. In that paper, however, the crucial question of how to efficiently approximate the action of $(A_\alpha + \gamma B^T W^{-1} B)^{-1}$ was left open. In this paper we propose some reasonably effective ways to address this difficult problem.

**3. Eigenvalue estimates.** In this section we analyze the eigenvalues of the submatrix $A + \gamma B^T W^{-1} B$ in the augmented problem (2.1) corresponding to the Stokes problem (with $\alpha = 0$). Information about its eigenvalue distribution is of interest since it helps to understand the performance of the (inexact) multigrid solver for the (1,1) block of (2.1), which is an essential component of the entire approach. In particular, we will show that under certain assumptions the eigenvalues of the problem:

$$(A + \gamma B^T W^{-1} B)u = \lambda_\gamma\, u \tag{3.1}$$

tend for $\gamma \to \infty$ to the (generalized) eigenvalues of the problem

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \lambda\, u \\ 0 \end{bmatrix}. \tag{3.2}$$

To show this we need the following assumptions. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite, i.e.,

$$A = A^T \qquad \text{and} \qquad c_1 I \le A \tag{3.3}$$

with some $c_1 > 0$. In (3.3), we have used "$\le$" to denote the usual positive semidefinite ordering. Let $B \in \mathbb{R}^{m\times n}$. Assume that the matrix $S = -BA^{-1}B^T$ is also nonsingular. Owing to (3.3), this is equivalent to assuming that

$$S = S^T \qquad \text{and} \qquad c_2 I \le -S\,, \tag{3.4}$$

with some $c_2 > 0$. Note that these two assumptions together imply that the block matrix on the left-hand side of (3.2) is also nonsingular and $n \geq m$. Also assume that $W \in \mathbb{R}^{m\times m}$ is symmetric positive definite. Finally, let $c_3$ be a positive constant from the estimate

$$\|Bv\| \leq c_3 \|A^{\frac{1}{2}} v\| \qquad \forall\, v \in \mathbb{R}^n\,. \tag{3.5}$$

We note that the above estimate is quite natural for the Stokes problem, where $A$ is a discrete vector Laplacian and $B$ a discrete divergence.

With the above assumptions the main result of this section is the following theorem on the generalized eigenvalues of (3.2). Here and throughout the paper, the matrix norm used is the spectral norm.

THEOREM 3.1. *The problem* (3.2) *has* $n-m$ *real finite eigenvalues* $0 < \lambda_1 \leq \cdots \leq \lambda_{n-m}$. *There are* $n-m$ *eigenvalues* $0 < \lambda_{\gamma,1} \leq \cdots \leq \lambda_{\gamma,n-m}$ *of* (3.1) *such that*

$$|\lambda_k^{-1} - \lambda_{\gamma,k}^{-1}| \leq C\,\gamma^{-1}\|W\| \tag{3.6}$$

*with* $C = (1 + c_1^{-\frac{1}{2}} c_3)^2 c_2^{-2}$. *The remaining* $m$ *eigenvalues of* (3.1) *can be estimated from below as*

$$\lambda_{\gamma,k} \geq C^{-1}\,\gamma \|W\|^{-1}. \tag{3.7}$$

*Proof.* From the assumption (3.4) we conclude that $B$ has full rank and thus $\dim(\ker(B)) = n-m$. Let $P : \mathbb{R}^n \to \ker(B)$ be the orthogonal projector. The problem (3.2) is equivalent to the eigenvalue problem: $PAu = \lambda u$ for $u \in \ker(B)$. Since the operator $PA$ is self-adjoint and positive definite on the kernel of $B$, the problem has $n-m$ positive real eigenvalues.

Denoting $p = Bu$ we rewrite (3.1) as

$$\begin{bmatrix} A & B^T \\ B & -\gamma^{-1} W \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \lambda_\gamma\, u \\ 0 \end{bmatrix}. \tag{3.8}$$

We will also use the following notations for the block matrices:

$$\mathcal{A}_\gamma := \begin{bmatrix} A & B^T \\ B & -\gamma^{-1} W \end{bmatrix}, \qquad \mathcal{I}_\delta := \begin{bmatrix} I_n & 0 \\ 0 & \delta\, I_m \end{bmatrix}.$$

Letting $\mu_k = \lambda_k^{-1}$, $\mu_{\gamma,k} = \lambda_{\gamma,k}^{-1}$, and

$$\mathcal{A} := \mathcal{A}_\infty = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix},$$

we can rewrite (3.2) and (3.8) in the form

$$\mathcal{A}^{-1}\mathcal{I}_0 x = \mu x, \qquad \mathcal{A}_\gamma^{-1}\mathcal{I}_0 x = \mu_\gamma x. \tag{3.9}$$

All eigenvalues of (3.9) are real and non-negative. Positive $\mu_k$ and $\mu_{\gamma,k}$ correspond to finite real eigenvalues of (3.2) and (3.8), while zero $\mu_k$ and $\mu_{\gamma,k}$ correspond to infinite eigenvalues of (3.2) and (3.8). Proving (3.6) and (3.7) is equivalent to showing the upper bound

$$|\mu_k - \mu_{\gamma,k}| \leq C\,\gamma^{-1}\|W\| \tag{3.10}$$

with $C = (1 + c_1^{-\frac{1}{2}} c_3)^2 c_2^{-2}$ for all eigenvalues of $\mathcal{A}^{-1}\mathcal{I}_0$ and $\mathcal{A}_\gamma^{-1}\mathcal{I}_0$. Consider the following auxiliary eigenvalue problem:

$$\mathcal{A}_\gamma^{-1}\mathcal{I}_\delta x = \mu_\gamma^\delta x \tag{3.11}$$

with some $\delta > 0$. As before the case $\gamma = \infty$ will denote the matrix $\mathcal{A}_\infty = \mathcal{A}$ with zero (2,2) block. The matrix $\mathcal{A}_\gamma^{-1}\mathcal{I}_\delta$ is nonsingular and self-adjoint in the $\langle \mathcal{I}_\delta \cdot, \cdot \rangle$ scalar product. By the triangle inequality we get:

$$|\mu_k - \mu_{\gamma,k}| \le |\mu_k - \mu_{\infty,k}^\delta| + |\mu_{\infty,k}^\delta - \mu_{\gamma,k}^\delta|. \tag{3.12}$$

For the second term on the right-hand side of (3.12) we will prove the estimate

$$|\mu_{\infty,k}^\delta - \mu_{\gamma,k}^\delta| \le C\,\gamma^{-1}\|W\| \tag{3.13}$$

with $C = (1 + c_1^{-\frac{1}{2}} c_3)^2 c_2^{-2}$ *independent* of $\delta$ (for small enough values $\delta$). Since the eigenvalues are continuous functions of the matrix elements, the first term on the right-hand side of (3.12) vanishes as $\delta \to 0$. Therefore, passing to the limit in (3.12) with $\delta \to 0$ we obtain the desired bound (3.10).

It remains to prove (3.13). Consider the block factorization

$$\mathcal{A}_\gamma^{-1} = \begin{bmatrix} I_n & -A^{-1}B^T \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S_\gamma^{-1} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -A^{-1}B^T & I_m \end{bmatrix},$$

where $S_\gamma = -BA^{-1}B^T - \gamma^{-1}W$. Using this factorization we obtain

$$\|\mathcal{A}^{-1} - \mathcal{A}_\gamma^{-1}\| \le (1 + \|A^{-1}B^T\|)^2 \|S^{-1} - S_\gamma^{-1}\| \tag{3.14}$$

for any $\gamma \in (0, \infty]$. Using (3.3) and (3.5) we immediately get $\|A^{-1}B^T\| \le c_1^{-\frac{1}{2}} c_3$. For the last term in (3.14) we obtain

$$\begin{aligned} \|S^{-1} - S_\gamma^{-1}\| \le \|S^{-1}S_\gamma - I_m\| \|S_\gamma^{-1}\| &= \gamma^{-1}\|S^{-1}W\|\|S_\gamma^{-1}\| \\ &\le \gamma^{-1}\|S^{-1}\|\|S_\gamma^{-1}\|\|W\| \le \gamma^{-1}c_2^{-2}\|W\|. \end{aligned} \tag{3.15}$$

In the last inequality we used the symmetry and positive definiteness of $W$ and (3.4) to conclude that $c_2 I \le -S \le -S + \gamma W = -S_\gamma$ and thus $\|S^{-1}\| \le c_2^{-1}$ and $\|S_\gamma^{-1}\| \le c_2^{-1}$. Substituting the bound (3.15) into (3.14) we get

$$\|\mathcal{A}^{-1} - \mathcal{A}_\gamma^{-1}\| \le (1 + c_1^{-\frac{1}{2}} c_3)^2 c_2^{-2} \gamma^{-1} \|W\|. \tag{3.16}$$

The Courant–Fischer Theorem gives for the $k$-th eigenvalue of problem (3.11) the characterization

$$\mu_{\gamma,k}^\delta = \max_{S \in \mathcal{V}_{k-1}} \min_{0 \ne y \in S^\perp} \frac{\langle \mathcal{A}_\gamma^{-1}\mathcal{I}_\delta y, \mathcal{I}_\delta y\rangle}{\langle \mathcal{I}_\delta y, y\rangle},$$

where $\mathcal{V}_{k-1}$ denotes the family of all $(k-1)$-dimensional subspaces of $\mathbb{R}^{n+m}$. Using $\min_y(a(y) + b(y)) \le \min_y a(y) + \max_y b(y)$ we estimate, assuming $\delta \in (0, 1]$:

$$\begin{aligned} \mu_{\infty,k}^\delta - \mu_{\gamma,k}^\delta &\le \max_{S \in \mathcal{V}_{k-1}} \max_{y \in S^\perp} \frac{\langle (\mathcal{A}^{-1} - \mathcal{A}_\gamma^{-1})\mathcal{I}_\delta y, \mathcal{I}_\delta y\rangle}{\langle \mathcal{I}_\delta y, y\rangle} \le \max_{y \in \mathbb{R}^{n+m}} \frac{\langle (\mathcal{A}^{-1} - \mathcal{A}_\gamma^{-1})\mathcal{I}_\delta y, \mathcal{I}_\delta y\rangle}{\langle \mathcal{I}_\delta y, y\rangle} \\ &\le \|\mathcal{A}^{-1} - \mathcal{A}_\gamma^{-1}\| \max_{y \in \mathbb{R}^{n+m}} \frac{\langle \mathcal{I}_\delta y, \mathcal{I}_\delta y\rangle}{\langle \mathcal{I}_\delta y, y\rangle} = \|\mathcal{A}^{-1} - \mathcal{A}_\gamma^{-1}\| \le (1 + c_1^{-\frac{1}{2}} c_3)^2 c_2^{-2} \gamma^{-1}\|W\|. \end{aligned}$$

One can estimate the difference $\mu^\delta_{\gamma,k} - \mu^\delta_{\infty,k}$ in the same way, so we have the desired bound on $|\mu^\delta_{\infty,k} - \mu^\delta_{\gamma,k}|$, i.e., inequality (3.13). The theorem is proved. □

REMARK 3.2. Assume that the matrix on the left-hand side of (3.2) results from an LBB-stable finite element (or finite difference) discretization of the Stokes problem and $W$ is the diagonal approximation to the mass matrix (or the $n \times n$ identity). Then the assumptions of Theorem 3.1 are satisfied. Depending on the boundary conditions for the Stokes problem, the matrix $S$ may have a one-dimensional kernel. This singularity of $S$ can be overcome by restricting the discrete pressure to lie in the subspace of all functions $p_h$ satisfying $(p_h, 1) = 0$. Moreover, if the mesh is quasi-uniform then for finite element discretization one has in (3.6) and (3.7) that $C\|W\| = O(h^{-d})$, where $h$ denotes the mesh size. Indeed, it can be easily shown that $c_1 = O(h^d)$ , $c_2 = O(h^d)$, $c_3 = O(h^{\frac{d}{2}})$, and $\|W\| = O(h^d)$. For (MAC finite difference discretizations, one has $C\|W\| = O(1)$; the same holds true for finite elements if the problem is scaled in such a way that $\lambda_1 = O(1)$ and $\lambda_{1,\gamma} = O(1)$.

REMARK 3.3. Two conclusions can be based on Theorem 3.1. First of all, solving (3.1) for large enough $\gamma$ can be used as a penalty method for finding the eigenvalues of the saddle-point problem (3.2). The theorem shows convergence of the first order with respect to the small parameter $\gamma^{-1}$ for the eigenvalues. In the literature one can find results on the first order convergence of the solution of the penalized problem to the solution of saddle point problem with zero (2,2) block, e.g., [3, 16] and [11, Thm. 7.2], but—to our knowledge—no result about eigenvalue convergence was known. Secondly, the $k$-th eigenvalue of the augmented problem (3.1) is in general larger then the $k$-th eigenvalue of the non-augmented problem ($\gamma = 0$), since

$$\max_{S\in\mathcal{V}_{k-1}} \min_{0\neq y\in S^\perp} \frac{\langle (A + \gamma BW^{-1}B^T)y, y\rangle}{\langle y, y\rangle} \geq \max_{S\in\mathcal{V}_{k-1}} \min_{0\neq y\in S^\perp} \frac{\langle Ay, y\rangle}{\langle y, y\rangle} .$$

Therefore, for a fixed $\alpha$ the problem $(A + \gamma BW^{-1}B^T - \alpha I_n)y = f$ is in general "less indefinite" for $\gamma > 0$. This property is advantageous for the multigrid solves. For example, in the case of the Stokes problem with Dirichlet boundary conditions in the unit square it is known that (after appropriate scaling) $\lambda_{\min}(A) \approx 2\pi^2 \approx 20$, whereas for the minimal eigenvalue of (3.2) one has $\lambda_{\min} \approx 52.3$; see [12, Section 36.3].

**4. Analysis of the Schur complement for the augmented system.** In [2] it was shown that the clustering of the eigenvalues of the augmented matrix in (2.1) preconditioned by the matrix $\mathcal{P}$ in (2.2) depends on the distribution of the eigenvalues of the Schur complement $B(A_\alpha + \gamma B^TWB)^{-1}B^T$. (Although the analysis in [2] was done for $\alpha = 0$, the same holds true for $\alpha \neq 0$.) In this section we study the spectrum of the Schur complement operator for several model problems.

**4.1. Analysis for Stokes-type problem.** Consider the indefinite Stokes periodic problem in two or three space dimensions with an additional "grad-div" term (augmentation):

$$\begin{aligned} -\nu\Delta\mathbf{u} - \alpha\mathbf{u} - \gamma\nabla\mathrm{div}\,\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \mathbb{R}^d, \\ -\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \mathbb{R}^d. \end{aligned}$$

Denote by $\mathcal{S}_\gamma$ the Schur complement operator for this problem. For the harmonic $q(\mathbf{x}) = \exp(\mathrm{i}(\mathbf{c}\cdot\mathbf{x}))$, where $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{c} \in \mathbb{N}^d$, $d = 2, 3$, it is straightforward to compute

$$(\nu + \gamma)\mathcal{S}_\gamma q = \lambda_{\mathbf{c}} q \qquad \text{with} \quad \lambda_{\mathbf{c}} = \frac{(\nu+\gamma)|\mathbf{c}|^2}{(\nu+\gamma)|\mathbf{c}|^2 - \alpha} , \tag{4.1}$$

where $|\mathbf{c}|^2 = \mathbf{c}\cdot\mathbf{c}$. It is clear that for large enough $|\mathbf{c}|$ we have $\lambda_{\mathbf{c}} \approx 1$. Let us estimate the number of "poor" eigenvalues, such that $|\lambda_{\mathbf{c}}| \le \varepsilon$ or $|\lambda_{\mathbf{c}}| \ge \varepsilon^{-1}$ for some (reasonably small) $\varepsilon \in (0,1)$. It is easy to check that

$$|\lambda_{\mathbf{c}}| \le \varepsilon \;\Leftrightarrow\; |\mathbf{c}|^2 \le \frac{\alpha}{(\varepsilon^{-1}+1)(\nu+\gamma)},$$
$$|\lambda_{\mathbf{c}}| \ge \varepsilon^{-1} \;\Leftrightarrow\; (1-\varepsilon) \le |\mathbf{c}|^2\alpha^{-1}(\nu+\gamma) \le (1+\varepsilon).$$

Thus, assuming $\nu$ and $\varepsilon$ are fixed we have that the number of "poor" eigenvalues is $O(\sqrt{\alpha/\gamma})$ for $\alpha,\gamma\to\infty$.

**4.2. Analysis for non-periodic problem.** Consider the augmented indefinite Stokes problem in two or three dimensions with nonstandard boundary conditions in a bounded domain $\Omega$ with Lipschitz boundary $\partial\Omega$:

$$-\nu\Delta\mathbf{u} - \alpha\mathbf{u} - \gamma\nabla\mathrm{div}\,\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{4.2}$$
$$-\mathrm{div}\,\mathbf{u} = g \quad \text{in } \Omega, \tag{4.3}$$
$$\mathbf{u}{\cdot}\mathbf{n} = 0, \quad \mathbf{n}\times\mathrm{rot}\,\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{4.4}$$

where $\mathbf{n}$ is a normal vector on $\partial\Omega$. For $\Omega\subset\mathbb{R}^2$ the boundary conditions are slightly different: $\mathbf{u}{\cdot}\mathbf{n} = \mathrm{rot}\,\mathbf{u} = 0$. For $\gamma = 0$ and $\alpha = 0$ (or if the term $\alpha\mathbf{u}$ enters the momentum equation with the positive sign) the well-posedness of the problem was shown in [9, 17]. Denote by $\mathcal{S}_\gamma$ the Schur complement operator for this problem. Using the technique developed in [17] one obtains the following representation for $\mathcal{S}_\gamma^{-1}$.

LEMMA 4.1. *Assume that the problem* (4.2)–(4.4) *is well-posed, then*

$$\mathcal{S}_\gamma^{-1} = (\nu+\gamma)I + \alpha\Delta_N^{-1} \quad \text{on } L_0^2(\Omega), \tag{4.5}$$

*where* $\Delta_N^{-1}$ *is the solution operator for the Poisson problem with Neumann boundary conditions.*

*Proof.* Define the following space:

$$\mathbf{V} = \mathbf{H}_0(\mathrm{div}\,)\cap\mathbf{H}(\mathrm{rot}\,)$$
$$= \{\mathbf{u}\in\mathbf{L}^2(\Omega)\,|\,\mathrm{div}\,\mathbf{u}\in\mathbf{L}^2(\Omega),\ \mathrm{rot}\,\mathbf{u}\in\mathbf{L}^2(\Omega)^{2d-3},\ \mathbf{u}{\cdot}\mathbf{n}|_{\partial\Omega}=0\}.$$

We note that properties of $\mathbf{V}$ are well studied in [10]. The weak form of (4.2)–(4.4) reads (cf. [17]): Given $\mathbf{f}\in\mathbf{V}^{-1}$, $g\in L_0^2$ find $\{\mathbf{u},p\}$ in $\mathbf{V}\times L_0^2$ satisfying

$$(\nu+\gamma)(\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v}) + \nu(\mathrm{rot}\,\mathbf{u},\mathrm{rot}\,\mathbf{v}) - \alpha(\mathbf{u},\mathbf{v}) - (p,\mathrm{div}\,\mathbf{v}) - (\mathrm{div}\,\mathbf{u},q) = \langle\mathbf{f},\mathbf{v}\rangle + (g,q)$$

for any $\{\mathbf{v},q\}$ in $\mathbf{V}\times L_0^2$. Here, as usual, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{V}$ and $\mathbf{V}^{-1}$. The well-posedness of (4.2)–(4.4) implies that the Schur complement operator is nonsingular. For arbitrarily given $r\in L_0^2$, consider

$$p = \mathcal{S}_\gamma^{-1} r\,. \tag{4.6}$$

In weak form, equality (4.6) can be written as

$$(\nu+\gamma)(\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v}) + \nu(\mathrm{rot}\,\mathbf{u},\mathrm{rot}\,\mathbf{v}) - \alpha(\mathbf{u},\mathbf{v}) + (p,\mathrm{div}\,\mathbf{v}) - (\mathrm{div}\,\mathbf{u},q) = (r,q) \tag{4.7}$$

for any $\{\mathbf{v}, q\}$ in $\mathbf{V} \times L_0^2$ with some auxiliary velocity $\mathbf{u} \in \mathbf{V}$. Using $-\mathrm{div}\,\mathbf{u} = r$ in (4.7) one gets

$$-(\nu+\gamma)(r, \mathrm{div}\,\mathbf{v}) + \nu(\mathrm{rot}\,\mathbf{u}, \mathrm{rot}\,\mathbf{v}) - \alpha(\mathbf{u}, \mathbf{v}) = -(p, \mathrm{div}\,\mathbf{v})\,. \tag{4.8}$$

Furthermore, for arbitrary $\psi \in L_0^2$ consider $\mathbf{v} = \nabla\Delta_N^{-1}\psi$. Note that $\mathbf{v} \in \mathbf{V}$ and $\mathrm{rot}\,\mathbf{v} = 0$. Substituting this into (4.8) we get

$$-(\nu+\gamma)(r, \psi) - \alpha(r, \Delta_N^{-1}\psi) = -(p, \psi) \quad \text{for all } \psi \in L_0^2. \tag{4.9}$$

Relation (4.9) is the weak form of $(\nu+\gamma)r + \alpha\Delta_N^{-1}r = p$. Since the function $r \in L_0^2$ in (4.6) was taken to be arbitrary, equalities (4.6) and (4.9) yield (4.5). □

In [17] the relation (4.5) was shown for the case of $\gamma = 0$ and $\alpha\mathbf{u}$ entering the momentum equation with the positive sign (in that case the second term in (4.5) should be subtracted). Representation (4.5) shows that the formula (4.1) can be extended to the non-periodic problem by replacing $|\mathbf{c}|$ with the eigenvalues of the Poisson problem with Neumann boundary conditions.

**4.3. Analysis for Oseen type problem in unbounded domain.** The analysis in subsection 4.1 can be extended to the case of nonsymmetric problems posed in $\mathbb{R}^d$ under certain (standard) assumptions. Assume that the mean flow $U$ in the linearized Navier–Stokes problem is constant. In this case the term $(\mathbf{u}\cdot\nabla)U$ vanishes and the augmented equations read:

$$-\nu\Delta\mathbf{u} - \alpha\mathbf{u} - \gamma\nabla\mathrm{div}\,\mathbf{u} + (U\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \mathbb{R}^d, \tag{4.10}$$
$$-\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \mathbb{R}^d, \tag{4.11}$$
$$\mathbf{v} \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \tag{4.12}$$

For the definite case ($\alpha \le 0$), the system (4.10)–(4.12) is also known as the Oseen problem and the proper weak formulation of the problem can be found in [14].

Let us consider further the following problem: Given $f \in C^\infty(\mathbb{R}^d)$, $d = 2, 3$, with compact support find $\mathbf{u} \in H^1(\mathbb{R}^d)^d$, $p \in L^2(\mathbb{R}^d)$ satisfying in the weak sense:

$$-\nu\Delta\mathbf{u} - \alpha\mathbf{u} - \gamma\nabla\mathrm{div}\,\mathbf{u} + (U\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \mathbb{R}^d, \tag{4.13}$$
$$-\mathrm{div}\,\mathbf{u} = f \quad \text{in } \mathbb{R}^d, \tag{4.14}$$
$$\mathbf{v} \to 0 \quad \text{as } |\mathbf{x}| \to \infty. \tag{4.15}$$

We note that problem (4.13)–(4.15) can be interpreted as finding the pressure $p$ satisfying $S_\gamma^\infty\, p = f$, where $S_\gamma^\infty$ defines the pressure Schur complement operator for the problem (4.10)–(4.12).

LEMMA 4.2. *Assume that the problem* (4.13)–(4.15) *has a unique solution. Then*

$$p = \left(-(\nu+\gamma)\Delta + \alpha + (U\cdot\nabla)\right)\mathcal{G} * f, \tag{4.16}$$

*where* $\mathcal{G}(\mathbf{x}) = (4\pi)^{-1}|\mathbf{x}|^{-1}$ *for* $d = 3$ *and* $\mathcal{G}(\mathbf{x}) = (2\pi)^{-1}\ln|\mathbf{x}|^{-1}$ *for* $d = 2$ *is the Green's function (fundamental solution) for the Laplace operator, and* $*$ *stands for convolution.*

*Proof.* By definition, the fundamental solution $\{\mathbf{v}, q\}$ for (4.13)–(4.15) satisfies:

$$-\nu\Delta\mathbf{v} - \alpha\mathbf{v} - \gamma\nabla\mathrm{div}\,\mathbf{v} - (U\cdot\nabla)\mathbf{v} + \nabla q = \mathbf{0} \quad \text{in } \mathbb{R}^d, \tag{4.17}$$
$$-\mathrm{div}\,\mathbf{v} = \delta_{\mathbf{0}}(\mathbf{x}) \quad \text{in } \mathbb{R}^d, \tag{4.18}$$
$$\mathbf{v} \to 0, \quad q \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{4.19}$$

where $\delta_{\mathbf{0}}$ stands for the Dirac's delta at the origin. Denote by $\tilde{f}$ the Fourier transform of $f$:

$$\tilde{f}(\mathbf{c}) = \int_{\mathbb{R}^d} f(\mathbf{x}) \exp(-\mathrm{i}\mathbf{c}\cdot\mathbf{x})\, \mathrm{d}\mathbf{x}\,.$$

One easily finds from (4.17)–(4.19)

$$\nu|\mathbf{c}|^2\tilde{v}_i - \alpha\tilde{v}_i + \gamma c_i(\mathbf{c}\cdot\tilde{\mathbf{v}}) - \mathrm{i}(U\cdot\mathbf{c})\tilde{v}_i + \mathrm{i}c_i\tilde{q} = 0 \quad \text{for } i = 1,\dots,d, \qquad \mathbf{c}\cdot\tilde{\mathbf{v}} = \mathrm{i}.$$

Solving for $\tilde{q}$, we obtain

$$\tilde{q} = ((\nu+\gamma)|\mathbf{c}|^2 - \alpha - \mathrm{i}(U\cdot\mathbf{c}))|\mathbf{c}|^{-2}.$$

The inverse Fourier transform gives

$$q = (-(\nu+\gamma)\Delta + \alpha + (U\cdot\nabla))\,\mathcal{G}(\mathbf{x}).$$

Denote by $\mathcal{L}$ the differential operator on the left-hand side of (4.13)–(4.14) and by $\mathcal{L}^*$ its adjoint. A standard argument yields

$$p = \{\mathbf{0}, \delta_{\mathbf{0}}(\mathbf{x})\} * \{\mathbf{u}, p\} = \mathcal{L}^*\{\mathbf{v}, q\} * \{\mathbf{u}, p\} = \{\mathbf{v}, q\} * \mathcal{L}\{\mathbf{u}, p\} = q * f.$$

The proof is complete. □

A similar technique as in Lemma 4.2 was used by Kay, Loghin, and Wathen in [13] to construct a preconditioner for the discrete Schur complement of the Oseen problem. An analogous result can be obtained with even simpler arguments by treating (4.10)–(4.11) as a periodic problem and evaluating $\mathcal{S}_\gamma q$ for the given harmonic $q(\mathbf{x}) = \exp(\mathrm{i}(\mathbf{c}\cdot\mathbf{x}))$, where $\mathbf{x}\in\mathbb{R}^d$, $\mathbf{c}\in\mathbb{N}^d$, $d = 2, 3$. Straightforward computations give[1]

$$\mathcal{S}_\gamma q = \lambda_{\mathbf{c}} q \qquad \text{with} \quad \lambda_{\mathbf{c}} = \frac{|\mathbf{c}|^2}{(\nu+\gamma)|\mathbf{c}|^2 - \alpha + \mathrm{i}(U\cdot\mathbf{c})}\;. \tag{4.20}$$

As one may expect, the $\lambda_c^{-1}$'s in (4.20) are the eigenvalues of the periodic counterpart of the operator on the right-hand side of (4.16).

The expression for the eigenvalues $\lambda_{\mathbf{c}}$, as well as relations (4.5) and (4.16), show that *for the Schur complement, $\gamma$ plays the same role as the viscosity $\nu$.* This explains why setting $\gamma = O(1)$ (assuming $\|U\| = O(1)$ and $\alpha = O(1)$) is sufficient for providing convergence rates independent of Reynolds number if sufficiently accurate solvers (or preconditioners) are used for the (1,1) block. This effect is not recovered by the purely algebraic analysis of the augmented system (Theorem 4.2 in [2]), where, under similar assumptions, $\nu$-independent bounds for the Schur complement of the Oseen system were proved only for $\gamma = O(\nu^{-1})$. Furthermore, the analysis of the periodic problem in subsection 4.1 shows that increasing $\gamma$ leads to a reduction in the number of "poor" eigenvalues of the Schur complement.

[1] The simplest way to show (4.20) is to look for $\mathbf{v}$ solving $-\nu\Delta\mathbf{v} - \alpha\mathbf{v} - \gamma\nabla\mathrm{div}\,\mathbf{v} + (U\cdot\nabla)\mathbf{v} = q(\mathbf{x}) = \exp(\mathrm{i}(\mathbf{c}\cdot\mathbf{x}))$ in the form $\mathbf{v} = \mathbf{k}\exp(\mathrm{i}(\mathbf{c}\cdot\mathbf{x}))$. This gives a $2\times 2$ or $3\times 3$ system for the vector $\mathbf{k}$. Solving this system for $\mathbf{k}$, one finds $\mathcal{S}_\gamma q = \mathrm{div}\,\mathbf{v} = \mathrm{i}(\mathbf{c}\cdot\mathbf{k})q$.

Table 5.1
*Results for the indefinite Helmholtz-type problem.*

| $\alpha$ and $h$ | parameter $\gamma$ | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | 10 | $10^2$ | $10^3$ |
| $\alpha = 100$ | | | | | |
| 1/256 | 4 (12s) | 6 (19s) | 5 (16s) | 7 (22s) | 7 (22s) |
| 1/512 | 4 (52s) | 6 (79s) | 5 (65s) | 7 (92s) | 7 (94s) |
| $\alpha = 400$ | | | | | |
| 1/256 | 4 (14s) | 6 (14s) | 7 (24s) | 6 (21s) | 6 (21s) |
| 1/512 | 4 (54s) | 4 (54s) | 5 (64s) | 4 (54s) | 6 (81s) |
| $\alpha = 1600$ | | | | | |
| 1/256 | 21 (101s) | 157 (792s) | 14 (68.6s) | 9 (44s) | 25 (124s) |
| 1/512 | 10 (148s) | 26 (394s) | 9 (134s) | 7 (104s) | 10 (150s) |
| $\alpha = 6400$ | | | | | |
| 1/256 | $> 200$ | $> 200$ | 115 (1247s) | 20 (214s) | 74 (798s) |
| 1/512 | $> 200$ | $> 200$ | 25 (520s) | 13 (269s) | 67 (1442s) |

Number of preconditioned FGMRES iterations and CPU times in seconds.

**5. Numerical experiments.** In our numerical experiments we use iso$P_2$-$P_0$ or iso$P_2$-$P_1$ finite elements on uniform grids. In all experiments for the Stokes-like problem we set $\nu = 1$. First we tested a multigrid method to solve a linear system of equations with the matrix $A_\alpha + \gamma B^T M_p^{-1} B$ from the (1,1) block in matrix (2.1). We need this multigrid further to define the preconditioner $\hat{A}$ in (2.2). Note that the problem in the (1,1) block can be interpreted as a discrete Helmholtz-type problem augmented with the term $\gamma B^T M_p^{-1} B$. To solve the system we consider a multigrid V(1,1) cycle. Since the problem is indefinite, we have to ensure the coarsest mesh is fine enough. We use the same criterion as the one suggested in [6]: Perform the coarsening while the mesh size satisfies $h\sqrt{\alpha} \le 0.5$. If the inequality fails to hold, then the mesh is treated as the coarsest one. For smoother we consider a block Gauss–Seidel method similar to those proposed by Schöberl in [21] for the linear elasticity problem; see also [2]. The restriction operator is canonical, and the prolongation operator is the one based on the solution of local subproblems, see again [21, 2]. This multigrid method was proved to be robust for the case of $\alpha = 0$ with respect to the variation of $\gamma$, see [21]. One smoothing step consists of one forward and one backward sweep of the block Gauss–Seidel method. On the coarsest grid we do not solve the problem exactly; rather, we perform 30 iterations of left-preconditioned GMRES with the same block Gauss-Seidel iteration as a preconditioner, and we use FGMRES for the outer iteration.

In Table 5.1 we give the number of iterations (and timings) for the preconditioned flexible GMRES method applied to the system

$$(A_\alpha + \gamma B^T M_p^{-1} B)\mathbf{v} = \mathbf{f},$$

where $A_\alpha = A - \alpha I_n$. As a preconditioner in FGMRES we use one V(1,1) cycle of the multigrid described above. We use the zero right-hand side ($\mathbf{f} = 0$) and a vector with random entries uniformly distributed in $[0, 1]$ as the initial guess. The stopping criterion was a drop of the 2-norm of the residual by $10^{-6}$.

Table 5.2
*Results for the indefinite Stokes-type problem, $\hat{S}^{-1} = (\nu + \gamma)M_p^{-1}$.*

| $\alpha$ and $h$ | parameter $\gamma$ | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | 10 | $10^2$ | $10^3$ |
| $\alpha = 100$ | | | | | |
| 1/256 | 18 (58s) | 15 (49s) | 10 (29s) | 9 (29s) | 9 (29s) |
| 1/512 | 14 (191s) | 13 (178s) | 9 (123s) | 9 (124s) | 8 (109s) |
| $\alpha = 400$ | | | | | |
| 1/256 | 151 (627s) | 91 (360s) | 40 (149s) | 10 (36s) | 10 (36s) |
| 1/512 | 127 (2113s) | 89 (1431s) | 43 (638s) | 9 (127s) | 8 (112s) |
| $\alpha = 1600$ | | | | | |
| 1/256 | $> 600$ | $> 600$ | 130 (708s) | 26 (130s) | 25 (124s) |
| 1/512 | $> 600$ | $> 600$ | 136 (2310s) | 22 (344s) | 11 (173s) |
| $\alpha = 6400$ | | | | | |
| 1/256 | $> 600$ | $> 600$ | $> 600$ | 220 (1755s) | 105 (1173s) |
| 1/512 | $> 600$ | $> 600$ | $> 600$ | 145 (3603s) | 45 (1004s) |

Number of preconditioned FGMRES iterations and CPU times in seconds.

Note that for fixed $\alpha$ the method scales perfectly with $h$. When $\alpha$ becomes considerably larger, the number of iteration increases. The dependence of the number of iterations on $\gamma$ is not significant for smaller $\alpha$; however, for larger $\alpha$ an appropriate choice of the augmentation parameter $\gamma$ can reduce the number of iterations considerably. This phenomenon can be explained with the help of the analysis of section 3, which predicts that the problem becomes less indefinite for $\gamma > 0$.

In Table 5.2 we show the number of iterations (and timings) for preconditioned FGMRES applied to the system (2.1). The method was restarted after every 200 iterations, if necessary. As a preconditioner in FGMRES we use the block triangular matrix $\mathcal{P}$ from (2.2), with $\hat{A}^{-1}$ implicitly defined through the execution of one V(1,1) cycle of the above-mentioned multigrid method, and $\hat{S}$ defined through $\hat{S}^{-1} = (\nu + \gamma)M_p^{-1}$. We use again a zero right-hand side and a vector $\{\mathbf{u}^0, p^0\}$ with entries randomly distributed in $[0, 1]$ as the initial guess. The stopping criterion was the same as before.

We can see that similar to the Helmholtz case, for fixed $\alpha$ the method scales perfectly with $h$. When $\alpha$ becomes considerably larger, the growth in the number of iteration is more noticeable than for the Helmholtz case. An appropriate choice of $\gamma > 0$ is now even more crucial than for the Helmholtz case.

In Table 5.3 we show iteration counts for the same problem as in Table 5.2 and a slightly different preconditioner $\mathcal{P}$. We now use (2.4) to define the action of the approximate inverse of the Schur complement, with $\hat{A}^{-1}$ defined as before. The matrix $B\hat{M}_u^{-1}B^T$ from (2.4) is not inverted exactly; instead, we use four V(0,4) cycles of a standard geometric multigrid method to define an approximate inverse of $B\hat{M}_u^{-1}B^T$. Since the number $m$ of pressure degrees of freedom is small compared to the total number $n + m$, this does not increase considerably the cost of applying the block preconditioner $\mathcal{P}$. The table shows results for two pairs of finite elements: iso$P_2$-$P_0$ and iso$P_2$-$P_1$. Poor convergence for the case of $\gamma = 10^3$ with iso$P_2$-$P_1$ occurs because the multigrid solver for the (1,1) block is not effective in this case. As discussed in

TABLE 5.3
*Indefinite Stokes-type problem, $\hat{S}^{-1} = (\nu+\gamma)M_p^{-1} + \alpha\big(B\hat{M}_u^{-1}B^T\big)^{-1}$. Results for isoP$_2$-P$_1$ / isoP$_2$-P$_0$ finite elements.*

| $\alpha$ and $h$ | parameter $\gamma$ | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | 10 | $10^2$ | $10^3$ |
| $\alpha = 100$ | | | | | |
| 1/256 | 24 / 18 | 13 / 13 | 10 / 8 | 17 / 9 | > 600 / 8 |
| 1/512 | 23 / 11 | 13 / 11 | 9 / 8 | 9 / 9 | > 600 / 8 |
| $\alpha = 400$ | | | | | |
| 1/256 | 138 / 93 | 53 / 44 | 48 / 28 | 24 / 10 | > 600 / 10 |
| 1/512 | 135 / 66 | 42 / 40 | 49 / 22 | 19 / 9 | > 600 / 8 |
| $\alpha = 1600$ | | | | | |
| 1/256 | > 600 / > 600 | 159 / > 600 | 48 / 124 | 24 / 19 | > 600 / 27 |
| 1/512 | > 600 / > 600 | 153 / 379 | 49 / 58 | 19 / 13 | > 600 / 11 |
| $\alpha = 6400$ | | | | | |
| 1/256 | > 600 / > 600 | > 600 / > 600 | 337 / > 600 | 42 / 200 | > 600 / 125 |
| 1/512 | > 600 / > 600 | > 600 / > 600 | 109 / > 600 | 36 / 85 | > 600 / 70 |

Number of preconditioned FGMRES iterations.

[2], the multigrid method we used for the (1,1) block is more sensitive to the ratio $\gamma/\nu$ for this finite element pair.

The results show that although including the $\alpha\big(B\hat{M}_u^{-1}B^T\big)^{-1}$ term in the Schur complement preconditioner leads to some improvement for the case of small $\gamma$ and large $\alpha$, it does not have as dramatic an effect on the performance as it does in the positive definite case ($\alpha < 0$, see [4]). This observation deserves further comments. In terms of Fourier analysis, the preconditioner $\hat{S}^{-1} = (\nu+\gamma)M_p^{-1} + \alpha\big(B\hat{M}_u^{-1}B^T\big)^{-1}$ is optimal for the Schur complement of the matrix from (2.1) for all values of $\alpha$. For the positive definite case ($\alpha < 0$), using this choice of $\hat{S}^{-1}$ together with a good preconditioner $\hat{A}^{-1}$ in (2.2) is well known to ensure the fast convergence of the preconditioned iterative method to solve (2.1). Comparing results in tables 5.1 and 5.3, we conclude that this is *not* the case if the (1,1)-block in (2.1) becomes indefinite ($\alpha > 0$) and no augmentation is used ($\gamma = 0$). This may indicate that without augmentation, finding a proper block preconditioner for the indefinite linearized Navier–Stokes problem can be a very difficult task.

**5.1. Newton linearization.** In this subsection we consider equations (1.1)–(1.3) with two examples of the mean flow in $\Omega = (0,1)^2$: a Poiseuille flow

$$U = (8y(1-y), 0)$$

and a flow mimicking a rotating vortex

$$U = (4(2y-1)(1-x)x,\ -4(2x-1)(1-y)y)\,.$$

The instability of the Poiseuille flow is a well-studied problem in the literature [18, 24]. The common definition of the Reynolds number for this problem is $Re =$

Table 5.4
*Results for linearized Navier–Stokes problems with indefinite term; $\alpha = 1$, iso$P_2$-$P_0$ elements, $\gamma = 1$.*

| $h$ | Reynolds number | | | |
|---|---|---|---|---|
| | 1 | 10 | $10^2$ | $10^3$ |
| $U$=Poiseuille flow | | | | |
| 1/256 | 13 (57s) | 13 (57s) | 16 (71s) | 31 (140s) |
| 1/512 | 13 (268s) | 13 (269s) | 16 (339s) | 26 (545s) |
| $U$=rotating vortex | | | | |
| 1/256 | 13 (56s) | 12 (53s) | 18 (79s) | 45 (203s) |
| 1/512 | 13 (264s) | 12 (242s) | 18 (370s) | 46 (976s) |

Number of preconditioned FGMRES iterations and CPU times.

$0.5H|U(0, 0.5)|\nu^{-1}$, where $H$ is the height of the channel. In our settings it holds $Re = \nu^{-1}$. It is well-known (see, e.g. [24, 18]) that the eigenvalue of (1.1)–(1.3) with minimal real part approaches the imaginary axis as $O(Re^{-1})$ for $Re \to \infty$. Thus, in our experiments we set $\alpha = 1$. This leads to increasingly indefinite problems as $Re \to \infty$.

Both problems are discretized by finite elements. In our experiments we use both iso$P_2$-$P_0$ and iso$P_2$-$P_1$ elements. Since the SUPG-type stabilization technique applied to (1.1)–(1.3) in the context of finite element methods leads to a bulk of additional terms in the matrix of the resulting system of algebraic equations, we apply the SUPG stabilization *only* in the preconditioner (see [2] for the details of the stabilization used). The latter is done on every grid level of our geometric multigrid and is known to be necessary to ensure that the multigrid method for the (1,1) block is efficient for the case of small diffusion coefficient $\nu$. For the stiffness matrix, which enters the residual part of our iterative method, we are using fine enough meshes to keep all local mesh Reynolds numbers reasonably small. Moreover, we do not incorporate the discretization of the term $(\mathbf{u} \cdot \nabla)U$ in the preconditioner. Numerical experiments show that when the mean velocity field $U$ is smooth (this is typical for a mean flow in a linear stability analysis), adding the discrete term $(\mathbf{u} \cdot \nabla)U$ only in the residual does not affect the convergence of the preconditioned solver in any substantial way. A similar experience for Newton-type Navier–Stokes linearizations is reported in [7].

The block triangular preconditioner $\mathcal{P}$ is used with FGMRES to solve linear systems of the form (2.1) corresponding to Newton linearizations of these problems with additional negative reaction terms. Now the inverse of $\hat{A}$ consists of one $V(1, 1)$ cycle of the multigrid method used in [2]. Also, we use $\hat{S}^{-1} = (\nu+\gamma)M_p^{-1}$ where $M_p^{-1}$ is an approximate solver for the pressure mass matrix, see again [2] for details. Iteration counts and timings are given in Tables 5.4 and 5.5. Once again, we observe the perfect scaling with respect to $h$, and a relatively mild degradation in the performance of the preconditioner for increasing Reynolds numbers. Note that $\gamma = 1$ works very well for all cases except for the iso$P_2$-$P_1$ elements with Reynolds number $Re = 10^3$, where a smaller value of $\gamma$ is needed.

**6. Conclusions.** In this paper we have extended the augmented Lagrangian-based preconditioner described in [2] to the case where the $(1, 1)$ block in the saddle point problem is indefinite, an important subproblem in the linear stability analy-

Table 5.5
*Results for linearized Navier–Stokes problems with indefinite term; $\alpha = 1$, iso$P_2$-$P_1$ elements.*

| $h$ | Reynolds number | | | |
|---|---|---|---|---|
| | 1 | 10 | $10^2$ | $10^3$ |
| | Parameter $\gamma$ | | | |
| | 1. | 1. | 1. | 0.1 |
| $U$=Poiseuille flow | | | | |
| 1/256 | 14 (59s) | 14 (59s) | 22 (92s) | 35 (148s) |
| 1/512 | 15 (271s) | 14 (254s) | 24 (444s) | 30 (554s) |
| $U$=rotating vortex | | | | |
| 1/256 | 14 (58s) | 14 (59s) | 24 (102s) | 51 (221s) |
| 1/512 | 15 (273s) | 14 (253s) | 25 (458s) | 52 (995s) |

Number of preconditioned FGMRES iterations and CPU times.

sis of solutions to the Navier–Stokes equations. We have derived estimates for the eigenvalues of various operators and matrices of interest. In particular we have shown that the augmentation influences the system in two ways: it makes the (1,1)-block of the system less indefinite, and it improves the numerical properties of the Schur complement matrix exactly in the way additional viscosity would. We have tested the preconditioner on some challenging model problems. Our results indicate that the augmented Lagrangian-based block triangular preconditioner is effective and robust over a wide range of problem parameters, including highly indefinite cases.

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