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A 1/2-Integral Relaxation for the General Factor Problem

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Abstract

The general factor problem generalizes matching problems by stipulating that the degree of each vertex lies in a set of admissible values. If these sets have no large gaps, the decision problem is in P, while the complexity of the optimization version is open. We present a compact 1/2-integral linear relaxation for this case.

1 Introduction

The *general factor problem* is a generalization of the b-matching problem in which for each vertex v we are given a set of admissible degree values $A_v \subseteq \{1, \ldots, |V|\}$ rather than a single upper bound b_v. More precisely, given a (weighted) graph $G = (V, E)$, the general factor problem is that of finding a subgraph H of G (of optimal weight) such that for each vertex v, the degree of v in H lies in A_v. This problem is perhaps too general in that it is NP-complete even for graphs that are planar, bipartite, and only possess nodes of degree 2 or 3 [2]. However this can be remedied by considering the gaps in the sets A_v: A_v has a gap of length p if $k, k+p+1 \in A_v$ but $k+1, \ldots, k+p \notin A_v$ for some integer k. In the case when there are no gaps of length 2 or more, Lovàsz [6, 7] provided a structural description in the vein of the Gallai-Edmonds [5, 3] decomposition, noting that the description did not imply an immediate polynomial time algorithm for the decision problem. Cornuéjols [2] addressed this gap by providing a polynomial time Edmonds-like [3] algorithm for deciding whether a graph has a general factor when all gaps are of length 1. As our intent is to design a polyhedral relaxation, we only focus on the case when gaps have length 1 and define the general factor problem as such.

Even this restricted version is quite rich in that it generalizes the matching problems known to be in P. Although Cornuéjols's result is inspiring, the questions of a polyhedral description and efficient optimization algorithm still remain open. Matching problems are notorious for disguises, and we might hope that the general factor problem is only garnishing the garb of a wolf; however, it seems that although techniques and results from matching theory seem to generalize to this problem, it does indeed have more complicated facets as well.

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In our enquiry it will be useful to think of *inadmissible* values for the degree of a vertex rather than the opposite; that is, given a graph G and for each vertex v, a set $B_v \subset \{0, \dots, |\delta_G(v)|\}$, we seek a subgraph (of optimal cost) H of G such $|\delta_H(v)| \notin B_v$ for any vertex. that $i \in B_v$ implies $i + 1 \notin B_v$. Note that this problem, the *general antifactor problem* is a specialization of the general factor problem; however, if in addition, we allow lower and upper bounds a_v and b_v on the degree of of v, the two are equivalent. Sebö [9] presented a refined algorithm and structural characterization for the general antifactor problem, noting that the questions of a polyhedral description and efficient optimization algorithm remain open for this special case as well. In fact a polyhedral description is not known even in the *antifactor* case, when $B_v = \{1\}$ for all v.

Our main result is a $\frac{1}{2}$-integral relaxation, one whose extreme points are integral when multiplied by 2, for the general factor problem, one of the richest matching problems that P(partially?) admits. Although our relaxation has an exponential number of inequalities for each vertex, we provide both a separation algorithm and a compact reformulation. In addition we present a class of facet-containing valid inequalities that generalize the blossom constraints of matching polytopes.

1.1 Preliminaries

We use mostly standard notation; however, we list some of our conventions here. For a graph $G = (V, E)$, we define three operators on a set $S \subseteq V$: $\delta(S) := \{e \in E \mid |e \cap S| = 1\}$, $E(S) := \{e \in E \mid |e \cap S| = 2\}$, and $\nabla(S) := \{e \in E \mid |e \cap S| \geq 1\}$. Given a vector $v \in R^{|E|}$ and a set $F \subseteq E$, we let $v(F) := \sum_{e \in F} v_e$. We also assume basic knowledge of polyhedral combinatorics.

2 Developing a relexation

We assume that we are given as input a loopless multigraph $G = (V, E)$, and for each vertex v, a set of inadmissible degrees B_v and an upper bound b_v and lower bound a_v on the degree. We are also given lower and upper bounds l_e and u_e on the capacity of each edge e and a special vertex $t \in V$, which is the only vertex such that $B_t = \emptyset$. We may always had such a dummy vertex, and if more than one candidate exists, edges between such vertices are immaterial, and we may identify all such vertices with t. In addition we insist that $u_e > l_e$ for all e, since we may remove edges which violate this condition and update the instance accordingly. The general factor problem is that of finding a subgraph, whose incidence vector is x, such that:

$$a_v \leq x(\delta(v)) \leq b_v, \text{ for all } v \in V \tag{1}$$

$$x(\delta(v)) \notin B_v, \text{ for all } v \in V \tag{2}$$

$$x_e \in \{l_e, \dots, u_e\} \text{ for all } e \in E$$

Our first concern is to find a linear representation for the constraints (2). Let $x^u_e := u_e - x_e$, and $x^l_e := x_e - l_e$. For a set $U \subseteq S$, $\overline{U}$ refers to $S \setminus U$, where S ought to be clear from context. Let $\mathcal{U}_v := \{U \subseteq \delta(v) \mid u(U) + l(\overline{U}) \in B_v\}$. For a vertex $v \in V$ consider the inequalities,

$$x^u(U) + x^l(\overline{U}) \geq 1, \text{ for all } U \in \mathcal{U}_v, \tag{3}$$

where x is the incidence vector of a general factor of G. Since $x^u_e, x^l_e \geq 0$ for all e, if an inequality above is violated for a set $U \subseteq \delta(v)$ then we have $(u(U) - x(U)) + (x(\overline{U}) - l(\overline{U})) = 0$. This is a contradiction since it implies $x(\delta(v)) = x(U) + x(\overline{U}) = u(U) + l(\overline{U}) \in B_v$, hence the inequalities above are valid for incidence vectors of general factors. We note that although our inequalities resemble the inequalities used in defining the convex hull of T-joins of a given graph, they do not seem to appear in the literature.

Unfortunately these inequalities do not suffice in defining the convex hull of general factors — not even when $B_v = \emptyset$ for all $v \neq t$. For instance when $l_e = 0$ and $u_e = 1$ for each e, and $a_v = 1$ and $b_v = \infty$, for all $v \neq t$, we have the edge cover problem, that of finding a (minimal cost) set of edges which is incident upon each vertex in $V \setminus \{t\}$. Edmonds and Johnson [4] showed that additional inequalities are necessary in defining the convex hull of edge covers:

$$x(\nabla(S)) \geq \frac{|S| + 1}{2}, \text{ for all } S \subseteq V : |S| \text{ odd} \tag{4}$$

Variants of these inequalities yield integral hulls for more general matching problems as well.

Fortunately analogues of (4) are valid for the general factor problem. Suppose we are given a vertex set $S \subseteq V \setminus \{t\}$ and an edge set $U_v \in \mathcal{U}_v$ for each $v \in S$. We consider exactly one valid inequality (3), $x^u(U_v) + x^l(\overline{U}_v) \geq 1$ for each $v \in S$. We employ these inequalities and the fact that incidence vectors of general factors are integral in deriving a valid inequality for S which is not implied by (3).

For notational convenience in addressing a technicality that arises for edges in $\delta(S)$, we imagine contracting $V \setminus S$ to a vertex $\bar{s}$ so that $\delta(\bar{s}) = \delta(S)$; we set $B_{\bar{s}} := \emptyset$ so that $U_{\bar{s}}$ may be an arbitrary subset of $\delta(S)$. The definitions above induce a natural partition of the edges of $\nabla(S)$ into three sets:

$$\begin{aligned} U_S &:= \{uv \in \nabla(S) \mid uv \in U_u \cap U_v\}, \\ L_S &:= \{uv \in \nabla(S) \mid uv \notin U_u \cup U_v\}, \text{and} \\ I_S &:= \nabla(S) \setminus (U_S \cup L_S). \end{aligned}$$

Lemma 1. *If $|S| - (u(I_S) - l(I_S))$ is odd, then the inequality below is valid.*

$$x^u(U_S) + x^l(L_S) \geq \frac{|S| - (u(I_S) - l(I_S)) + 1}{2} \tag{5}$$

Proof. Summing the appropriate inequality (3) for each $v \in S$ yields:

$$\begin{aligned} |S| &\leq \sum_{v \in S} x^u(U_v) + x^l(\overline{U_v}) \\ &\leq x^u(U_{\bar{s}}) + x^l(\overline{U_{\bar{s}}}) + \sum_{v \in S} x^u(U_v) + x^l(\overline{U_v}) \\ &= 2(x^u(U_S) + x^l(L_S)) + u(I_S) - l(I_S) \end{aligned}$$

Thus we have $2(x^u(U_S)+x^l(L_S)) \geq |S| - (u(I_S) - l(I_S))$, but when x is the incidence vector of a general factor the LHS is even, and the RHS is odd, hence $2(x^u(U_S) + x^l(L_S)) \geq |S| - (u(I_S) - l(I_S)) + 1$ must also be valid. $\square$

Note that (5) generalizes both (3) ($|S| = 1$ and $I_S = \emptyset$) and (4) ($U_S, I_S = \emptyset$). We also note that we may strengthen the above derivation by including inequalities among (1). In fact we know that there are additional facets of the convex hull of general factors which we have not discussed, and that they have coefficients that seem to grow with the size of the instance. In the next section we show that relaxation using (3) is 1/2-integral.

3 1/2-Integrality

The polyhedra for matching problems that are known to be in Ptend to have at most one inequality of (5) when $S = \{v\}$. Although in the general factor case the same assumption is not true, we show below that a similar assumption may replace it.

Lemma 2. *For a vertex $v \in V$ there is at most one non-redundant tight member of the system,*

$$\hat{x}^u(U) + \hat{x}^l(\overline{U}) \geq 1, \text{ for all } U \in \mathcal{U}_v.$$

Proof. Suppose there exist $U_1, U_2 \in \mathcal{U}_v$ such that $U_1 \neq U_2$ and $\hat{x}^u(U_1) + \hat{x}^l(\overline{U_1}) = \hat{x}^u(U_2) + \hat{x}^l(\overline{U_2}) = 1$. Summing these two equations we get

$$2(\hat{x}^u(U_1 \cap U_2) + \hat{x}^l(\overline{U_1} \cap \overline{U_2})) + u(U_1 \Delta U_2) - l(U_1 \Delta U_2) = 2,$$

hence $u(U_1 \Delta U_2) - l(U_1 \Delta U_2) \in [1, 2]$ since $U_1 \neq U_2$.

Let $b_i := u(U_i) + l(\overline{U_i})$ for $i = 1, 2$, and recall that $u_e - l_e \geq 1$ for all $e \in E$. If $u(U_1 \Delta U_2) - l(U_1 \Delta U_2) = 1$ then $|b_1 - b_2| = 1$, but this cannot occur since B_v does not contain consecutive integers.

We have only to consider $u(U_1 \Delta U_2) - l(U_1 \Delta U_2) = 2$. In this case $\hat{x}^u(U_1 \cap U_2) + \hat{x}^l(\overline{U_1} \cap \overline{U_2}) = 0$, and we find that the equations $x^u(U_1 \setminus U_2) + x^l(U_2 \setminus U_1) = 1$ and $x^u(U_2 \setminus U_1) + x^l(U_1 \setminus U_2) = 1$ are redundant. $\square$

We will also need the fact that the inequalities, (1) mingle well with (2).

Lemma 3. *For a vertex $v \in V$ there is at most one non-redundant tight member of the system,*

$$a_v \leq \hat{x}(\delta(v)) \leq b_v,$$
$$\hat{x}^u(U) + \hat{x}^l(\overline{U}) \geq 1, \text{ for all } U \in \mathcal{U}_v.$$

Proof. Suppose there is a $U \in \mathcal{U}_v$ such that $\hat{x}^u(U) + \hat{x}^l(\overline{U}) = 1$. By the above lemma we may assume such a U is unique. We have, $|\hat{x}(\delta(v)) - (u(U) + l(\overline{U}))| \leq \hat{x}^u(U) + \hat{x}^l(\overline{U}) = 1$. If $|\hat{x}(\delta(v)) - (u(U) + l(\overline{U}))| = 1$, then any inequality of the form $\hat{x}(\delta(v)) = c$, where c is an integer, is either redundant or inconsistent.

This leaves only the case in which $|\hat{x}(\delta(v)) - (u(U) + l(\overline{U}))| < 1$; however, in this case $\hat{x}(\delta(v))$ cannot be an integer, hence if $\hat{x}$ is feasible then we must have, $a_v < \hat{x}(\delta(v)) < b_v$. $\square$

We have accumulated the background information and tools necessary to state and prove our main result.

Theorem 4. *Given an instance of General Factor: G, t, a, b, $B_v : \forall v \in V \setminus \{t\}$, l, and u, the extreme points of the polyhedron, $\mathcal{GF}$,*

$$a_v \leq x(\delta(v)) \leq b_v, \text{ for all } v \in V \setminus \{t\} \tag{6}$$

$$x^u(U) + x^l(\overline{U}) \geq 1, \text{ for all } U \in \mathcal{U}_v : v \in V \setminus \{t\} \tag{7}$$

$$u_e \geq x_e \geq l_e, \text{ for all } e \in E, \tag{8}$$

are $\frac{1}{2}$-integral with non-integral components ocurring in vertex-disjoint cycles.

Proof. Let $G := (V, E)$ be a counterexample minimal with respect to $|V| + |E|$, and let $\hat{x}$ be an extreme point of $\mathcal{GF}$ that is not $\frac{1}{2}$-integral. We say that an edge e is *slack* if $u_e > \hat{x}_e > l_e$; an edge that is not slack is *tight*.

If some edge, $e = uv$, is integral then we may remove e, deleting any isolated nodes, and decrease a_u, a_v, b_u, b_v, and each element of B_u and B_v by $\hat{x}_e$ (deleting negative elements if necesary) to obtain a smaller instance. If this process renders B_u empty for some u, we contract u with t. We may verify without difficulty that the $\hat{x} \mid_{E \setminus \{e\}}$ is feasible for the $\mathcal{GF}$ on the smaller instance, hence $\hat{x} \mid_{E \setminus \{e\}}$ is a convex combination of solutions of the smaller instance. We can augment each of these solutions with e at a value of $\hat{x}_e$ to obtain a convex decomposition of the extreme point, $\hat{x}$ — a contradiction. This also implies that each edge must be slack.

We call a vertex $v \in V \setminus \{t\}$ *tight* if an inequality of either (6) or (7) corresponding to v is tight. For each tight v for which an inequality of (7) is tight, let $U_v \in \mathcal{U}_v$ be some set such that $\hat{x}^u(U_v) + \hat{x}^l(\overline{U_v}) = 1$. For each of the remainder of the tight vertices, we have that $\hat{x}(\delta(v)) = c_v \in \{a_v, b_v\}$. For the sake of convenience in addressing each such tight vertex v, we impose the convention that $U_v = \emptyset$ and $l(\delta(v)) = c_v - 1$, so that we have $\hat{x}^u(U_v) + \hat{x}^l(\overline{U_v}) = \hat{x}(\delta(v)) - (c_v - 1) = 1$. Although there may be several choices for U_v, lemmas 2 and 3 allow us the assumption that there is a unique tight inequality for each tight vertex. As with edges, a vertex that is not tight is *slack*. For each slack vertex v we select an arbitrary set $U_v \in \mathcal{U}_v$. Following the discussion preceding Lemma 1 we partition $E_t := E(V \setminus \{t\})$ into the sets, $U := \{e \in E_t \mid e \in U_{e_1} \cap U_{e_2}\}$, $L := \{e \in E_t \mid e \notin U_{e_1} \cup U_{e_2}\}$, and $I := E_t \setminus (U \cup L)$. An edge $e \in E_t$ is *inconsistent* if $e \in I$, and is called *consistent* otherwise.

We will appeal to the following tools in our analysis.

Proposition 5. *No tight vertex may have exactly one edge incident upon it.*

Proof. If e is the edge incident upon a v which contradicts the hypothesis, we have that one of $\hat{x}_e = a_v$, $\hat{x}_e = b_v$, $\hat{x}_e = u_e - 1$, or $\hat{x}_e = 1 - l_e$ must hold, contradicting the non-integrality of $\hat{x}_e$. □

A walk is *even* if it contains t or an even number of consistent edges; it is *odd* otherwise.

Lemma 6. *G does not contain even closed walks or paths between slack vertices.*

Proof. We show that the contradiction of the hypothesis implies that there is a vector $\hat{\epsilon}$ such that both $\hat{x}+\hat{\epsilon}$, $\hat{x}-\hat{\epsilon} \in \mathcal{GF}$, demonstrating that $\hat{x}$ is not an extreme point. For $e \in U \cup L$ let

$$\hat{y}_e := \begin{cases} \hat{x}^u_e & \text{if } e \in U, \\ \hat{x}^l_e & \text{if } e \in L. \end{cases}$$

Suppose G contains an even closed walk $W \subseteq U \cup L$ such that $t \notin V(W)$. We walk along the edges $e \in W$, alternating between increasing and decreasing $\hat{y}_e$ by an appropriately chosen ϵ. This augmentation leaves $\hat{y}(\delta(v)) = \hat{x}^u(U_v) + \hat{x}^l(\overline{U}_v)$ unaltered for $v \in W$. Observe that by our selection of U_v for each vertex, the augmentation does not affect the LHS of constraints among (7) and (6). This augmentation of $\hat{y}$ induces an augmentation $\hat{\epsilon}$ of $\hat{x}$, and we choose ϵ small enough in magnitude so that no slack constraints are violated. Thus we arrive at the contradiction that both $\hat{x}+\hat{\epsilon}$ and $\hat{x}-\hat{\epsilon}$ are feasible for $\mathcal{GF}$.

Now we extend the above argument to include edges in $E \setminus (U \cup L)$. If $t \in W$, then we may simply choose U_t so as to induce an even number of consistent edges in W. This only leaves inconsistent edges, those $e = \{u, v\} \in I$. Suppose that $e \in U_u \cap \overline{U_v}$. Increasing $\hat{x}_e$ by ϵ increases $\hat{x}^u(U_v) + \hat{x}^l(\overline{U_v})$ and decreases $\hat{x}^u(U_u) + \hat{x}^l(\overline{U_u})$ by ϵ, thus we may treat e as if it were a path of even length, say 0, between u and v.

For the case of a path between slack vertices, u and v, note that we can perform an augmentation as above that will preserve the LHS of the inequalities of (6) and (7) corresponding to all internal vertices of the path. Since u and v are slack we can choose ϵ small enough so that we do not violate any inequality of (6) or (7) corresponding to u or v. □

Since $\hat{x}$ is an extreme point, and all of (8) are slack, we have that $\hat{x}$ is the solution of tight inequalities exclusively among (6) and (7). By lemmas 2 and 3, there can be at most $|V|-1$ of these that are independent, hence $|E| \leq |V|-1$. Let T be the component of G which contains t. We must have that either $|T| = 1$ or $|T| = |V|$, for otherwise $\hat{x} \mid_{E(T)}$ is a convex combination of general factors of T, implying $\hat{x}$ cannot be an extreme point.

If $|T| = |V|$, since $|E| \leq |V|-1$, we must have two slack leaves by Proposition 5; however, Lemma 6 precludes this possibility.

If $|T| = 1$ then $V \setminus T$ must be connected, otherwise a component (augmented with a vertex to replace t) would serve as a smaller counterexample. Since $|E| \leq |V|-1$, Proposition 5 and Lemma 6 imply that $V \setminus T$ is unicyclic with at most one slack node. The existence of a slack node s implies a path with slack endpoints (both s), hence this only leaves us the option of $V \setminus T$ being a $\frac{1}{2}$-integral cycle. □

We note that adding (5) to $\mathcal{GF}$ is not sufficient to define the integral hull, which possesses additional facets which do not seem to fall into this framework. Now that we have established our relaxation, we may turn to the issue of efficiently computing an optimal solution.

Algorithm 1 Separation algorithm for $\mathcal{GF}$

Input: A candidate solution $\hat{x}$ and a vertex, $v \in V$

Output: A set, $U \subseteq \delta(v)$ such that $\hat{x}$ violates (7) for U, or "feasible"

(0) If there is no $b \in B_v$ such that $|\hat{x}(\delta(v)) - b| < 1$ then return "feasible"; otherwise, let b be the unique integer in B_v such that $|\hat{x}(\delta(v)) - b| < 1$.

(1) Let $D := \{e \in \delta(v) \mid u_e - l_e = 1\}$, $D_+ := \{e \in \delta(v) \setminus D \mid \hat{x}_e \geq \frac{u_e - l_e}{2}\}$, $D_- := \{e \in \delta(v) \setminus D \mid \hat{x}_e \leq \frac{u_e - l_e}{2}\}$, and $b' = b - u(D_+) - l(D_-)$.

(2) If $D_- \cap D_+ \neq \emptyset$ or $b' < l(D)$ then return "feasible"; else, let S consist of the $b' - l(D)$ elements, $e \in D$, which minimize $(u_e - \hat{x}_e) - (\hat{x}_e - l_e)$.

(3) Let $U := S \cup D_+$. If $\hat{x}^u(U) + \hat{x}^l(\overline{U}) \geq 1$ then return "feasible"; else, return U.

4 Polynomial-time solvability

The polyhedron $\mathcal{GF}$ consists of inequalities whose number is exponential in the size of the instance over which $\mathcal{GF}$ is defined, hence our only hope of optimizing over this polyhedron efficiently for all input instances is to provide a polynomial-time separation algorithm which we may use in conjunction with the ellipsoid method. The following proposition inspires hope that such an algorithm might exist.

Proposition 7. *At most one inequality among* (7) *corresponding to a vertex* $v \in V$ *may be violated by an* $\hat{x} \in R^{|E|}$.

Proof. The proof follows from that of Lemma 2. Suppose there exist $U_1, U_2 \in \mathcal{U}_v$ such that $U_1 \neq U_2$, $\hat{x}^u(U_1) + \hat{x}^l(\overline{U_1}) < 1$, and $\hat{x}^u(U_2) + \hat{x}^l(\overline{U_2}) < 1$. We have $2(\hat{x}^u(U_1 \cap U_2) + \hat{x}^l(\overline{U_1} \cap \overline{U_2})) + u(U_1 \Delta U_2) - l(U_1 \Delta U_2) < 2$, hence $u(U_1 \Delta U_2) - l(U_1 \Delta U_2) = 1$; however, this leads to a contradiction as in the proof of Lemma 2. □

Algorithm 1 is a separation algorithm for all the inequalities among (7) corresponding to a particular vertex, $v \in V$. We may easily separate over (6) and the nonnegativity constraints in polynomial time, hence running Algorithm 1 for each vertex in V yields a separation algorithm for $\mathcal{GF}$.

Theorem 8. *Algorithm 1 is correct and runs in polynomial time.*

Proof. Suppose there exists a $U \subseteq \mathcal{U}_v$ such that $\hat{x}^u(U) + \hat{x}^l(\overline{U}) < 1$. To justify **(0)** we note that $|\hat{x}(\delta(v)) - (u(U) + l(\overline{U}))| \leq \hat{x}^u(U) + \hat{x}^l(\overline{U}) < 1$. We observe that we must have $D_+ \subseteq U$ and $U \cap D_- = \emptyset$, which implies that D, D_+, and D_- partition $\delta(v)$. Thus **(2)** returns a correct result, since in order for a violation we must also have $u(D_+) + l(D_-) + l(D) \leq b$.

We have reduced the algorithm's goal to determining whether there exists a set $S \subseteq D$ such that $U := S \cup D_+$ is a violator and $u(S) + l(D \setminus S) + u(D_+) + l(D_-) = b$. Thus S must contain exactly $b' - l(D)$ elements of D, since $u_e - l_e = 1$ for all $e \in D$. We choose precisely those $b' - l(D)$ elements which minimize $\hat{x}^u(S \cup D_+) + \hat{x}^l(\overline{S \cup D_+})$, hence the algorithm is correct. The algorithm can be easily implemented to run in $O(|\delta(v)| \log |\delta(v)|)$ time. □

Another way in which we might address the issue of finding a solution in polynomial time is to present a compact formulation for $\mathcal{GF}$. Algorithm 1 is simple enough that we might hope to capture it is as a linear program which could use in compactly formulating $\mathcal{GF}$. This type of technique has been used in particular to show that compact separation and compact optimization are equivalent [8, 1].

Although we do believe that $\mathcal{GF}$ admits a compact formulation for all instances, we have not looked into this in detail. We present a compact formulation for instances in which $u_e - l_e = 1$ for all $e \in E$. For a particular vertex $v \in V$ and $b \in B_v$, our approach is to find a compact set of inequalities to represent the potentially exponential family of inequalities:

$$x^u(U) + x^l(\overline{U}) \geq 1, \text{ for all } U \subseteq \delta(v) : u(U) + l(\overline{U}) = b. \tag{9}$$

This suffices since $\mathcal{GF}$ requires at most $\sum_{v \in V} |B_v|$ of the above families. Given a candidate solution $\hat{x}$, separating over (9) amounts to solving a knapsack problem, which we can formulate as an integer program:

$$\begin{aligned} w^* = \hat{x}(\delta(v)) - l(\delta(v)) + \min \sum_{e \in \delta(v)} & (u_e + l_e - 2\hat{x}_e) y_e \\ \sum_{e \in \delta(v)} & (u_e - l_e) y_e = b - l(\delta(v)) \\ & y_e \in \{0, 1\}, \text{ for all } e \in \delta(v). \end{aligned}$$

We may interpret the variable y_e as indicating whether $e \in U$, that is $y_e = 1$ iff $e \in U$. Under this interpretation, the equality constraint ensures that $u(U) + l(\overline{U}) = b$, and the objective is precisely $\bar{x}^u(U) + \bar{x}^l(\overline{U})$; hence, we have a violation if and only if there is a feasible solution with $w^* < 1$.

When $u_e - l_e = 1$ for all e, it is not difficult to see that we may relax each integrality constraint to $y_e \in [0, 1]$ to yield an equivalent linear program whose extreme points are integral. The dual of this linear program is,

$$\begin{aligned} \hat{x}(\delta(v)) - l(\delta(v)) + \max & z(\delta(v)) + (b - l(\delta(v)))\alpha \\ & z_e + (u_e - l_e)\alpha \leq u_e + l_e - 2\hat{x}_e, \text{ for all } e \in \delta(v) \\ & z_e \leq 0, \text{ for all } e \in \delta(v). \end{aligned}$$

For any given $\hat{x}$, there is a feasible solution (z, α) to the above program such that $\hat{x}(\delta(v)) - l(\delta(v)) + z(\delta(v)) + (b - l(\delta(v)))\alpha \geq 1$, if and only if $w^* \geq 1$, thus (9) is the projection of the inequalities below on $\hat{x}$, yielding a compact formulation for $\mathcal{GF}$ with at most $O(|V|^3)$ constraints and variables when $u_e - l_e = 1$ for all e.

$$\begin{aligned} \hat{x}(\delta(v)) - z(\delta(v)) + (b - l(\delta(v)))\alpha &\geq 1 + l(\delta(v)) \\ 2\hat{x}_e + z_e + \alpha &\leq u_e + l_e, \text{ for all } e \in \delta(v) \\ z_e &\leq 0, \text{ for all } e \in \delta(v) \end{aligned}$$

Although we arrived at the above formulation somewhat mechanically, we can use duality and complementary slackness to arrive at interpretations for the variables z_e and α. For instance, we may interpret $z_e < 0$ as indicating $e \in U$.

5 Conclusion

We presented a 1/2-integral relaxation for the general factor problem which can be solved in polynomial time. We used generalizations of standard techniques from the study of matching polyhedra, and we had to take care in handling the potentially exponential number of inequalities necessary for each vertex. In this case when $u_e - l_e = 1$ for all $e \in E$, we derive a compact formulation for $\mathcal{GF}$ and believe that this should extend to the general case.

Outstanding open problems include characterizations of additional facets of the convex hull of general factors of an instance. We know that facets exist with coefficients larger than 1, which is not true of other matching problems. Ultimately the goal is to produce a polynomial-time algorithm for optimizing over this hull, or determining that the problem of deciding whether a general factor of cost at most k is NP-complete.

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