Technical Report

TR-2004-031

Gianfranco Cimmino's Contributions to Numerical Mathematics

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GIANFRANCO CIMMINO'S CONTRIBUTIONS TO NUMERICAL MATHEMATICS*

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Abstract. Gianfranco Cimmino (1908-1989) authored several papers in the field of numerical analysis, and particularly in the area of matrix computations. His most important contribution in this field is the iterative method for solving linear algebraic systems that bears his name, published in 1938. This paper reviews Cimmino's main contributions to numerical mathematics, together with subsequent developments inspired by his work. Some background information on Italian mathematics and on Mauro Picone's *Istituto Nazionale per le Applicazioni del Calcolo*, where Cimmino's early numerical work took place, is provided. The lasting importance of Cimmino's work in various application areas is demonstrated by an analysis of citation patterns in the broad technical and scientific literature.

Key words. Cimmino's method, history of numerical linear algebra

AMS subject classifications. Primary 0108, 01A60. Secondary 65F10, 65R30.

1. Introduction. Gianfranco Cimmino was a distinguished Italian mathematician who made important contributions to the theory of partial differential equations and to other branches of mathematical analysis. Side-by-side with his main research area, Cimmino cultivated other mathematical interests, including numerical analysis. Under the influence of his teacher, Mauro Picone (1885-1977), Cimmino developed an early interest in numerical questions, some of which he will repeatedly revisit in the course of his long career. Outstanding among his contributions is an elegant iterative method for the solution of linear algebraic systems. This method was published in 1938 and is widely known as "Cimmino's method." As we shall see, this algorithm has withstood the test of time and is still widely used, albeit in modified form, in a wide variety of scientific and technical applications.

This paper surveys Cimmino's contributions to numerical mathematics and describes some of the circumstances that led him to work in this area. In order to do so, it is necessary to take a quick look at certain aspects of numerical analysis in the early decades of the 20th century. In particular, Italy's rather exceptional role in this arena, due to Mauro Picone's *Istituto Nazionale per le Applicazioni del Calcolo* (INAC), will be highlighted.

The paper is organized as follows. Section 2 contains a brief overview of Cimmino's career and scientific production. Section 3 is devoted to Mauro Picone and the INAC. Section 4 addresses the status of pre-WWII numerical analysis and discusses some early work in numerical linear algebra, with particular attention to work done at the INAC in the Thirties. Cimmino's method of 1938 is the subject of section 5, while later developments by Cimmino himself and by other researchers are discussed in section 6. The lasting influence of Cimmino's work in scientific computing is assessed in section 7, using in part citation data. Some general reflections on Cimmino's numerical work are given in section 8, which concludes the paper.

2. A brief overview of Cimmino's career. Gianfranco Cimmino was born in Naples on March 12, 1908. His father, Francesco, was a renowned Orientalist and Sanskrit scholar. His mother belonged to an aristocratic family from Novara (in the North of Italy), the Gibellini Tornielli

*This draft dated 17 December 2004.

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Boniperti. Cimmino died in Bologna on May 30, 1989.

After receiving his *Laurea* degree in Mathematics at the University of Naples under Mauro Picone in 1927 (at age 19), Cimmino became Picone's assistant in Analytic Geometry in 1928. *Libero docente* in 1931, he was in charge of the courses of Higher Analysis and Analytic Geometry until 1938, when he won the chair of Mathematical Analysis at the University of Cagliari, in Sardinia. At the end of 1939 he moved permanently to the University of Bologna to occupy the chair of Mathematical Analysis. In the course of his long career, Cimmino was awarded a number of prestigious awards and distinctions [71]. In addition, he was called to occupy leadership positions both at the University of Bologna (where he served as Dean of the Science Faculty for several years) and in the Italian mathematical community (INdAM and UMI).

Cimmino's scientific production, covering a period of nearly 60 years, consists of seven books (mostly lecture notes for his courses), fifty-six papers, and several additional minor works, including obituaries and writings devoted to the popularization of certain aspects of mathematics. The latter include the text of a talk on *Dante and Mathematics*, which shows Cimmino's broad cultural interests. Cimmino's scientific papers are written in Italian, German, French and English. Furthermore, Cimmino authored hundreds of reviews for the *Zentralblatt für Mathematik*, most of them in German.

Cimmino's first paper, 24 pages long, appeared in 1928; his last paper is a note of 4 pages for the proceedings of the *Accademia dei Lincei*, published posthumously in 1990. None of Cimmino's publications are in collaboration with others. With the exception of two "dry spells" (in 1943-1947, obviously due to the War and its immediate aftermath, and again in 1976-1981 due to the consequences of a serious car accident), Cimmino's scientific output appears to have been remarkably steady, even during his term as Dean of the Faculty of Science. The most fruitful years in Cimmino's career are those from 1937 to 1939, with nine papers, including some truly remarkable ones. Furthermore, there are eight publications from 1982 until his death.

A majority of Cimmino's papers concern the theory of linear partial differential equations, in which he obtained important results. Other topics treated by Cimmino include:

- Two-point boundary value problems for ordinary differential equations;
- Infinite systems of linear differential and integral (Fredholm) equations in infinitely many unknowns;
- The calculus of variations;
- Differential geometry;
- Conformal and quasi-conformal mappings;
- Topological vector spaces;
- The theory of (ultra-)distributions;
- Numerical analysis, especially matrix computations.

Cimmino's publications in numerical analysis amount to 5-6 short papers, which will be discussed below. Some of Cimmino's most remarkable papers date to the period 1937-1938 and concern the theory of partial differential equations of elliptic type. In particular, Cimmino was the first mathematician to study the *Dirichlet problem with generalized boundary conditions* [31, 33]. In this problem one considers Laplace's equation $\Delta u = 0$ (or Poisson's, $\Delta u = f$) on a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary Γ, subject to the condition that the trace of u on Γ is a prescribed function $g \in L^2(\Gamma)$; see, e.g., [69, Chapter IV]. Cimmino will revisit this topic several times in the subsequent decades, achieving various generalizations and improvements of his original results from this period. Incidentally, this problem was found many years later to be of central importance in optimal control theory; see [67, page vi]. For detailed information on Cimmino's work in analysis and partial differential equations (PDEs), we refer the reader to [71].

3. Mauro Picone and the INAC. Gianfranco Cimmino was one of Picone's first students, together with Giuseppe Scorza Dragoni (1908-1996) and Carlo Miranda (1912-1982). The star of this group was Renato Caccioppoli (1905-1959) who, while not formally a student of Picone's (having received his degree under Ernesto Pascal in 1925), may well be considered a member of his school [77]. All four members of this remarkable group were bound by a strong sense of camraderie and friendship [79]. Picone's Neapolitan period goes from 1925 until 1932, when he transferred to the University of Rome. For several years (precisely, since his military service in WWI as an artillery officer, during which he became deeply involved with questions in ballistics), Picone nurtured the dream of creating a research institute devoted to numerical analysis and its applications. The dream became reality in 1927, thanks to a grant (amounting to 50,000 Liras) from the *Banco di Napoli*, made possible by the intervention of the economist Luigi Amoroso, a friend of Picone's and a fellow alumn of the *Scuola Normale* in Pisa. The *Istituto di Calcolo*, initially attached to the chair of Mathematical Analysis at the University of Naples, grew into the *Istituto Nazionale per le Applicazioni del Calcolo* after its move to Rome in 1932. The INAC is considered by many historians to have been the first research institute specifically devoted to numerical analysis, in the modern sense of the phrase; see, e.g., [15, 44, 46, 58].

Picone was, among other things, an excellent talent scout, and was very good at identifiying and attracting promising students. Once he had become convinced that a budding mathematician had the necessary attributes, he did everything in his power to encourage and promote the young researcher's work. And his power was considerable: Picone was highly influential and politically well-connected. Especially in the later years of the Fascist regime, he and Francesco Severi (the famous algebraic geometer) were practically in control of much of Italy's mathematical scene. Over the years the INAC became the first workplace for an impressive assembly of mathematicians, including several who were to become among the leading exponents of Italian mathematics. Among these we mention R. Caccioppoli, G. Cimmino, G. Scorza Dragoni, C. Miranda, T. Viola, L. Cesari, S. Faedo, G. Mammana, D. Caligo, L. Sobrero, R. Einaudi, G. Krall, C. Tolotti, G. Grioli, M. Salvadori, F. Conforto, C. Minelli, G. Doetsch, W. Gröbner, and others. In addition to this group of researchers, the Institute employed a total of eleven *computers* and *draftsmen*. These were highly skilled men and women, many with university degrees, who carried out all the necessary numerical calculations using the computing equipment available at the time, including various electro-mechanical and graphical devices.[1]

What kind of mathematical work was the INAC actually concerned with? Besides fundamental research in mathematical analysis, differential and integral equations, functional analysis and numerical analysis, the INAC staff was also engaged in a wide variety of applied research projects. These took the form of consulting agreements and research contracts with government agencies (both Italian and foreign), public utility companies, branches of the military, and a number of private companies ranging from major shipyards to small engineering firms. In addition, there were frequent collaborations with university researchers in various scientific and technical fields. One such collaboration with Enrico Fermi resulted in a detailed study by Miranda (1934) of the Fermi-Thomas equation of atomic physics. See [3] for a description of the manifold activities carried out at the INAC during the 1930s.

Among the topics treated by INAC researchers we find, in addition to "pure" mathematics, problems in classical mechanics (including celestial mechanics), fluid dynamics, structural analysis, elasticity theory (especially the study of beams), atomic physics, electromagnetism, aeronautics, hydraulics, astronomy, and so forth. One of the strong points of INAC researchers was their penchant

[1] It is worth mentioning that in the aftermath of World War II, the research group working at the INAC included mathematicians of the caliber of Gaetano Fichera and Ennio De Giorgi.

for developing and applying sophisticated techniques of mathematical analysis to solve problems stemming from concrete and urgent applications. Among the preferred tools we find: variational methods, including variants of the Rayleigh-Ritz method (a precursor of the modern finite element techniques); fixed point theorems in function spaces; the reformulation of boundary value problems in terms of systems of integral equations; and techniques from asymptotic analysis. Although most of the papers produced by INAC researchers were of a rather theoretical nature, the motivation often came from practical questions that had been submitted to INAC by one or another of its many "customers."[2]

Much of the work done at INAC embodied Picone's philosophy, according to which the mathematical analysis of a problem should not be confined to the study of the existence, uniqueness, and regularity of the solution, but should also supply tools for the (approximate) numerical solution of the problem together with rigorous bounds, in the appropriate norm, of the error incurred. Not surprisingly, Cimmino's early works (including his first published paper [30]) are informed by this way of thinking. Even later in his career Cimmino always showed a strong preference for constructive proofs, whenever possible. Even in his important paper [31] on the generalized Dirichlet problem, Cimmino observes that the method he used to prove the existence of a solution suggests a numerical procedure to compute an approximate solution (see [31, footnote 3], or [39, page 258]). This philosophy is of course not unique to Picone's school: suffices to mention the celebrated 1928 paper by Courant, Friedrichs and Lewy on the partial difference equations of mathematical physics [40].

Within the realm of numerical analysis proper, the problems studied at the INAC included the numerical tratment of initial and boundary value problems for ordinary and partial differential equations, optimization problems, integral and integro-differential equations, the approximation and interpolation of functions, least-squares problems, computational harmonic analysis, the determination of the eigenvalues and eigenvectors of matrices, the numerical solution of linear and nonlinear systems of equations, and so forth. For a more thorough discussion of Picone's Institute and of Italian mathematics in general in the period between the two world wars, we refer the reader to [44, 47, 55].

4. Numerical analysis in the Thirties. The presence of an institute devoted to numerical analysis in a major mathematical research center like the University of Rome was entirely due to Picone's tenacity and farsightedness. The successes of INAC notwithstanding, it would be a mistake to conclude that numerical analysis had gained by the 1930s the status of a mature mathematical discipline, recognized as important by the wider mathematical establishment and well grounded at the institutional level. Suffices to say that the leading exponent of Italian "pure" mathematics of that time, Francesco Severi, was quite against the notion of an institute devoted to what was unglamorously described as the "ancillary roles" of mathematics. It is also worth mentioning that Picone, who already in the early Thirties had introduced a course on *Numerical and graphical computations* (*Calcoli numerici e grafici*) for students of Statistics and Actuarial Sciences at the University of Rome, failed in his attempts to have the course included among the electives for the degree in Mathematics.

Both at the national and international level, numerical analysis was considered a minor field, a Cinderella among the mathematical sciences. One result of the very modest standing of numerics among mathematicians is that many of the most important advances that took place in this field during the second half of the 19th century and the first half of the 20th were due to scientists and engineers, in particular physicists, astronomers and geodesists. At the beginning of the 1930s

[2]Because of the strong theoretical flavor of the papers coming out of it, not all applied mathematicians were favorably disposed towards the INAC. See [80] for some of the opinions circulating among contemporary German mathematicians, especially pages 91, 102–103, and 318.

there existed only a handful of treatises devoted to numerical analysis. At the international level we mention those by Whittaker and Robinson [89] and by Scarborough [78]. At the national level there were the tract by Cassina [19], mostly devoted to elementary questions, and the one by Cassinis [21]. See also [20] for a complete bibliography up to the 1940s (especially page 16). This situation was due, at least in part, to the increasing specialization among scientists beginning with the first half of the 1800s. Until then most mathematicians were also keenly interested in physics and astronomy, and there are many numerical methods associated with the names of Newton, Lagrange, Euler, Laplace, Gauss, Jacobi, Cauchy, Fourier, and so forth. The picture began to change only after World War II, with the advent of high-speed electronic computers. Numerical analysis also benefitted greatly from the fact that several prestigious mathematicians (especially John von Neumann, but also Alan Turing, Stanislaw Ulam, Eduard Stiefel and others) devoted a great deal of attention to this discipline, at first because of war needs but then with increasing enthusiasm and conviction [54, 58].

One of the basic problems in scientific computation consists in the solution of systems of linear algebraic equations. Almost all problems of computational mathematics boil down, in the end, to the solution of such systems, often of very large dimensions. A classical example is the numerical solution of boundary value problems for the equations of mathematical physics, such as Poisson's equation or the Stokes system. Upon discretization by—say—finite difference or finite element methods, these problems are reduced to linear algebraic systems of the form $A\mathbf{x} = \mathbf{b}$ where A is a large, sparse matrix and $\mathbf{b}$ a given right-hand side vector. Over time, a large number of techniques have been developed for solving linear systems. Such methods are traditionally grouped in two main categories: *direct* methods, which (barring rounding errors) are guaranteed to return the exact solution $\mathbf{x}_* = A^{-1}\mathbf{b}$ in a finite number of steps, and *iterative* methods, which produce a sequence of successive approximations $\mathbf{x}^{(k)}$ which, under appropriate conditions, converge to $\mathbf{x}_*$ as $k \to \infty$. Direct methods include Gaussian elimination, Cholesky factorization, and various orthogonalization schemes. These algorithms are the preferred method of solution for linear systems of small or moderate size. Iterative methods are better suited to solve large-scale linear systems. They form a much larger class, and many new such methods are still being proposed. Among the "classical" iterations we mention those of Jacobi [61], Seidel [81] and Richardson [74]. See [12, 59, 76, 85] for historical notes on classical iterative methods.

In 1929, Richard von Mises and Hilda Pollaczek-Geiringer published an important study [86] that provided the foundation for the development of a general theory of iterative methods for linear systems. A few INAC researchers, including Picone and Lamberto Cesari (1910–1990), who after World War II went on to a brilliant career in the United States, knew this paper well and made good use of it. In 1937 *La Ricerca Scientifica*, the journal of the *Consiglio Nazionale delle Ricerche*, published a paper [28] by Cesari on iterative methods (an extended abstract of the paper was also published in [27]). As customary, the paper was preceded by a brief introduction by Picone, which is worth reporting:

> The present work by Dr. Lamberto Cesari, collaborator of the Chair of the *Istituto per le Applicazioni del Calcolo*, makes a most remarkable contribution to the problem of solving systems of linear algebraic equations. With the aid brought by the present paper, the methods for the numerical solution of linear algebraic systems receive a rather advanced treatment in which, one can well expect, the Institute will always find what it needs in order to carry out, with a sufficient degree of approximation, the difficult final numerical evaluations that are required by the challenging scientific and technical problems it faces.[3]

[3]The Italian original reads: "Il presente lavoro del dott. Lamberto Cesari, coadiutore della Direzione dell'Istituto

Cesari's paper is truly remarkable. In it, the fundamental concept of *preconditioning* of a linear system is explicitly introduced, apparently for the first time (Cesari uses the word *trasformazione*). In addition, Cesari presents a general formalism, due in part to Picone, which allows him to give a unified convergence theory for the classical methods of Jacobi, Seidel, and von Mises (the latter being a stationary Richardson iteration). Subsequently, Cesari goes on to introduce a polynomial preconditioner $p(A)$ with the goal of reducing the spectral condition number of the linear system $A\mathbf{x} = \mathbf{b}$. The preconditioner is used to accelerate the rate of convergence of von Mises' method. The theory is illustrated by numerical experiments on a system of normal equations in three unknowns.

Cesari's paper, although not well known today, did not go unnoticed; see the references to it in [12, 45, 59, 87] as well as [76]. This paper is also important for the effect it had on Cimmino. The following year (1938) *La Ricerca Scientifica* published a short (8 pages) paper [32] by Cimmino titled "Calcolo approssimato per le soluzioni dei sistemi lineari" ("Approximate computation of the solutions of linear systems"). Again, the article is preceded by an introductory note by Picone:

> Prof. Gianfranco Cimmino can be considered, among other things, one of the founders of the Istituto per le Applicazioni del Calcolo, to which he lent his continuing and productive assistance during the embryonal stages of the Institute itself, in Naples, in the laboratory annexed to that university's Calculus chair, from 1928-VI to 1932-X. Towards the end of that period prof. Cimmino devised a numerical method for the approximate solution of systems of linear equations that he reminded me of in these days, following the recent publication by Dr. Cesari (...), which provides a systematic treatment of the above mentioned computing methods which, however, does not consider the one by Cimmino, a method which, in my opinion, is most worthy of consideration in the applications because of its generality, its efficiency and, finally, because of its guaranteed convergence which can make the method practicable in many cases. Therefore, I consider it useful to publish in this journal Prof. Cimmino's note on the above mentioned method, note that he has accepted to write upon my insistent invitation.[4]

Thus we learn that Cimmino's method dates back at least to 1932 and, moreover, that Cimmino may have never bothered to publish it had it not been for Picone's insistence. It is likely that at that time Cimmino was busy with his research on the generalized Dirichlet problem [31, 33] and reluctant to take time off to write on a topic which was widely regarded as one of lesser importance. Picone's insistence bears witness, once again, to his farsightedness. The content of Cimmino's paper [32] is examined next.[5]

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per le Applicazioni del Calcolo, apporta un notevolissimo contributo al problema della risoluzione numerica dei sistemi di equazioni lineari algebriche. Con gli apporti recati dal presente lavoro, i metodi di risoluzione numerica dei sistemi di equazioni lineari algebriche, ricevono una sistemazione assai progredita, nella quale, come è ben lecito presumere, l'Istituto potrà sempre trovare quanto occorre per le difficili definitive valutazioni numeriche, con un sufficiente grado di approssimazione, relative agli ardui problemi che gli sono assegnati dalla Scienza e dalla Tecnica."

[4] "Il prof. Gianfranco Cimmino è da considerarsi, anche, uno dei fautori dell'Istituto per le Applicazioni del Calcolo al quale prestò una assidua e proficua opera di assistenza durante il periodo embrionale dell'Istituto stesso, trascorso a Napoli, nel gabinetto annesso a quella Cattedra di Calcolo infinitesimale, dal 1928-VI al 1932-X. Verso la fine di tale periodo il prof. Cimmino escogitò un metodo numerico di approssimazione delle soluzioni dei sistemi di equazioni lineari che egli mi ha richiamato alla memoria in questi giorni in seguito alla pubblicazione del dott. Cesari, recentemente apparsa (...), nella quale è data una sistemazione dei sopradetti metodi di calcolo che, però, non contempla quello sopradetto del Cimmino, metodo che, secondo il mio avviso, è degnissimo di essere tenuto presente nelle applicazioni e per la sua generalità e per la sua rapidità di calcolo numerico delle successive approssimazioni, ed, infine, per la sua assicurata convergenza che, in molti casi, può dare al metodo il necessario carattere di praticità. Ritengo, perciò, utile pubblicare in questa Rivista la nota del prof. Cimmino relativa al suo sopradetto metodo, nota che egli ha accondisceso a redigere per mio insistente invito."

[5] Strangely, this important paper has not been included in the volume [39] of selected works of Cimmino.

5. Cimmino's method. In [32], Cimmino considers a system of linear algebraic equations $A\mathbf{x} = \mathbf{b}$ where A is a real $n \times n$ matrix, initially assumed to be nonsingular, and $\mathbf{b} \in \mathbb{R}^n$. If $\mathbf{a}_i^T = [a_{i1}, a_{i2}, \ldots, a_{in}]$ denotes the ith row of A, the solution $\mathbf{x}_* = A^{-1}\mathbf{b}$ is the unique intersection point of the n hyperplanes described by

$$\langle \mathbf{a}_i, \mathbf{x} \rangle = b_i, \quad i = 1, 2, \ldots, n\,. \tag{5.1}$$

Given an initial approximation $\mathbf{x}^{(0)} \in \mathbb{R}^n$, Cimmino takes, for each $i = 1, 2, \ldots, n$, the reflection (mirror image) $\mathbf{x}_i^{(0)}$ of $\mathbf{x}^{(0)}$ with respect to the hyperplane (5.1):

$$\mathbf{x}_i^{(0)} = \mathbf{x}^{(0)} + 2\,\frac{b_i - \langle \mathbf{a}_i, \mathbf{x}^{(0)} \rangle}{||\mathbf{a}_i||^2}\,\mathbf{a}_i \tag{5.2}$$

(here $||\cdot||$ denotes the Euclidean norm.) Given n arbitrarily chosen positive quantities $m_1, \ldots, m_n$, Cimmino constructs the next iterate $\mathbf{x}^{(1)}$ as the center of gravity of the system formed by placing the n masses m_i at the points $\mathbf{x}_i^{(0)}$ given by (5.2), for $i = 1, 2, \ldots, n$. Cimmino notes that the initial point $\mathbf{x}^{(0)}$ and its reflections with respect to the n hyperplanes (5.1) all lie on a hypersphere the center of which is precisely the point common to the n hyperplanes, namely, the solution of the linear system. Because the center of gravity of the system of masses $\{m_i\}_{i=1}^n$ must necessarily fall inside this hypersphere, it follows that the new iterate $\mathbf{x}^{(1)}$ is a better approximation to the solution than $\mathbf{x}^{(0)}$:

$$||\mathbf{x}^{(1)} - \mathbf{x}_*|| < ||\mathbf{x}^{(0)} - \mathbf{x}_*||\,.$$

At this point the procedure is repeated starting from the new approximation $\mathbf{x}^{(1)}$, and so forth. The sequence $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}_* = A^{-1}\mathbf{b}$ as $k \to \infty$.

In matrix form, Cimmino's method can be written as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{2}{\mu}\,A^T D^T D(\mathbf{b} - A\mathbf{x}^{(k)})$$

($k = 0, 1, \ldots$), where we have set

$$D = \begin{bmatrix} \frac{\sqrt{m_1}}{||\mathbf{a}_1||} & & & \\ & \frac{\sqrt{m_2}}{||\mathbf{a}_2||} & & \\ & & \ddots & \\ & & & \frac{\sqrt{m_n}}{||\mathbf{a}_n||} \end{bmatrix} \tag{5.3}$$

and $\mu = \sum_{i=1}^{n} m_i$. In particular, taking $m_i = ||\mathbf{a}_i||^2$ we obtain

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{2}{\mu}\,A^T(\mathbf{b} - A\mathbf{x}^{(k)}),$$

which is nothing but a special case of von Mises' mehod (stationary Richardson iteration) applied to the system of normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$. Moreover, if the rows of A are normalized so that $m_i = ||\mathbf{a}_i|| = 1$ for $i = 1, 2, \ldots, n$ then $\mu = n$ and Cimmino's method coincides with the under-relaxed Jacobi iteration for a special choice of the damping parameter (one that automatically guarantees convergence). Other choices of D besides the one in (5.3) are possible. In particular D need not be diagonal, and other methods can be obtained with different choices of D. However, this possibility is not noted by Cimmino, who does not appear to be aware of the relationship between his method and existing ones. It is noteworthy that the paper contains no bibliographic references.

Cimmino shows that the iterates converge to a solution of $A\mathbf{x} = \mathbf{b}$ even in the case of a singular (but consistent) system, provided that $\text{rank}\,(A) \geq 2$. He then notes that the sequence $\{\mathbf{x}^{(k)}\}$ converges even when the linear system is inconsistent, always provided that $\text{rank}\,(A) \geq 2$. Revisiting this topic many years later, Cimmino writes in [34]:

> The latter observation, however, is just a curiosity, being obviously devoided of any practical usefulness.[6]

Ironically, it is precisely this property that makes Cimmino's method useful in a variety of applications. We will return on this topic in the section 7 below.

In the singular case, Cimmino obtains a bound on the relative error in the Euclidean norm, showing the linear rate of convergence of his method. Cimmino concludes his note showing how, in principle, the method can be extended in a straightforward manner to the approximate solution of infinite-dimensional problems, in particular to Fredholm integral equations of the first kind. It is worth mentioning that Cimmino's note was reviewed for the *Zentralblatt* by Franz Rellich.

Cimmino's method is striking because of its simplicity and elegance. Unlike so many other algorithms for solving linear equations, it is based on a geometrical-mechanical construction rather than on algebraic manipulations. At the time it was conceived, the greatest attraction of the method was probably the fact that the method is always convergent: no restriction is imposed on the system matrix A except the extremely mild one of having rank at least 2.

It must be mentioned that an iterative method with some similarities (and many common features) to Cimmino's had been published in 1937 by the Polish mathematician Stefan Kaczmarz, a close collaborator of Hugo Steinhaus;[7] see [63]. In this method, the current approximation $\mathbf{x}^{(k)}$ is orthogonally projected (instead of reflected) onto the hyperplanes (5.1), not simultaneously but instead sequentially. The projection onto the nth hyperplane is taken as the new approximation $\mathbf{x}^{(k+1)}$, and the process is repeated. It is easy to see (using the triangle inequality) that the sequence $\{\mathbf{x}^{(k)}\}$ constructed in this manner converges to the solution of $A\mathbf{x} = \mathbf{b}$ as $k \to \infty$.

The methods of Cimmino and Kaczmarz are closely related. Cimmino's algorithm has been found to be better suited for parallel computers, whereas Kaczmarz's method tends to converge somewhat faster. It is of course possible to combine the two ideas to obtain hybrids. For instance, the original reflections in Cimmino's method are often replaced by (simultaneous) orthogonal projections onto the hyperplanes, but this is only a minor modification. Another feature the two methods have in common is that they have both been rediscovered several times over the years. More on this later.

6. Later developments. For a long time (several decades) Cimmino's method, in spite of its virtues, did not see much use. This is probably due to the fact that linear systems of moderate size were more efficiently solved by Gaussian elimination, while large linear systems arising from the discretization of PDEs were solved faster using specialized iterative methods that could exploit the properties and structure of the matrix [42]. It was also soon realized that except for special cases (e.g., when A is nearly orthogonal), the convergence of the method tends to be very slow. After the advent of electronic computers, in the Fifthies and Sixties, Cimmino's method was referenced in several surveys and monographs on matrix computations [12, 45, 48, 59] but some authors explicitly

[6] "Quest'ultima osservazione ha tuttavia il valore di una semplice curiosità, riuscendo essa evidentemente priva di utilità pratica."

[7] Tragically, Kaczmarz was killed in action in September of 1939 when the German troops invaded Poland.

advised against its use; see, e.g., Caprioli's passionate but somewhat misguided defense of Cimmino's method from Bodewig's criticism in [18]. Beginning in the early Eighties, however, an ever increasing number of researchers has turned to Cimmino's method. Currently there exist a number of variants and extensions of the algorithm, to the point that in the literature it has become customary to talk of *Cimmino-type methods.* This phrase stands for a large class of algorithms which are based on the idea of projecting a current approximation $\mathbf{x}^{(k)}$ *simultaneously* on the manifolds defined by a system of (possibly nonlinear) equations or inequalities, and take a weighted average of these projections as the new iterate $\mathbf{x}^{(k+1)}$.

The following is a list of noteworthy developments, together with a few representative references:

- Chebyschev, Lanczos and conjugate gradient acceleration of the classical Cimmino method [5, 6, 8, 75, 82, 84];
- Block variants for parallel computers [2, 4, 8, 13];
- Extensions to systems of linear inequalities [25, 26, 43];
- Extensions to nonlinear systems [24, 60];
- Extensions to operator equations in infinite-dimensional Hilbert and Banach spaces [16, 17, 35, 65, 70];
- Non-deterministic (Monte Carlo) versions of Cimmino's method [34, 35];
- Numerous applications, especially in the medical field [9, 23, 26, 70, 90, 91].

A theoretical advance with important practical consequences has been the realization that Cimmino's method enjoys the *regularization* property when applied to problems that are ill-posed (in the sense of Hadamard), including all operator equations of the form

$$A\mathbf{u} = \mathbf{f}\,,$$

where A is a compact linear operator on an infinite-dimensional Banach space X. Here the continuous problem is ill-posed because A^{-1}, if defined, is unbounded. In practice, such equations are reduced to finite-dimensional problems by discretization: when the original (continuous) problem is ill-posed, the discrete problem is severely ill-conditioned. This means that the singular values of A decay very rapidly to zero and thus A^{-1}, if it exists, has a huge norm. Therefore, the solution will be generally very sensitive to perturbations in the data.

Simply put, the regularization property means that after a certain number k of iterations Cimmino's method returns a sufficiently good approximation $\mathbf{u}^{(k)}$, and subsequent iterations not only do not improve the quality of the computed solution, but actually result in worse and worse approximations. This is because the iterates become completely dominated by the errors, inevitably present in the data, which are "amplified" by the operator A^{-1} which, in turn, is being approximated better and better as the iteration progresses. More precisely, consider the linear problem with a perturbed right-hand side

$$A\mathbf{u} = \mathbf{f} + \mathbf{e}\,,$$

where $\mathbf{e}$ represents some kind of "noise" (which could be due to measurement error, to discretization error, etc.). The exact solution of the perturbed problem, assuming (for simplicity) that A is invertible and that both $\mathbf{f}$ and $\mathbf{e}$ are in the range of A, is $\hat{\mathbf{u}} = A^{-1}\mathbf{f} + A^{-1}\mathbf{e}$, and since A^{-1} is huge this may have nothing to do with the "true" solution $\mathbf{x}_* = A^{-1}\mathbf{f}$, no matter how small $\mathbf{e}$ is in norm. The iterates computed with Cimmino's method tend to reconstruct first the "good" part of the solution (the *signal*), and only later the unwanted part (the *noise*). Hence, Cimmino's method initially "filters out" the noise, which gets reconstructed only later. The same property is enjoyed by other iterative methods as well, including Kaczmarz's and the conjugate gradient method (applied to $A^*A\mathbf{u} = A^*\mathbf{f}$, where A^* is the adjoint of A); see [7, 57, 83].

It is clear that in this context, the fact that convergence is very slow is not necessarily a drawback, and in fact it may be advantageous. The main difficulty with iterative regularization methods is to decide when to stop the iteration process. Several empirical rules have been devised to this end, but no universally agreed criterion exists. It is clear that if a method converges slowly, with the approximations not differing very much from one step to the next, running a few iterations more than strictly necessary is not going to significantly ruin an approximate solution.

It is well-known that a very important class of ill-posed problem is represented by the Fredholm integral equations of the first kind:

$$\int_\Omega K(s,t)\,u(t)\,dt = f(s), \quad s \in \Omega$$

with suitable assumptions on the domain Ω, on the kernel K, and on the data f. Equations of this type frequently occur, e.g., in image processing—a fundamental problem in medicine, astronomy, microscopy, etc. It is interesting that in his original 1938 paper, Cimmino discussed the extension of his method *precisely to this class of functional equations.* Cimmino himself returned to this topic during the Sixties and again in his last publications. Of special interest is his extension in a probabilistic sense of his method. The gist of his reasoning is the following. In order to improve the quality of an initial guess, it is possible to proceed in one of two ways. The first consists in taking additional iterations: as we know, the method will eventually converge to the exact solution of the linear system. The alternative is to apply one step of the iteration but to an equivalent enlarged system $\hat{A}\mathbf{x} = \hat{\mathbf{b}}$ obtained from the original system $A\mathbf{x} = \mathbf{b}$ by adding a number of equations which are linear combinations of the original ones. Consider for instance the case of a $n \times n$ system with a nonsingular matrix A. It is possible to add to the hyperplanes (5.1) an arbitrary number of additional hyperplanes passing through the same point $\mathbf{x}_*$. The new linear system has the coefficient matrix $\hat{A} \in \mathbb{R}^{m\times n}$ with $m > n$. Cimmino shows that taking one step of Cimmino's method on this enlarged system gives a better approximation to $\mathbf{x}_*$ than taking one iteration on the original system. Furthermore the larger m, the better the approximation, and indeed in the limit as $m \to \infty$ we obtain the exact solution in one step. The question then is how to choose the linear combinations of equations to be added to the system. Choosing the coefficients of the linear combination at random effectively leads to a Monte Carlo-type method for solving linear systems; see [34, 35].

In [36] Cimmino went on to consider the situation where instead of a finite number m of masses, a continuous mass distribution is given on the hypersphere whose center is the solution point $\mathbf{x}_*$. When the distribution is uniform, the center of gravity of the system is precisely the solution point $\mathbf{x}_*$, and one step of Cimmino's extended method yields explicit formulas for the solution components in terms of ratios of (hyper-)spherical integrals:

$$\mathbf{x}_* = n\,\frac{\displaystyle\int_\omega ||A\mathbf{x}||^{-n-2}\langle \mathbf{x}, \mathbf{b}\rangle\, A\mathbf{x}\, d\omega}{\displaystyle\int_\omega ||A\mathbf{x}||^{-n}\, d\omega} \tag{6.1}$$

where $\omega = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| = 1\}$. Such integrals, for n large, can be evaluated by Monte Carlo methods. The expressions (6.1) offer an alternative to the usual Cramer's formulas. See also [37, 38]. It is interesting to note that in [36] Cimmino wrote:

> I didn't try to check whether formulas (6.1) are already known, as it seems rather likely.

Indeed, the expression of the solutions of a linear algebraic system in terms of $(n-1)$-dimensional (spherical) integrals had already been given by Jacobi in [62].

7. Cimmino's legacy. We can give an idea of the impact of Cimmino's work on numerical analysis and scientific computing by means of data bases such as MathSciNet and ZMATH (the online versions of *Mathematical Reviews* and *Zentralblatt für Mathematik*, respectively) as well as Web of Science (online citation index, managed by the Institute for Scientific Information). Concerning the years up to 1980 we were able to identify just eight journal articles citing Cimmino's method, four of which from the Seventies. After 1980, the number of papers that mention Cimmino's method increases very rapidly:

Period	No. of papers
1981-1990	32
1991-2000	56
2001-2004	34

These are, of course, only lower bounds on the true numbers. The above-mentioned data bases do not cover all journals, and (especially in recent years) many authors discuss Cimmino's method based on second-hand accounts and do not cite the original 1938 paper. Also, many conference proceedings papers are not included in the count. It is perhaps worth mentioning that the citations of [32] far outnumber all other citations of Cimmino's works, and that after 1980 there are almost no references to Cimmino's work in analysis and PDEs. Thus today, in the literature, Cimmino's name is mostly remembered because of his method for solving linear systems.

We are also aware of at least sixteen monographs that describe (or otherwise reference) Cimmino's method. The list includes books on matrix computations, linear algebra, optimization, numerical analysis, inverse problems, and economics: see [1, 9, 11, 12, 14, 26, 29, 45, 50, 51, 59, 68, 72, 73, 75, 84]. There are probably several more. In many of these books, the Cimmino and Kaczmarz methods are often discussed one after the other. Incidentally, the history of Kaczmarz's method is very similar to that of Cimmino's. Both methods have been generalized and extended in similar directions, and both methods have been rediscovered a number of times, especially by researchers in applied areas. A notable example is the field of computerized tomography, where the method of Kaczmarz was rediscovered around 1970 under the name of *ART* (for Algebraic Reconstruction Technique); see [53]. A class of algorithms closely related to Cimmino's, known as *SIRT* (for Simultaneous Reconstruction Techniques), was introduced around the same time in [52]; see also [83]. Closely related to Cimmino's methods are the iterative processes known as *Landweber's method* [66] and *Bialy's method*, see [10]. A similar method is the one proposed by Fridman [49]; see Grace Wahba's discussion of the "Richardson-Landweber-Fridman-Picard-Cimmino method(s)" in [88]. Several other authors have proposed variants of these methods, usually unaware of previous work.

The resurgence of interest in old algorithms like Cimmino's and Kaczmarz's is primarily due to two technological advances:

- The widespread diffusion, around 1970, of tomography (CAT scans) in radiology;
- The appearance, about a decade later, of parallel computers.

Concerning the first aspect, the methods of Kaczmarz and Cimmino were found to be especially well-suited because of their regularizing properties (image reconstruction from projections is an ill-posed problem) and because of their extremely low storage demand. The latter feature is especially evident for Kaczmarz's method, which only requires one row of the coefficient matrix (and the corresponding entry in the right-hand side vector) to perform one iteration: in other words, Kaczmarz's iteration is a *row-action method*, see [22]. However, Cimmino's method can also implemented in this way. As for parallel computing, we have already commented on the fact that Cimmino's method is ideally suited for parallel implementation, and such implementations have indeed proved to be efficient;

see, e.g, [8]. Parallel implementation of Kaczmarz's method seems to be less straightforward, but blocks versions of it can be efficiently parallelized; see for instance [41, 64]. Another attractive feature of Cimmino's method (not shared by Kaczmarz's method in its original form) is the fact that the method is always convergent even in the case of a rank-deficient and inconsistent linear system. When the system is inconsistent, the iterates generated by Cimmino's method converge to a weighted least-squares solution; if, moreover, $m_i = ||\mathbf{a}_i||_2^2$, Cimmino's method produces a solution to the least-squares problem, $||A\mathbf{x} - \mathbf{b}||_2 = \min$. Moreover, choosing $\mathbf{x}^{(0)} = \mathbf{0}$ results in the minimum norm solution of the least-squares problem; see [42, pages 628–629]. The latter property is of great importance in radiation therapy planning, where minimum norm solutions correspond to feasible treatment plans of minimum intensity.

At present, Cimmino's method (in one or another of its modern "reincarnations") finds application primarily in the following areas:

- Convex mathematical programming, especially convex feasibility problems (the problem of determining whether a family of convex sets has non-empty intersection, and if so to find a common point);
- Fast adaptation of radiation therapy planning;
- Solution of inverse problems in astronomy, medical physics and geophysics;
- Image reconstruction from projections;
- Training of neural networks;
- Solution of large linear systems arising from the discretization of Fredholm integral equations of the first kind.

In particular, fast adaptation of radiation therapy plans in oncology seems to be one of the most important areas of applications of Cimmino's method in the last few years; see, e.g., [90]. We notice that virtually all the above mentioned problems are ill-posed. Morover, these problems are often nonlinear and are expressed in terms of inequalities rather than equalities. As a matter of fact, both Cimmino's method and Kaczmarz's are rarely used for solving systems of linear equations—the type of problem they were originally conceived for. This is mostly because far more efficient methods exist nowadays for solving large linear systems. This can be best understood by observing that Cimmino's method is equivalent to a damped Jacobi iteration applied to the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$. Likewise, Kaczmarz's method is equivalent to the classical Gauss-Seidel method applied to the system $AA^T\mathbf{y} = \mathbf{b}$, with $A^T\mathbf{y} = \mathbf{x}$. The normal equations interpretation has led to the suggestion that block versions of the Cimmino and Kaczmarz algorithms may be used as preconditioners for the conjugate gradient method. Such preconditioned solvers have been developed and applied to the solution of large sparse systems arising from the discretization of partial differential equations and other problems [6, 8, 13, 75]. However, while very robust, this approach has been found to be generally inferior to preconditioned Krylov methods applied directly to $A\mathbf{x} = \mathbf{b}$. We mention that Cimmino-type and Kaczmarz-type iterations are also used as smoothers for multigrid methods applied to nonsymmetric and indefinite systems, see [56].

Some idea of the popularity enjoyed by Cimmino's method in various scientific and technical areas can be gained by noting that articles on the use of Cimmino's method have appeared in journals like *Physics in Medicine and Biology, Medical Physics, Inverse Problems, Int. J. of Radiation Oncology, Biology and Physics, IEEE Transactions on Medical Imaging, IEEE Transactions on Image Processing, IEEE Transactions on Signal Processing, Annals of Operations Research*, and so forth. In addition to these, dozens of articles on Cimmino's methods and its extensions have appeared in journals devoted to scientific computing, numerical analysis, parallel computing, optimization, functional analysis, convex analysis, linear algebra, etc.

8. Epilogue. The total number of Gianfranco Cimmino's published papers in the field of numerical analysis is small, not exceeding 30 printed pages. Yet, this work (primarily the 1938 paper [32]) has had a remarkable, if belated, impact and has proven to be of lasting importance for many areas of applied scientific computing.

While Cimmino did not live to see how extensive the influence of his numerical work would be, he did know that algorithms based on his 1938 paper were being developed and applied in the medical field. Indeed, one of his last reviews for the *Zentralblatt*, appeared in 1988, concerned a paper on the use of Cimmino's method in radiation therapy planning [23]. He must have been pleased to learn that his elegant, youthful intuition had been found to be useful in the medical field.

The story of Cimmino's method is interesting for several reasons. It shows that good mathematical ideas may take many years to become fully appreciated and to be brought to fruition. Major technological changes may completely transform the perception of an algorithm, from useless curiosity to brilliant invention. If an idea is indeed good, it is almost certain that it will eventually come to forefront (perhaps through re-discovery), no matter how obscure the original publication or how long the period of oblivion. Of course Cimmino's method is not unique in this respect, and one could give numerous other examples of this phenomenon.

It is also interesting to observe that Cimmino's impact on his main research area, the theory of partial differential equations, while nonnegligible, has not been as great and as lasting as his work in numerical mathematics. This can be explained perhaps by the fact that the theory of PDEs was already a fairly well-developed area by the time Cimmino started making contributions to it, whereas numerical linear algebra in the 1930s was still in a very primitive stage. In this sense, Cimmino (like Cesari), through his association with Picone's Institute, was well-positioned to make pioneering contributions to the numerical field. It should be noted that Cimmino's later papers on solving linear systems, written at a time when numerical linear algebra was maturing as an independent discipline, have had virtually no impact on the field.

It is clear from reading Cimmino's papers that he regarded his work on solving linear systems almost as a hobby, something he did as an amateur rather than as a professional mathematician. And indeed some of his statements, including ones made at a time when iterative methods for linear systems were already being vigorously developed (in Europe, USA and USSR), strike us as rather naïve. It is also clear from the lack of references in his papers that Cimmino did not follow the numerical analysis literature, in spite of his long-lasting interest in numerical questions. And the lack of numerical experiments suggests that Cimmino was not knowledgeable about (or even interested in) computers. This shows that valuable contributions to a specific discipline may well come from outsiders. However, this is probably unlikely to happen for those fields that have already reached a considerable level of maturity and/or which require a great deal of specialization.

Acknowledgment. This paper is based on the text of an invited talk I gave at the *Seminario di Analisi Matematica* at the University of Bologna on May 7, 2004. The talk was part of a series of lectures devoted to the memory of Gianfranco Cimmino. My sincere thanks to Professors Bruno Pini and Ermanno Lanconelli, coordinators of the *Seminario*, for their kind invitation and financial support.

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