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## A New Approach to Hilbert's Theorem on Ternary Quartics

by

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# A New Approach to Hilbert's Theorem on Ternary Quartics

# Une nouvelle approche du théorème de Hilbert sur les quartiques ternaires

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**Abstract**

Hilbert proved that a non-negative real quartic form $f(x, y, z)$ is the sum of three squares of quadratic forms. We give a new proof which shows that if the plane curve $Q$ defined by $f$ is smooth, then $f$ has exactly 8 such representations, up to equivalence. They correspond to those real 2-torsion points of the Jacobian of $Q$ which are not represented by a conjugation-invariant divisor on $Q$.

**Résumé**

Hilbert a démontré qu'une forme réelle non négative $f(x, y, z)$ de degré 4 est la somme de trois carrés de formes quadratiques. Nous donnons une nouvelle démonstration qui montre que si la courbe plane $Q$ definie par $f$ est non singulière, alors $f$ a exactement 8 telles représentations, à equivalence près. Elles correspondent aux points de 2- torsion du jacobien de $Q$ qui ne sont pas représentés par un diviseur de $Q$ invariant par conjugaison.

## 1. Introduction

A *ternary quartic form* is a homogeneous polynomial $f(x, y, z)$ of degree 4 in three variables. If $f$ has real coefficients, then $f$ is *non-negative* if $f(x, y, z) \geq 0$ for all real $x$, $y$, $z$. Hilbert [4] showed that every non-negative real ternary quartic form is a sum of three squares of quadratic forms. His proof (see [7], [8] for modern expositions) was non-constructive: The map $\pi\colon\ (p, q, r) \longmapsto p^2 + q^2 + r^2$ from triples of real quadratic forms to non-negative quartic forms is surjective, as it is both open and closed when restricted

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to the preimage of the (dense) connected set of non-negative quartic forms which define a smooth complex plane curve. An elementary and constructive approach to Hilbert's theorem was recently begun by Pfister [5].

A *quadratic representation* of a complex ternary quartic form $f = f(x, y, z)$ is an expression

$$f = p^2 + q^2 + r^2 \tag{1}$$

where $p$, $q$, $r$ are complex quadratic forms. A representation $f = (p')^2 + (q')^2 + (r')^2$ is *equivalent* to this if $p, q, r$ and $p', q', r'$ have the same linear span in the space of quadratic forms.

Powers and Reznick [6] investigated quadratic representations computationally, using the Gram matrix method of [1]. In several examples of non-negative real ternary quartics, they found 63 inequivalent representations as a sum of three squares of complex quadratic forms and 15 were sums or differences of squares of real forms. We explain these numbers, in particular the number 15, and show that precisely 8 of the 15 are sums of squares.

If the complex plane curve $Q$ defined by $f = 0$ is smooth, it has genus 3, and so the Jacobian $J$ of $Q$ has $2^6 - 1 = 63$ non-zero 2-torsion points. Coble [2, Chap 1,§14] showed that these are in one-to-one correspondence with equivalence classes of quadratic representations of $f$. If $f$ is real, then $Q$ and $J$ are defined over $\mathbb{R}$. The non-zero 2-torsion points of $J(\mathbb{R})$ correspond to *signed quadratic representations* $f = \pm p_1^2 \pm p_2^2 \pm p_3^2$, where $p_i$ are real quadratic forms. If $f$ is also non-negative, the real Lie group $J(\mathbb{R})$ has two connected components, and hence has $2^4 - 1 = 15$ non-zero 2-torsion points. We use Galois cohomology to determine which 2-torsion points give rise to sum of squares representations over $\mathbb{R}$.

**Theorem 1** *Suppose that $f(x, y, z)$ is a non-negative real quartic form which defines a smooth plane curve $Q$. Then the inequivalent representations of $f$ as a sum of three squares are in one-to-one correspondence with the eight 2-torsion points in the non-identity component of $J(\mathbb{R})$, where $J$ is the Jacobian of $Q$.*

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## 2. Quadratic representations of smooth ternary quartics

Let $f(x,y,z)$ be an irreducible quartic form over $\mathbb{C}$, and let $Q$ be the curve $f = 0$ in the complex projective plane. Assume that $Q$ is smooth. The Picard group $\mathrm{Pic}(Q)$ of $Q$ is the group of Weil divisors on $Q$, modulo divisors of rational functions. Let $J$ be the Jacobian of $Q$, so that $J$ is the identity component of $\mathrm{Pic}(Q)$. The following proposition is due to Coble [2, Chap 1,§14].

**Proposition 1** *The non-trivial 2-torsion points of $J$ are in one-to-one correspondence with the equivalence classes of quadratic representations of $f$.*

*Proof.* Given a quadratic representation (1), consider the map $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2, \quad x \mapsto \big(p(x) : q(x) : r(x)\big)$. The image of $Q$ under $\varphi$ is the conic $C$ defined by the equation $y_0^2 + y_1^2 + y_2^2 = 0$. Let $y$ be any point in $C$, then $\varphi^*(y)$ is an effective divisor of degree 4 that is not the divisor of a linear form. Indeed, after a linear change of coordinates we can assume $y = (0 : 1 : i)$. A linear form vanishing on $\varphi^*(y)$ would divide each conic $\alpha p + \beta(q + ir)$ through $\varphi^*(y)$, and thus would divide $f = p^2 + (q + ir)(q - ir)$, contradicting the irreducibility of $f$.

Fix a linear form $\ell$, then $L := \mathrm{div}(\ell)$ is an effective divisor of degree 4 on $Q$. Let $\zeta = [\varphi^*(y) - L]$. Since $2y$ is the divisor of a linear form (the tangent line to $C$ at $y$), $\varphi^*(2y)$ is the divisor on $Q$ of a quadratic form. Thus $2\zeta = 0$. Moreover, $\zeta \neq 0$ as $\varphi^*(y)$ is not the divisor of a linear form. The 2-torsion point $\zeta$ of $J$ depends only upon the map $\varphi$.

Conversely, suppose that $\zeta \in J(\mathbb{C})$ is a non-zero 2-torsion point. Let $D \neq D'$ be effective divisors which represent the class $\zeta + [L]$ in $\mathrm{Pic}(Q)$. As $Q$ has genus 3, the Riemann-Roch Theorem implies that there is a pencil of such divisors. Then $2D$, $2D'$ and $D + D'$ are effective divisors of degree 8, and are linearly equivalent to $2L$, the divisor of a conic. Again, the Riemann-Roch Theorem implies that there are quadratic forms $q_0$, $q_1$ and $q_2$ so that

$$\operatorname{div}(q_0) = 2D, \quad \operatorname{div}(q_1) = 2D' \quad \text{and} \quad \operatorname{div}(q_2) = D + D'.$$

Therefore, the rational function $g := q_0 q_1 / q_2^2$ on $Q$ is constant. Scaling $q_1$ and $q_2$ appropriately, we may assume that $g \equiv 1$ on $Q$ and also that $f = q_0 q_1 - q_2^2$. Diagonalizing the quadratic form $q_0 q_1 - q_2^2$ gives a quadratic representation for $f$. This defines the inverse of the previous map. □

### 3. Quadratic representations of real quartics

Suppose now that $f$ is a non-negative real quartic form defining a smooth real plane curve $Q$ with complexification $Q_{\mathbb{C}} = Q \otimes_{\mathbb{R}} \mathbb{C}$. The elements of $\operatorname{Pic}(Q)$ can be identified with those divisor classes in $\operatorname{Pic}(Q_{\mathbb{C}})$ that are represented by a conjugation-invariant divisor. Let $J$ be the Jacobian of $Q$.

If $\zeta \in J(\mathbb{C})$ is the 2-torsion point corresponding to a signed quadratic representation

$$f \;=\; \pm p^2 \pm q^2 \pm r^2$$

consisting of real polynomials $p$, $q$, $r$, then $\zeta = \overline{\zeta}$, i.e., $\zeta \in J(\mathbb{R})$.

Conversely, let $0 \neq \zeta \in J(\mathbb{R})$ with $2\zeta = 0$, and let $L$ be the divisor on $Q$ of a linear form $\ell$. We can choose an effective divisor $D \neq \overline{D}$ on $Q_{\mathbb{C}}$ representing the class $\zeta + [L]$. Then $2D$, $2\overline{D}$ and $D + \overline{D}$ are each equivalent to $2L$. Let $r$ be a real quadratic form with divisor $D + \overline{D}$, and let $g$ be a complex quadratic form with divisor $2D$ (both divisors taken on $Q_{\mathbb{C}}$).

Since $D \sim \overline{D}$, there is a rational function $h$ on $Q_{\mathbb{C}}$ with $\operatorname{div}(h) = \overline{D} - D$. Let $c = h\overline{h}$, a nonzero real constant on $Q$. Since $\operatorname{div}(r) = \operatorname{div}(g) + \operatorname{div}(h)$, there is a complex number $\alpha \neq 0$ with $\frac{r}{g} = \alpha h$ on $Q$, which implies that

$$c|\alpha|^2 \;=\; \frac{r}{g} \cdot \frac{\overline{r}}{\overline{g}} \;=\; \frac{r^2}{p^2 + q^2}$$

on $Q$, where $p$ and $q$ are the real and imaginary parts of $g = p + iq$. So the quartic form

$$u \;:=\; r^2 - c\,|\alpha|^2 (p^2 + q^2)$$

vanishes identically on $Q$. Since $u \neq 0$, $f$ is a constant multiple of $u$. If $c > 0$, we get a signed quadratic representation of $f$, with both signs $\pm$ occuring. If $c < 0$, $f$ must be a positive multiple of $u$ since $f$ is non-negative, and we get a representation of $f$ as a sum of three squares of real forms.

We now calculate the sign of $c$. For this we use the well-known exact sequence

$$0 \to \operatorname{Pic}(Q) \to \operatorname{Pic}(Q_{\mathbb{C}})^G \xrightarrow{\partial} \operatorname{Br}(\mathbb{R}) \to \operatorname{Br}(Q).$$

It arises from the Hochschild-Serre spectral sequence for étale cohomology with coefficients $\mathbb{G}_m$. Here $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\operatorname{Pic}(Q_{\mathbb{C}})$ by conjugation, and $\operatorname{Pic}(Q_{\mathbb{C}})^G$ is the group of $G$-invariant divisor classes. Moreover, $\operatorname{Br}(\mathbb{R})$ is the Brauer group of $\mathbb{R}$, which is of order 2, and $\operatorname{Br}(Q)$, the Brauer group of $Q$, can be identified with the subgroup of $\operatorname{Br}\mathbb{R}(Q)$ consisting of all Brauer classes which are everywhere unramified. The map $\operatorname{Br}(\mathbb{R}) \to \operatorname{Br}(Q)$ is the restriction map.

It is easy to see that $c < 0$ if and only if $\partial(\zeta)$ is the non-trivial class in $\operatorname{Br}(\mathbb{R})$.

By a theorem of Witt [11], every non-negative rational function on a smooth projective curve over $\mathbb{R}$ is a sum of two squares of rational functions. Since $Q$ is smooth and $f$ is non-negative, this forces $Q(\mathbb{R}) = \emptyset$. Hence $-1$ is a sum of two squares in $\mathbb{R}(Q)$. This means $(-1,-1) = 0$ in $\operatorname{Br}(Q)$, and hence the map $\partial$ is surjective.

Since the genus of $Q$ is odd (equal to 3), a theorem of Weichold [10,3] implies that all classes in $\operatorname{Pic}(Q_{\mathbb{C}})^G$ have even degree, and the real Lie group $J(\mathbb{R})$ has exactly two connected components. Thus the sequence

$$0 \to J(\mathbb{R})^0 \to J(\mathbb{R}) \xrightarrow{\partial} \operatorname{Br}(\mathbb{R}) \to 0$$

is (split) exact. Since $J(\mathbb{R})^0 \cong (S^1)^3$ as a real Lie group, there exist $2^4 - 1 = 15$ non-zero 2-torsion classes in $J(\mathbb{R})$. The 8 that do not lie in $J(\mathbb{R})^0$, or equivalently, which cannot be represented by a conjugation-invariant divisor on $Q_{\mathbb{C}}$, are precisely those that give rise to sums of squares representations of $f$. This completes the proof of Theorem 1.

We close with a few remarks about the singular case. Wall [9] studies quadratic representations of (possibly singular) complex ternary quartic forms $f$. If $f$ is irreducible, the non-trivial 2-torsion points on the generalized Jacobian of the curve $Q = \{f = 0\}$ again give equivalence classes of quadratic representations of $f$. These representations are special in that they have no basepoints.

By classifying all possibilities for quadratic representations for each possible base locus in the case that the form $f$ is real and non-negative, one arrives at the number of inequivalent quadratic representations of $f$. This classification, together with arguments from Galois cohomology, gives all inequivalent representations of $f$ as a sum of squares. If $f$ is reducible, different methods can be applied to complete the picture. This complete analysis will appear in an unabridged version.

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