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Concerning Cut Point Spaces of Order Three

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CONCERNING CUT POINT SPACES OF ORDER THREE

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Abstract. A point p of a topological space X is a cut point of X if $X - \{p\}$ is disconnected. Further, if $X - \{p\}$ has precisely m components for some natural number $m \geq 2$ we will say that p has cut point order m. If each point y of a connected space Y is a cut point of Y, we will say that Y is a cut point space. Herein we construct a space S so that S is a connected Hausdorff space and each point of S is a cut point of order three. We also note that there is no separable cut point space with each point a cut point of order three and therefore no such space may be embedded in a Euclidean space.

1. Introduction

The study of cut points in topological spaces has long been of interest. Whyburn (e.g. [8], [9], [10]) studied heavily the role of cut points of metric continua. In particular, he showed that all cut points of a separable metric continuum are of order two except for a countable number.

M. Shimrat [5] proved that the following are equivalent for a non-empty connected separable metric space X: (1) X is locally connected and every point of X is a cut point; (2) X is locally arcwise connected, contains no simple closed curves, and has no end-points; (3) X is an open ramification. The reader is also referred to Stone [6].

A J. Ward [7] showed that every metric space that is separable, connected and locally connected, and in which each point is a strong cut point (having cut point order two), is homeomorphic to the real line R. Franklin and Krishnarao [1] have shown that the same characterization does not hold for Hausdorff spaces. Klieber [3] has provided a characterization similar to that of Ward's, namely that a separable Hausdorff space X is homeomorphic to R if every $x \in X$ is a strong cut point and the set of components of complements of point sets forms a subbase for the space X.

A comprehensive study of cut point spaces in the most general setting has been done by Honari and Bahrtampour [2]; the work is done without the assumption of any separation axioms. It is shown that each cut-point is either open or closed and that every cut-point space has infinitely many closed points and is non-compact. It is also shown that there is just one irreducible cut-point space, to within homeomorphism, namely the "Khalimsky line". This is a topology on the set $\mathbf{Z}$ of all integers, in which each odd integer is isolated and each even integer n has a smallest neighborhood $\{n-1, n, n+1\}$.

A natural question is whether a connected space may have each point be a cut point of fixed order greater than or equal to three. Herein we complement the studies mentioned above by constructing a space S so that S is a connected Hausdorff space and each point of S is a cut point of order three. We also demonstrate in Section 4 that no cut point space with each point a cut point of order

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three may be embedded in a Euclidean space, and indeed that no such space can be embedded in a hereditarily separable Hausdorff space.

2. Preliminaries

We will say that a point p of a topological space X is a cut point of X if $X - \{p\}$ is disconnected. Further, if $X - \{p\}$ has precisely m components for some natural number $m \geq 2$ we will say that p has cut point order m. If each point y of a connected space Y is a cut point of Y, we will say that Y is a cut point space. If N is a natural number greater than or equal to two and each point y of a cut point space Y has cut point order N, we will say that Y is a cut point space of order N.

For a space X and $A \subseteq X$, $\mathrm{Cl}(A)$ will denote the closure of A in X. For subsets A and B of space X, we will say that A and B are mutually separated if and only if $\mathrm{Cl}(A) \cap B = \emptyset$ and $A \cap \mathrm{Cl}(B) = \emptyset$. For points x and y in the Euclidean space $\mathcal{R}^2$, let $d(x, y)$ denote the Euclidean distance between x and y and, for $\epsilon > 0$, let $N(x, \epsilon)$ denote the open neighborhood $\{y : d(x, y) < \epsilon\}$.

3. Construction of Cut Point space S

We first construct a connected set in the plane each point of which is a cut point of order two or three. The closure of this set is a well known dendrite.

Consider the open interval $G_0 = (0, 1) \times \{0\}$ on the x-axis in $\mathcal{R}^2$. Although not itself an element of the space, the origin will play a special role when we define the topology for our space and will be denoted by $\mathcal{O}$. Let D be the set of all dyadic rational numbers in $(0, 1)$. That is, let $x \in D$ if and only if there is a positive integer n and a positive integer k such that $k \leq n$ and $x = (2k - 1)/2^n$. For each $x = (2k - 1)/2^n \in D$, let I_x denote the open vertical interval $\{x\} \times (0, 1/2^n)$. Let G_1 be the set of all these intervals I_x. Next, for each interval g in G_1, add a collection of open horizontal intervals as was done for G_0. The midpoint p of each $g \in G_1$ should have an interval added of length half the length of g with left endpoint at p. Call this collection of open intervals G_2. Next add a collection of open vertical intervals for each interval in G_2 in the same manner. Call this collection of open intervals G_3. Continue this process inductively. No two intervals in $\bigcup_{i>0} G_i$ should intersect. Let M_0 be the connected union of all these intervals; Figure 1 gives an indication of the first few steps in the construction of M_0.

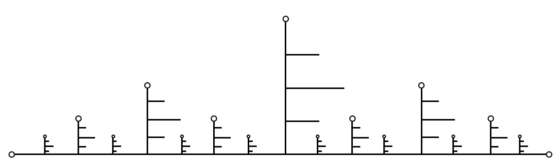

Figure 1

Let T_0 be the set of cut points of M_0 of order three and let C_0 be the set of cut points of M_0 of order two. For each whole number n, let M_n denote the set of all sequences $(p_0, p_1, \ldots, p_n)$ such that $p_n \in M_0$ and if $n > 0$, then $p_i \in C_0$ for each i such that $0 \leq i < n$. If $(p_0) \in M_0$, we may refer to (p_0) simply as p_0.

Let $S = \cup_{i=0}^{\infty} M_i$; S is the set of points (finite sequences) on which we will define a topology $\mathcal{T}$. If $p \in S$, then for each positive number ϵ we will define a subset $R(p, \epsilon)$ of S containing p. Let

$B_p = \{R(p,\epsilon) : \epsilon > 0\}$. The members of B_p will be called regions and the union of all of the sets B_p for $p \in S$ will form a basis for $\mathcal{T}$.

Let $p \in S$. Then $p = (p_0, p_1, \cdots, p_n)$ is in M_n, $p_n \in M_0$ and if $n > 0$ then for each i such that $0 \leq i < n$, $p_i \in C_0$. Let $\epsilon > 0$.

We next define our regions $R(p,\epsilon)$.

1) If $p_n \in T_0$, then
if $n = 0$, $R(p,\epsilon) = N(p_0,\epsilon) \cap T_0$, and
if $n > 0$, $R(p,\epsilon) = (p_0, p_1, \cdots, p_{n-1}) \times (N(p_n,\epsilon) \cap T_0)$.

2) If $p_n \in C_0$, then
if $n = 0$, $R(p,\epsilon) = \{p_0\} \cup (N(p_0,\epsilon) \cap T_0) \cup (p_0 \times (N(\mathcal{O},\epsilon) \cap T_0))$, and
if $n > 0$, $R(p,\epsilon) = (\{p\} \cup (p_0, p_1, \cdots, p_{n-1}) \times (N(p_n,\epsilon) \cap T_0)) \cup ((p_0, p_1, \cdots, p_n) \times (N(\mathcal{O},\epsilon) \cap T_0))$.

The next two lemmas are direct applications of our definitions.

Lemma 1. *If $p \in S$, $\epsilon > 0$, $\delta > 0$, and $\epsilon < \delta$ then $R(p,\epsilon) \subseteq R(p,\delta)$.*

Lemma 2. *If $p \in S$, $\epsilon > 0$, and $q \in R(p,\epsilon)$ then there is a positive number δ such that $R(q,\delta) \subseteq R(p,\epsilon)$.*

Theorem 3. *$B = \{B_p = R(p,\epsilon) : p \in S, \epsilon > 0\}$ is a basis for a topology $\mathcal{T}$ on S.*

Here we must show that if a point p is in each of the regions U and V, there is a region containing p that is a subset of $U \cap V$. The proof is a direct application of Lemmas 1 and 2.

Theorem 4. *$(S,\mathcal{T})$ is Hausdorff.*

Proof. Suppose $p = (p_0, p_1, \ldots, p_n)$ and $q = (q_0, q_1, \ldots, q_m)$ are distinct elements of S. We consider two cases:

Case 1. Assume $m = n$. Select ϵ to be one-third of the distance between p_n and q_n. Then $N(p_n,\epsilon) \cap N(q_n,\epsilon) = \emptyset$ and therefore $R(p,\epsilon) \cap R(q,\epsilon) = \emptyset$.

Case 2. Assume without loss of generality that $m > n$. Since for any point $x \in S$ and any $\epsilon > 0$, if $x \in M_n$ then $R(x,\epsilon) \subseteq M_n \cup M_{n+1}$, then $R(p,\epsilon) \cap R(q,\epsilon) = \emptyset$ unless $m = n+1$. In this case we set ϵ to be less than one-third of the distance from q_m to $\mathcal{O}$. Thus we have that $N(q_m,\epsilon) \cap N(\mathcal{O},\epsilon) = \emptyset$. It follows that $R(p,\epsilon) \cap R(q,\epsilon) = \emptyset$. □

Theorem 5. *$(S,\mathcal{T})$ is connected.*

Proof. We begin by showing that M_0 with the subspace topology of S is connected. Assume not. Then there is a non-empty set $U \neq M_0$ open relative to M_0 such that no point is a boundary point of U. If $x \in U$, then there exists an $\epsilon_x > 0$ such that $N(x,\epsilon_x) \cap T_0 \subseteq U$. Moreover, if $p \in N(x,\epsilon_x) \cap C_0$, $p \in U$ since otherwise p is a boundary point of U. Thus $U = [\cup_{x \in U}(N(x,\epsilon_x) \cap T_0)] \cup (C_0 \cap U) = \cup_{x \in U}[N(x,\epsilon_x)]$. Then U is a non-empty open set in M_0 with the subspace topology of R^2 such that no point is a boundary point of U, a contradiction.

We next show that $M_0 \cup M_1$ with the subspace topology of S is connected. If $p_0 \in C_0$ then p_0 is a limit point of $M_1(p_0) = p_0 \times M_0$ and $M_1(p_0)$ is connected since M_0 is connected. Now $M_1 = \bigcup_{x \in C_0} M_1(x)$ so $M_0 \cup M_1$ is the union of a collection of connected sets one of which, M_0, contains a limit point of each of the others so $M_0 \cup M_1$ is connected.

By a similar argument and by induction $\bigcup_{i=0}^{k} M_k$ is connected for each natural number k. It then follows that $S = \bigcup_{i=0}^{\infty} M_k$ is connected. □

Lemma 6. *With M_0 having the subspace topology of S, each point of T_0 is a cut point of order three in M_0 and each point of C_0 is a cut point of order two in M_0.*

Proof. Suppose $t \in T_0$. If M_0 were to have the subspace topology of the plane, it is clear that t would have cut point order three with $M_0 - \{t_0\} = K_1 \cup K_2 \cup K_3$ such that K_1, K_2 and K_3 are pairwise mutually separated and each is connected. We claim that K_1, K_2 and K_3 are also the pairwise mutually separated components of $M_0 - \{t_0\}$, where M_0 has the subspace topology of S.

We show that $\mathrm{Cl}(K_1) \cap K_2 = \emptyset$. Assume that $s \in \mathrm{Cl}(K_1) \cap K_2$. Then for each natural number j, $R(s, \frac{1}{j}) \cap K_1 \neq \emptyset$. Then $N(s, \frac{1}{j}) \cap K_1 \neq \emptyset$ and K_1 and K_2 are not mutually separated with M_0 having the subspace topology of the plane, a contradiction. In a similar way, $K_1 \cap \mathrm{Cl}(K_2) = \emptyset$ and K_1 and K_2 are mutually separated. By parallel arguments, the pairs K_1 and K_3 and K_2 and K_3, respectively, are mutually separated.

By a proof similar to that of Theorem 5, each of K_1, K_2, and K_3 is connected in S, and therefore $t \in T_0$ is a cut point of order three in $M_0 \subset S$.

Suppose $c \in C_0$. If M_0 were to have the subspace topology of the plane, it is clear that c would have cut point order two with $M_0 - \{c_0\} = K_1 \cup K_2$ such that K_1 and K_2 are mutually separated and each is connected. By an argument like that above, K_1 and K_2 are also the mutually separated components of $M_0 - \{c_0\}$, where M_0 has the subspace topology of S. Therefore $c \in C_0$ is a cut point of order two in $M_0 \subset S$. □

Lemma 7. *If q_0 is a fixed element of C_0 then the collection of sequences $Q_0 = \{(q_0, p_1, \cdots, p_n)\}$ in S for all whole numbers n is connected. Furthermore, $Q_0 - \{q_0\}$ is connected.*

Proof. Note that $N_1' = \{q_0\} \times M_0$ is connected since M_0 is connected. Since q_0 is a limit point of N_1', $N_1 = N_1' \cup \{q_0\}$ is also connected. Similarly, $N_2'(x) = (q_0, x) \times M_0$ is connected for each $x \in C_0$. As before, (q_0, x) is a limit point of $N_2'(x)$ and a point of N_1'. Thus $N_2' = \bigcup_{x \in C_0} N_2'(x)$ is the union of a collection of connected sets each having a limit point in N_1'. So we have that $N_2 = N_2' \cup N_1'$ is connected. Next define for each $(x_1, x_2) \in C_0 \times C_0$, $N_3'(x_1, x_2) = (q_0, x_1, x_2) \times M_0$. $N_3'(x_1, x_2)$ is connected and has a limit point $(q_0, x_1, x_2) \in N_2'$. Thus $N_3' = \bigcup_{(x_1, x_2) \in C_0 \times C_0} N_3'(x_1, x_2)$ is the union of a collection of connected sets each having a limit point in the connected set $N_2' \cup N_1'$ so $N_3' \cup N_2' \cup N_1'$ is connected. This process can be continued to define N_n' for each positive integer n to be the union of a collection of connected copies of M_0 each having a limit point in N_{n-1}' so that $N_1' \cup N_2' \cup \cdots \cup N_n'$ is connected and contains all points of Q_0 having $n+1$ or fewer coordinates. Thus Q_0 and $Q_0 - \{q_0\} = \bigcup_{i>0} N_i'$ is connected. □

Theorem 8. *Each point of $(S, \mathcal{T})$ is a cut point of order three.*

Proof. If C is a component of $M_0 - \{p_0\}$ for some $p_0 \in M_0$, let C' denote $\{p = (x_0, p_1, p_2, \ldots, p_n) \in S : n \text{ is a whole number, and } x_0 \in C\}$.

Let $p = (p_0, p_1, p_2, \ldots, p_n)$ be a point of $(S, \mathcal{T})$. We now consider four cases:

Case 1: Suppose $n = 0$ and $p_0 \in T_0$. From Lemma 6, we have $M_0 - \{p_0\} = S_1 \cup S_2 \cup S_3$ so that S_i is a component of $M_0 - \{p_0\}$ for each $1 \leq i \leq 3$. Then $S - \{p_0\} = S'_1 \cup S'_2 \cup S'_3$. Note that each S'_i, $1 \leq i \leq 3$ is connected follows from Lemma 7.

We show that $\mathrm{Cl}(S'_1) \cap S'_2 = \emptyset$ and $S'_1 \cap \mathrm{Cl}(S'_2) = \emptyset$. Assume that $t \in \mathrm{Cl}(S'_1) \cap S'_2$. We now consider three cases.

Case 1a: Assume $t = (t_0) \in S'_2$ with $t_0 \in S_2 \cap T_0$. Let U be an open set in S with $t \in U$ that contains no point of S_1. Then $U \cap S'_1 \neq \emptyset$ and $U \cap S'_1 \subseteq T_0$. This implies that U contains a point $s = (s_0)$ with $s_0 \in S_1$ contrary to the definition of U.

Case 1b: Assume $t = (t_0) \in S'_2$ with $t_0 \in C_0$. Let ϵ be a positive number such that $N(t_0, \epsilon)$ contains no point of S_1 in R^2. Let $U = R(t_0, \epsilon) = \{t_0\} \cup (N(t_0, \epsilon) \cap T_0) \cup (t_0 \times (N(\mathcal{O}, \epsilon) \cap T_0))$. $U \cap S'_1$ must contain a point p in S. But if $p = (p_0)$ then $p \in N(t_0, \epsilon) \cap S_1$ contrary to the definition of ϵ. Also if $p = (p_0, p_1)$ then $p_0 = t_0 \notin S_1$ so $p \notin S'_1$.

Case 1c: Assume $t = (t_0, t_1, \cdots, t_n) \in S'_2$ with $n > 0$ and $t_0 \in S_2$. If $U = R(t, \epsilon)$, and $q \in U$, then $q = (q_0, q_1, \cdots, q_m) \in U$ where $m = n$ or $m = n + 1$. In either case $q_0 = t_0$ so $q \notin S'_1$, contrary to the assumption that $\mathrm{Cl}(S'_1) \cap S'_2 \neq \emptyset$.

Therefore, $\mathrm{Cl}(S'_1) \cap S'_2 = \emptyset$. By a parallel argument, $S'_1 \cap \mathrm{Cl}(S'_2) = \emptyset$. By similar arguments, $\mathrm{Cl}(S'_1) \cap S'_3 = \emptyset$ and $S'_1 \cap \mathrm{Cl}(S'_3) = \emptyset$, and $\mathrm{Cl}(S'_2) \cap S'_3 = \emptyset$ and $S'_2 \cap \mathrm{Cl}(S'_3) = \emptyset$. Therefore, S'_1, S'_2 and S'_3 are pairwise mutually separated and p_0 is a cut point of order three.

Case 2. Suppose $n = 0$ and $p_0 \in C_0$. Suppose $M_0 - \{p_0\} = S_1 \cup S_2$ so that S_i is a component of $M_0 - \{p_0\}$ for each $1 \leq i \leq 2$. Then $S - \{p_0\} = S'_1 \cup S'_2 \cup T'$ where $T' = \{p = (p_0, p_1, \cdots, p_n) : p \in S,\ n \geq 1\}$. S'_1, S'_2 and T' are pairwise mutually separated by arguments similar to those used in Case 1, and each of S'_1, S'_2 and T' is connected by Lemma 7. Thus (p_0) is a cut point of order three.

Case 3. Suppose $n > 0$, $p = (p_0, p_1, \ldots, p_n)$, and $p_n \in T_0$. Suppose $M_0 - \{p_n\} = S_1 \cup S_2 \cup S_3$ and without loss of generality assume that S_1 has $\mathcal{O}$ in its closure (if S_1 were to have the subspace topology of the plane). Let A_0 be the set of all points of S having a point of $M_0 - \{p_0\}$ as its first coordinate. For each positive integer $j < n$, let A_j be the set of all points of S whose first $j + 1$ coordinates are $p_0, p_1, \cdots, p_{j-1}, x$ where x is a point of $M_0 - \{p_j\}$. Let $A = \bigcup_{i=0}^{i=n-1} A_i$. If $i \in \{1, 2, 3\}$, let B_i be the set of all points of S whose first $n + 1$ coordinates are $p_0, p_1, \cdots, p_{n-1}, x$ where $x \in S_i$. A direct argument shows that $S - \{p\} = A \cup B_1 \cup B_2 \cup B_3$. We will show that $A \cup B_1$, B_2 and B_3 are mutually separated.

We show that $\mathrm{Cl}(A \cup B_1) \cap B_2 = \emptyset$. Assume that $t \in \mathrm{Cl}(A \cup B_1) \cap B_2$. We consider two cases.

Case 3a: Assume $t = (t_0, t_1, \ldots, t_n)$. Since $t \in B_2$, $t_n \in S_2$ and there is an $\epsilon > 0$ such that $N(t_n, \epsilon) \cap S_1 = \emptyset$. Let $U = R(t, \epsilon)$. If $x \in U$, $x = (x_0, x_1, \cdots, x_k)$ for $k = n$ or $k = n + 1$. In either case $x_n \in N(t_n, \epsilon)$ so $x_n \notin S_1$ and $x \notin B_1$. It remains to show that $A_1 \cap U = \emptyset$. If $x \in U$, $x_i = t_i = p_i$ for $0 \leq i < n$. But if $x \in A$, there is an i, $0 \leq i < n$ such that $x \in A_i$ and $x_i \neq p_i$.

Case 3b: $t = (p_0, p_1, \cdots, p_{n-1}, t_n, \cdots, t_k)$ with $k > n$ and $t_n \in S_2 \cap C_0$. If U is a region containing t and x is in U, then x has the same first $k-1$ coordinates as t. But this means that $x_n = t_n \in S_2$ so x is not in S_1. As before $x \notin A$ since $x_i = t_1 = p_i$ for $0 \le i < n$.

We now show that $(A_1 \cup B_1) \cap \mathrm{Cl}(B_2) = (A_1 \cap \mathrm{Cl}(B_2)) \cup (B_1 \cap \mathrm{Cl}(B_2)) = \emptyset$. Assume that $t \in (A_1 \cup B_1) \cap \mathrm{Cl}(B_2)$. We consider two cases.

Case 3a$'$: $t \in A_1 \cap \mathrm{Cl}(B_2)$. Then $t = (t_0, t_1, \ldots, t_j)$ for some whole number j, and since $t \in A$, there is an integer k such that $0 \le k < n$ such that $t_k \ne p_k$. If x is in the region $R(t, \epsilon)$, then $x_i = t_i$ for $0 \le i < n$. But this implies that $x_k = t_k \ne p_k$ and $x \notin B_2$, contrary to our assumption that $t \in Cl(B_2)$.

Case 3b$'$: $t \in B_1 \cap \mathrm{Cl}(B_2)$. Then $t = (t_0, t_1, \ldots, t_{n-1}, t_n, t_{n+1} \ldots, t_k)$ with $t_n \in S_1$, $k \ge n$, and $t_n \ne p_n$. Since S_1 and S_2 are mutually separated, there is a positive number ϵ such that $N(t_n, \epsilon) \cap S_2 = \emptyset$. It follows that $R(t, \epsilon) \cap B_2 = \emptyset$, contrary to the assumption that $t \in Cl(B_2)$.

Therefore, $(A_1 \cup B_1)$ and B_2 are mutually separated. In a similar way, the pairs $(A_1 \cup B_1)$ and B_2 and B_2 and B_3, respectively, are mutually separated. Furthermore, it follows from Lemma 7 that each of $(A_1 \cup B_1)$, B_2, and B_3 is connected. Therefore, $p = (p_0, p_1, \ldots, p_n)$ with $n > 0$ and $p_n \in T_0$ is a cut point of order three.

Case 4. Suppose $n > 0$, $p = (p_0, p_1, \cdots, p_n)$, and $p_n \in C_0$. Suppose $M_0 - \{p_n\} = S_1 \cup S_2$ and without loss of generality assume that S_1 has $\mathcal{O}$ in its closure (if S_1 were to have the subspace topology of the plane). Let A be defined exactly as was done in Case 3. For $j \in \{1, 2\}$, let B_j be the set of all points of S whose first $n+1$ coordinates are $p_0, p_1, \cdots, p_{n-1}, x$ where $x \in S_1$. Let B_3 be the set of all points of S whose first $n+1$ coordinates are $p_0, p_1, \cdots, p_n$. Using arguments entirely similar to those already given it can be shown that each of $(A_1 \cup B_1)$, B_2, and B_3 is connected and that they are pairwise mutually separated. Therefore, $p = (p_0, p_1, \ldots, p_n)$ with $n > 0$ and $p_n \in C_0$ is a cut point of order three. □

4. Embedding Cut Point Spaces

In Kuratowski (Theorem 1, page 160, of [4]), it is shown that for a connected separable metric space Z, the set $Z - \{z\}$ is connected or is the union of two connected sets for every $z \in Z$ except for a countable set of points of Z. See also Theorem 3.2 of [9]. The following is therefore immediate.

Theorem 9. *If X is a cut point space and each point p of X has cut point order m where $m \ge 3$, then X may not be separable and metric and thus may not be embedded in $\mathcal{R}^n$ for any $n \ge 2$.*

We now provide an analogue of the theorems of Kuratowski and Whyburn in the setting of hereditarily separable spaces.

Theorem 10. *If X is a separable connected Hausdorff space, then X does not contain uncountably many points that separate X into three mutually separated connected sets.*

Proof. Assume that there is an uncountable set of points T of X that separate X into 3 mutually exclusive connected sets. Let $P = \{p_1, p_2, p_3, \ldots\}$ be a countable dense subset of X with $p_i \ne p_j$ if and only if $i \ne j$. For each two positive integers m and n, let $C_{m,n}$ be the set of all points of X

that separate p_m from p_n. Note that if $x \in T$ then $X - \{x\}$ is the union of two mutually exclusive open sets so x separates two points of P. Thus each point of T is in $C_{m,n}$ for some choice of m and n. Thus there exist integers i and j such that $M = T \cap C_{i,j}$ is uncountable. If $x \in M$, then $X - \{x\}$ is the union of three mutually separated sets, and x separates p_i from p_j so these points belong to different components of $X - \{x\}$. For each $x \in M$, let A_x be the component containing p_i, B_x the component containing p_j, and C_x the other component. Note that C_x is open in X for each $x \in M$.

We now show that if x and y are two points of M, then C_x does not intersect C_y. Assume to the contrary that there exist points x and y in M such that $C_x \cap C_y \neq \emptyset$. Now $X - \{x\} = A_x \cup B_x \cup C_x$. Note that $y \neq C_x$ since if it were, then $X - \{y\}$ would contain $A_x \cup B_x \cup \{x\}$ which is connected so y would not separate p_i from p_j, contrary to the definition of M. So y is in A_x or B_x. First assume $y \in B_x$. Then $X - \{y\}$ contains $\{x\}$, A_x, C_x and C_y and the union of these sets is connected and thus a subset of A_y. Thus we have that $C_y \subseteq A_y$, but these sets are mutually exclusive. Next assume that $y \in A_x$. In this case we have $\{x\} \cup B_x \cup C_x \cup C_y$ is a connected subset of $X - \{y\}$ and thus of B_y. This is again a contradiction since C_y and B_y are mutually exclusive.

Therefore, the set of all C_x for all $x \in M$ is an uncountable collection of mutually exclusive open sets in X, contrary to the separability of X. □

Corollary 11. *If X is a connected cut point space and each point p of X has cut point order 3, then X may not be Hausdorff and thus may not be embedded in $\mathcal{R}^n$ for any $n \geq 2$ or indeed in any hereditarily separable Hausdorff space.*

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