

Technical Report

TR-2004-018

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by

Brendan Nagle, Vojtech Rodl, Mathias Schacht

MATHEMATICS AND COMPUTER SCIENCE

EMORY UNIVERSITY

A SHORT PROOF OF THE 3-GRAPH COUNTING LEMMA

BRENDAN NAGLE, VOJTĚCH RÖDL, AND MATHIAS SCHACHT

ABSTRACT. Szemerédi’s Regularity Lemma proved to be a powerful tool in extremal graph theory. Many of its applications are based on a so-called ‘Counting Fact’ which allows the embedding of small subgraphs in the context of an appropriately given ε -regular partition.

Frankl and Rödl [3] extended Szemerédi’s Regularity Lemma to 3-graphs and Nagle and Rödl [11] established an accompanying Counting Lemma. In this paper, we provide an alternative proof to the 3-graph Counting Lemma far shorter and more transparent than the original.

1. INTRODUCTION

Szemerédi’s Regularity Lemma [19] is one of the most powerful tools in combinatorics with significant applications in Extremal Graph Theory, Combinatorial Number Theory and Theoretical Computer Science (see the excellent surveys [9, 10] for many of these applications). Szemerédi’s Regularity Lemma makes the beautiful assertion that all large graphs can be decomposed into constantly many edge-disjoint subgraphs, almost all of which behave in a “random-like” way (see Theorem 1.1 below).

The broad applicability of Szemerédi’s lemma to graph problems suggests that a regularity lemma for hypergraphs might render many applications. Frankl and Rödl [3] established an extension of Szemerédi’s Regularity Lemma to 3-graphs (hereafter referred to as the *FR-Lemma* for short), guaranteeing that all large triple systems admit a decomposition into constantly many edge-disjoint subsystems, almost all of which behave in a “random-like” way. Applications of the FR-Lemma to 3-graphs ensued [3, 5, 6, 8, 18, 12, 15, 17].

That the FR-Lemma is perhaps a useful extension of Szemerédi’s Lemma follows from the fact that this lemma admits a structural counterpart, a so-called *Counting Lemma* (see Theorem 3.3.1, [11]), required for the applications above. As a cogent example, the Counting Lemma, working within the framework of the FR-Lemma, gives a quick but non-obvious proof (cf. [3]) that every set $Z \subseteq \{1, \dots, n\}$ without arithmetic progressions of length 4 satisfies $|Z| = o(n)$.

The first proof of the 3-graph Counting Lemma was obtained by the first two authors in [11] (see also [14] for a different approach). The aim of this note is to give a substantially shorter proof of the same result. To be more precise about our work here requires some discussion. We begin by making Szemerédi’s Regularity Lemma more precise.

1.1. Szemerédi’s Regularity Lemma. Szemerédi’s lemma pivots on the concept of an ε -regular pair. Let bipartite graph G_{XY} be given with bipartition $X \cup Y$. We say the pair X, Y is (d, ε) -regular (with respect to G_{XY}) if for all $X' \subseteq X$ and $Y' \subseteq Y$ where $|X'| > \varepsilon|X|$ and $|Y'| > \varepsilon|Y|$, we have $d_{G_{XY}}(X', Y') = d \pm \varepsilon$ where $d_{G_{XY}}(X', Y') = |G_{XY}[X', Y']|/|X'|^{-1}|Y'|^{-1}$ is the *density* of the bipartite graph $G_{XY}[X', Y']$ induced on $X' \cup Y'$. We say the pair X, Y is ε -regular if it is (d, ε) -regular for some d .

In this paper, we use a well-known invariant of Szemerédi’s Regularity Lemma for k -partite graphs $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$, and therefore present Szemerédi’s lemma in this context. Let k -partite graph $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ be given with k -partition $V = V(G) = V_1 \cup \dots \cup V_k$. We say a partition $V_i = W_1^i \cup \dots \cup W_t^i$, $1 \leq i \leq k$, is t -equitable if $|W_1^i| \leq \dots \leq |W_t^i| \leq |W_1^i| + 1$. We say a t -equitable partition $V_i = W_1^i \cup \dots \cup W_t^i$, $1 \leq i \leq k$, is ε -regular if for all $1 \leq i < j \leq k$, all but εt^2 pairs W_a^i and W_b^j , $1 \leq a, b \leq t$, are ε -regular. Szemerédi’s Regularity Lemma (for k -partite graphs) is then stated precisely as follows.

Theorem 1.1 (Szemerédi’s Regularity Lemma). *Let integer k and $\varepsilon > 0$ be given. There exist positive integers $N_0 = N_0(k, \varepsilon)$ and $T_0 = T_0(k, \varepsilon)$ such that any k -partite graph $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$, $V = V(G) = V_1 \cup \dots \cup V_k$, $|V_1|, \dots, |V_k| \geq N_0$, admits an ε -regular t -equitable partition $V_i = W_1^i \cup \dots \cup W_t^i$, $1 \leq i \leq k$, where $t \leq T_0$.*

Central to many applications of Szemerédi’s Regularity Lemma is the assertion that any subgraph F of constant size may be embedded into an appropriately given collection of “dense and regular” pairs $\{W_a^i\}_{(i,a) \in I}$, $I = I(F)$, from an ε -regular t -equitable partition. This observation is due to the following “Counting Fact” for graphs where, for sake of presentation, $\mathcal{K}_s^{(2)}(P)$ denotes the set of all s -tuples from $V(P)$ spanning cliques $K_s^{(2)}$ in a graph P .

Fact 1.2 (Counting Fact). *For every integer s and $d > 0$ there exists $\varepsilon > 0$ so that whenever $P = \bigcup_{1 \leq i < j \leq s} P^{ij}$ is an s -partite graph with s -partition $V_1 \cup \dots \cup V_s$ satisfying that all P^{ij} , $1 \leq i < j \leq s$, are (d, ε) -regular and $|V_1| = \dots = |V_s| = n$, then $|\mathcal{K}_s^{(2)}(P)| = d^{\binom{s}{2}} n^s (1 \pm \theta(\varepsilon))$ where $\theta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

Both the Szemerédi Regularity Lemma and Fact 1.2 may be generalized to 3-uniform hypergraphs. This was done in the form of the FR-Lemma [3] and its accompanying Counting Lemma [11], respectively. While we omit a formulation of the FR-Lemma¹, we proceed to present its accompanying Counting Lemma, the main topic of this paper.

1.2. 3-uniform hypergraphs. The following definition originates from [3] and provides an extension to 3-graphs of the concept of an ε -regular pair.

Definition 1.3 ((δ, r) -regularity). *Let a positive integer r and $d, \delta > 0$ be given along with 3-graph \mathcal{H} and graph P . We say that \mathcal{H} is (d, δ, r) -regular with respect to P if for any r -tuple of subgraphs $\mathbf{Q} = (Q(1), \dots, Q(r))$, $Q(s) \subseteq P$, $1 \leq s \leq r$:*

$$\left| \bigcup_{s=1}^r \mathcal{K}_3^{(2)}(Q(s)) \right| > \delta \left| \mathcal{K}_3^{(2)}(P) \right| \implies |d_{\mathcal{H}}(\mathbf{Q}) - d| < \delta$$

where

$$d_{\mathcal{H}}(\mathbf{Q}) = \frac{\left| \mathcal{H} \cap \bigcup_{s=1}^r \mathcal{K}_3^{(2)}(Q(s)) \right|}{\left| \bigcup_{s=1}^r \mathcal{K}_3^{(2)}(Q(s)) \right|}$$

is the density of \mathbf{Q} . We say \mathcal{H} is (δ, r) -regular with respect to P if it is (d, δ, r) -regular with respect to P for some d .

The Counting Lemma takes place in the following technical environment in which we work for the remainder of this paper. The environment below is produced by an application of the FR-Lemma when, as in Fact 1.2, there is a coincidence of “dense parts” within the partition the FR-Lemma guarantees.

Setup 1.4. *Let integers k, r and n be given. Let positives d_3, δ_3, d_2 and δ_2 be given. Suppose*

- (1) $V = V_1 \cup \dots \cup V_k$, $|V_1| = \dots = |V_k| = n$, is a partition of vertex set $V = V(P) = V(\mathcal{H})$.
- (2) $P = \bigcup_{1 \leq i < j \leq k} P^{ij}$ is a k -partite graph where all P^{ij} , $1 \leq i < j \leq k$, are (d_2, δ_2) -regular.
- (3) $\mathcal{H} = \bigcup_{1 \leq h < i < j \leq k} \mathcal{H}^{hij} \subseteq \mathcal{K}_3^{(2)}(P)$ is a k -partite 3-graph where all \mathcal{H}^{hij} , $1 \leq h < i < j \leq k$, are (d_3, δ_3, r) -regular with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$.

In the environment above, we wish to estimate the number of hypercliques $K_k^{(3)}$ in \mathcal{H} . This estimation is precisely the content of the Counting Lemma, and for the sake of this presentation, let $\mathcal{K}_k^{(3)}(\mathcal{H})$ denote the set of all k -tuples from $V(\mathcal{H})$ spanning hypercliques $K_k^{(3)}$.

¹The FR-Lemma is centered about Definition 1.3, however, to present this lemma fully requires some rather technical auxiliary considerations. As we don’t need the FR-Lemma to present the Counting Lemma, we omit the formulation of FR-Lemma for simplicity and refer the reader to [3].

Theorem 1.5 (Counting Lemma, [11]). *For all integers k and for all $d_3 > 0$ there exists $\delta_3 > 0$ so that for all $d_2 > 0$ there exist integer r and $\delta_2 > 0$ so that with n sufficiently large, if \mathcal{H} and P are as in Setup 1.4 with these constants, then*

$$|\mathcal{K}_k^{(3)}(\mathcal{H})| = d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k (1 \pm f(\delta_3))$$

where $f(\delta_3) \rightarrow 0$ as $\delta_3 \rightarrow 0$.

Proving Theorem 1.5 is the content of this note and we proceed to this proof in Section 2. We conclude our Introduction with the following remarks which address certain aspects of proving Theorem 1.5.

The sole technicality of proving Theorem 1.5 is working with the given quantification of constants:

$$\forall d_3 \exists \delta_3 : \forall d_2 \exists \delta_2 \exists r : \dots$$

This quantification, consistent with the output of the FR-Lemma, allows for the hypergraph \mathcal{H} to be very ‘‘sparse’’. If, on the other hand, the quantification of constants were allowed as $\forall d_3, d_2 \exists \delta_3, \delta_2$, then this ‘‘dense’’ version of Theorem 1.5 is easier to prove and, in fact, is done in [7] (see Theorem 6.5). We mention that no appeal to parameter r is required in the dense case.

The sparse case, which we consider here, seems to resist standard approaches of proof. Two different approaches of proving Theorem 1.5 were considered in [11] and [14], but both were rather technical. In this paper, we pursue a third approach, more appealing than the former two being fairly simple and short.

As in the dense case, a natural place to start with proving Theorem 1.5 is looking into the *links* of \mathcal{H} . Fix vertex $v \in V_1$ and fix $1 < i < j \leq k$. The (i, j) -link graph L_v^{ij} is defined as $L_v^{ij} = \{\{v_i, v_j\} \in P^{ij} : \{v, v_i, v_j\} \in \mathcal{H}\}$. Note that L_v^{ij} has vertex set $N_{P^{1i}}(v) \cup N_{P^{1j}}(v)$, $1 < i < j \leq k$, and is a subgraph of $P_v^{ij} = P^{ij}[N_{P^{1i}}(v), N_{P^{1j}}(v)]$. The link graph L_v of v is then set as $L_v = \bigcup_{1 < i < j \leq k} L_v^{ij}$. Since

$$|\mathcal{K}_{k-1}^{(3)}(\mathcal{H})| = \sum_{v \in V_1} \left| \mathcal{K}_{k-1}^{(3)} \left(\mathcal{H} \cap \mathcal{K}_3^{(2)}(L_v) \right) \right|, \quad (1)$$

one would like to count $|\mathcal{K}_{k-1}^{(3)}(\mathcal{H} \cap \mathcal{K}_3^{(2)}(L_v))|$, $v \in V_1$, by induction. However, as one can show (see upcoming Fact 1.2 and also [3, 11]), for almost all $v \in V_1$, with $1 < i < j \leq k$ fixed, the links L_v^{ij} are ‘regular’, but not nearly as regular as the original graphs P^{ij} were, as would be required by an induction hypothesis.

We overcome this difficulty outlined above by simply applying the Szemerédi Regularity Lemma, Theorem 1.1, to the graphs L_v , $v \in V_1$. The Regularity lemma partitions the graphs L_v , $v \in V_1$, into extremely regular classes. While we must take a few technical details into consideration, the advantage here is that Regularity Lemma then allows the outlined induction approach above to work. *The regularization of the links is the central message of our current paper.*

2. PROOF OF THEOREM 1.5

We begin by saying that we focus only on the lower bound of Theorem 1.5. It shall be fairly transparent that our method for proving the lower bound also gives the (asymptotically) matching upper bound. As well, it may be mentioned that it was shown in [11] that for proving Theorem 1.5 it suffices to prove the lower bound only.

Our proof of Theorem 1.5 proceeds by induction on $k \geq 3$. The base case $k = 3$ is trivial. Indeed, by Definition 1.3, $\mathcal{H} = \mathcal{H}^{123}$ has density $(d_3 \pm \delta_3)$ with respect to $P = P^{12} \cup P^{23} \cup P^{13}$. Fact 1.2 asserts $|\mathcal{K}_3^{(2)}(P)| = d_3^3 n^3 (1 \pm \theta(\delta_2))$, $\theta(\delta_2) \rightarrow 0$ as $\delta_2 \rightarrow 0$. The Counting Lemma for $k = 3$ then follows.

We assume Theorem 1.5 holds through $k - 1$ and consider Theorem 1.5 for k . We first describe the constants $k, d_3, \delta_3, d_2, \delta_2$ and r involved. With integer k given in our induction, let $d_3 > 0$ be given. We take $0 < \delta_3 = \delta_3(k, d_3) \ll 1/k, \delta_3$ to be a small constant implicitly defined by the calculations in the sequel. Let $d_2 > 0$ be given. We take auxiliary constant $\delta_2 = \delta_2(k, d_3, \delta_3, d_2, \delta_2) \ll \min\{\delta_3, d_2\}$ very

small and auxiliary integer $r = r(k, d_3, \delta_3, d_2, \delta_2) \gg \max\{1/\delta_3, 1/d_2\}$ very large, although to precisely describe the orders of constants δ_2 and r requires brief discussion.

With k, d_3, δ_3 and d_2 given as above, we introduce small auxiliary constant $\varepsilon_0 = \varepsilon_0(k, d_3, \delta_3, d_2) \ll \min\{\delta_3, d_2\}$ and large integer $r' = r'(k, d_3, \delta_3, d_2) \gg \max\{1/\delta_3, 1/d_2\}$. We will use the constant ε_0 when applying the Regularity Lemma to the link graphs L_v , $v \in V_1$, as indicated in the Introduction. The role of parameter r' will be easily understood in context. Let $T_0 = T_0(\varepsilon_0)$ be the constant guaranteed by Szemerédi's Regularity Lemma, Theorem 1.1.

We take $0 < \delta_2 = \delta_2(k, d_3, \delta_3, d_2, \varepsilon_0, r', T_0) \ll \min\{\varepsilon_0, 1/r', 1/T_0\}$ very small and take integer $r = r(k, d_3, \delta_3, d_2, \varepsilon_0, r', T_0) \gg \max\{1/\varepsilon_0, r', T_0\}$ very large. The following hierarchy then summarizes all constants we use in our proof.

$$\frac{1}{k}, d_3 \gg \delta_3 \geq \min\{\delta_3, d_2\} \gg \varepsilon_0, \frac{1}{r'}, \frac{1}{T_0} \gg \delta_2, \frac{1}{r} \gg \frac{1}{n}. \quad (2)$$

All constants $k, d_3, \delta_3, d_2, \varepsilon_0, r', T_0, \delta_2, r$ and n are fixed for the remainder of the paper.

Suppose \mathcal{H} and P are as in Setup 1.4. We show $|\mathcal{K}_k^{(3)}(\mathcal{H})| \geq d_3^{(3)} d_2^{(2)} n^k (1 - f(\delta_3))$ where $f(\delta_3) \rightarrow 0$ as $\delta_3 \rightarrow 0$. We now proceed to the main idea of our proof.

2.1. Regularizing the links. Fix vertex $v \in V_1$.

- With constant $\varepsilon_0 > 0$ in (2), apply Szemerédi's Regularity Lemma, Theorem 1.1, to the $((k-1)$ -partite) link graph $L_v = \bigcup_{1 < i < j \leq k} L_v^{ij}$ to obtain ε_0 -regular, t_v -equitable partition \mathbf{P}_v , $t_v \leq T_0$, where T_0 appears in (2).
- Let partition \mathbf{P}_v be given by $N_{P^{1i}}(v) = W_1^{v,i} \cup \dots \cup W_{t_v}^{v,i}$, $1 < i \leq k$, where for each $1 < i < j \leq k$, all but $\varepsilon_0 t_v^2$ pairs $1 \leq a, b \leq t_v$ satisfy that $L_v^{ij}[W_a^{v,i}, W_b^{v,j}]$ is ε_0 -regular.

As we show in this paper, the advantage of regularizing the links is that for a “typical” vertex $v \in V_1$, we may apply our induction assumption on Theorem 1.5 (for $k-1$) within “typical $(k-1)$ -sausages” (cf. Figure 1) of the partition \mathbf{P}_v . This point is crucial, but to discuss it more precisely, we need a bit more notation.

2.1.1. Sausages of \mathbf{P}_v . As above, let $v \in V_1$ be fixed and fix $(k-1)$ -length vector $\vec{a}_v = (a_2, \dots, a_k) \in [t_v] \times \dots \times [t_v] = [t_v]^{k-1}$ (cf. Figure 1).

- Set

$$L_v(\vec{a}_v) = \bigcup_{1 < i < j \leq k} L_v^{ij} [W_{a_i}^{v,i}, W_{a_j}^{v,j}] = L_v [W_{a_2}^{v,2}, \dots, W_{a_k}^{v,k}] \quad (3)$$

to be the subgraph of $L_v = \bigcup_{1 < i < j \leq k} L_v^{ij}$ induced on the sets $W_{a_2}^{v,2}, \dots, W_{a_k}^{v,k}$.

- Set

$$\mathcal{H}(\vec{a}_v) = \mathcal{H} \cap \mathcal{K}_3^{(2)}(L_v(\vec{a}_v)) = \bigcup_{1 < h < i < j \leq k} \mathcal{H}^{hij}(\vec{a}_v) \quad (4)$$

where $\mathcal{H}^{hij}(\vec{a}_v) = \mathcal{H}^{hij} \cap \mathcal{K}_3^{(2)}(L_v[W_{a_h}^{v,h}, W_{a_i}^{v,i}, W_{a_j}^{v,j}])$, $1 < h < i < j \leq k$.

We refer to the objects $\mathcal{H}(\vec{a}_v)$ and $L_v(\vec{a}_v)$ as *sausages*.

Observe that (1) can be reformulated as

$$|\mathcal{K}_k^{(3)}(\mathcal{H})| = \sum_{v \in V_1} \sum_{\vec{a}_v \in [t_v]^{k-1}} |\mathcal{K}_{k-1}^{(3)}(\mathcal{H}(\vec{a}_v))|. \quad (5)$$

Note also that as L_v was regularized, all but $\varepsilon_0 \binom{k-1}{2} t_v^{k-1} < \varepsilon_0 k^2 t_v^{k-1}$ vectors $\vec{a}_v = (a_2, \dots, a_k) \in [t_v]^{k-1}$ satisfy that all $\binom{k-1}{2}$ bipartite graphs $L_v^{ij}[W_{a_i}^{v,i}, W_{a_j}^{v,j}]$, $1 < i < j \leq k$, are ε_0 -regular.

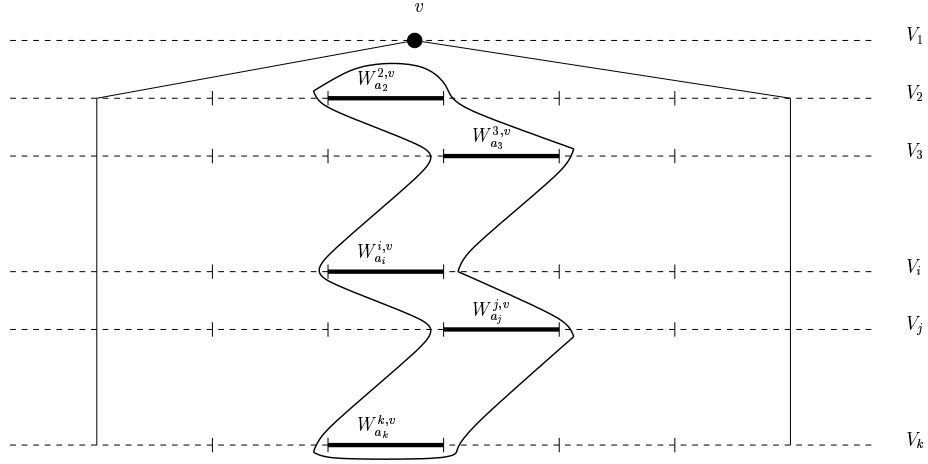


FIGURE 1. A sausage of the partition \mathbf{P}_v .

2.2. Properties of sausages in \mathbf{P}_v . We present the following assertions about “typical” sausages of \mathbf{P}_v , $v \in V_1$.

Fact 2.1. *For all but $2k\delta_2 n$ vertices $v \in V_1$, $|N_{P^{1i}}(v)| = (d_2 \pm \delta_2)n$ for all $1 < i \leq k$. Consequently, for each such vertex,*

$$(d_2 - 2\delta_2) \frac{n}{t_v} \leq \left\lfloor \frac{|N_{P^{1i}}(v)|}{t_v} \right\rfloor \leq |W_{a_i}^{v,i}| \leq \left\lceil \frac{|N_{P^{1i}}(v)|}{t_v} \right\rceil \leq (d_2 + 2\delta_2) \frac{n}{t_v}.$$

Claim 2.2. *For all but $\delta_3^{1/4} n$ vertices $v \in V_1$, all but $\delta_3^{1/20} t_v^{k-1}$ vectors $\vec{a}_v = (a_2, \dots, a_k) \in [t_v]^{k-1}$ yield sausages $L_v(\vec{a}_v)$ and $\mathcal{H}(\vec{a}_v)$ satisfying that*

- ★ **Part A.** *all $\binom{k-1}{2}$ bipartite graphs $L_v^{ij}[W_{a_i}^{v,i}, W_{a_j}^{v,j}]$, $1 < i < j \leq k$, have density $d_2 d_3 (1 \pm \delta_3^{1/4})$.*
- ★ **Part B.** *all $\binom{k-1}{3}$ 3-partite graphs $L_v^{hi}[W_{a_h}^{h,v}, W_{a_i}^{i,v}] \cup L_v^{ij}[W_{a_i}^{i,v}, W_{a_j}^{j,v}] \cup L_v^{hj}[W_{a_h}^{h,v}, W_{a_j}^{j,v}]$, $1 < h < i < j \leq k$, are $(d_3, \delta_3^{1/20}, r')$ -regular with respect to $\mathcal{H}^{hij}(\vec{a}_v)$ where r' is given in (2).*

Fact 2.1 is folklore (cf. [10]) and we therefore omit its proof. While technical looking, Part A of Claim 2.2 is actually easy to prove and we address its proof in Section 3.1. Part B, on the other hand, is non-trivial to prove and may be considered the central burden in the current proof of Theorem 1.5. We devote Section 4 to its proof.

Let V_1^{typ} denote the set of “typical” vertices $v \in V_1$ to which Fact 2.1 and Claim 2.2 refer. For each $v \in V_1^{\text{typ}}$, let $[t_v]_{\text{typ}}^{k-1}$ denote the set of “typical” vectors $\vec{a}_v \in [t_v]^{k-1}$ to which Claim 2.2 refers.

2.3. Fundamental conclusion. We conclude from Fact 2.1 and Claim 2.2 above that for any vertex $v \in V_1^{\text{typ}}$, all sausages $\mathcal{H}(\vec{a}_v)$ and $L_v(\vec{a}_v)$, $\vec{a}_v \in [t_v]_{\text{typ}}^{k-1}$, satisfy the hypothesis of Setup 1.4 with the constants² $k-1$, d_3 , $\delta_3^{1/20}$, $d_2 d_3 (1 \pm \delta_3^{1/4})$, ε_0 , r' and $\frac{d_2 n}{t_v} (1 \pm 2\delta_2)$. Indeed, as in Setup 1.4, observe that

²We admit that symbol $d_2 d_3 (1 \pm \delta_3^{1/4})$ allows slightly different densities over distinct (but typical) graphs $L_v^{ij}[W_{a_i}^{i,v}, W_{a_j}^{j,v}]$. Similarly, the symbol $\frac{d_2 n}{t_v} (1 \pm 2\delta_2)$ allows slightly different sizes over distinct (but typical) vertex sets $W_{a_i}^{i,v}$. These small fluctuations only superficially make our reduced setup $\mathcal{H}(\vec{a}_v)$ and $L_v(\vec{a}_v)$ slightly different from our

$\mathcal{H}(\vec{a}_v)$ replaces \mathcal{H} , $L_v(\vec{a}_v)$ replaces P , $k-1$ replaces k , d_3 remains d_3 , $\delta_3^{1/20}$ replaces δ_3 , $d_2d_3(1 \pm \delta_3^{1/4})$ replaces d_2 , ε_0 replaces δ_2 , r' replaces r and $\frac{d_2n}{t_v}(1 \pm 2\delta_2)$ replaces n .

We make the *critical* observation from (2) that the constants above satisfy

$$\frac{1}{k-1}, d_3 \gg \delta_3^{1/20} \geq \min\{\delta_3^{1/20}, d_2d_3\} \gg \varepsilon_0, \frac{1}{r'} \gg \frac{t_v}{d_2n}. \quad (6)$$

As such, for fixed $v \in V_1^{\text{typ}}$ and $\vec{a}_v = (a_2, \dots, a_k) \in [t_v]_{\text{typ}}^{k-1}$, we may apply the induction hypothesis to the sausages $\mathcal{H}(\vec{a}_v)$ and $L_v(\vec{a}_v)$. Upon doing so, we find

$$\begin{aligned} \left| \mathcal{K}_{k-1}^{(3)}(\mathcal{H}(\vec{a}_v)) \right| &\geq d_3^{\binom{k-1}{3}} \left(d_2d_3(1 \pm \delta_3^{1/4}) \right)^{\binom{k-1}{2}} \left(\frac{d_2n}{t_v}(1 \pm 2\delta_2) \right)^{k-1} (1 - f_{k-1}(\delta_3)) \\ &= d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} \frac{n^{k-1}}{t_v^{k-1}} \left(1 - \hat{f}_{k-1}(\delta_3) \right) \end{aligned} \quad (7)$$

where $f_{k-1}(\delta_3) \rightarrow 0$ as $\delta_3 \rightarrow 0$ is inductively guaranteed and $\hat{f}_{k-1}(\delta_3) \rightarrow 0$ as $\delta_3 \rightarrow 0$ is implicitly defined from $f_{k-1}(\delta_3)$ and remaining multiplicative error terms.

Now the proof of Theorem 1.5 for k is nearly complete. We return to (5) and apply Fact 2.1 and Claim 2.2 to see

$$\begin{aligned} \left| \mathcal{K}_k^{(3)}(\mathcal{H}) \right| &\geq d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^{k-1} \left(1 - \hat{f}_{k-1}(\delta_3) \right) \sum_{v \in V_1^{\text{typ}}} \frac{|[t_v]_{\text{typ}}^{k-1}|}{t_v^{k-1}} \\ &\geq d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k \left(1 - \hat{f}_{k-1}(\delta_3) \right) \left(1 - \delta_3^{1/20} \right) \left(1 - \delta_3^{1/4} - 2k\delta_2 \right) = d_3^{\binom{k}{3}} d_2^{\binom{k}{2}} n^k \left(1 - f_k(\delta_3) \right) \end{aligned}$$

for an implicitly defined function $f_k(\delta_3) \rightarrow 0$ as $\delta_3 \rightarrow 0$. Thus, our proof of Theorem 1.5 is complete upon justifying Claim 2.2.

3. PARTS A AND B OF CLAIM 2.2

In this section, we reformulate Parts A and B of Claim 2.2 in a precise language and then prove Part A. To begin, we find the following notation helpful to discuss both parts of Claim 2.2 and use it in the remainder of this paper.

Notation 3.1. Fix $v \in V_1$. For fixed $1 < i < j \leq k$, set

$$\mathbf{L}_{\text{good}}^{ij}(v) = \left\{ (a, b) \in [t_v]^2 : L_v^{ij}[W_a^{v,i}, W_b^{v,j}] \text{ is } \varepsilon_0\text{-regular with density } d_2d_3(1 \pm \delta_3^{1/4}) \right\}$$

and set $\mathbf{L}_{\text{bad}}^{ij}(v) = [t_v]^2 \setminus \mathbf{L}_{\text{good}}^{ij}(v)$. For fixed $1 < h < i < j \leq k$, set

$$\mathbf{L}_{\text{good}}^{hij}(v) = \left\{ (a, b, c) \in [t_v]^3 : (a, b) \in \mathbf{L}_{\text{good}}^{hi}(v), (b, c) \in \mathbf{L}_{\text{good}}^{ij}(v), (a, c) \in \mathbf{L}_{\text{good}}^{hj}(v) \right\}$$

and set $\mathbf{L}_{\text{bad}}^{hij}(v) = [t_v]^3 \setminus \mathbf{L}_{\text{good}}^{hij}(v)$. Similarly, set

$$\mathbf{H}_{\text{good}}^{hij}(v) = \left\{ (a, b, c) \in [t_v]^3 : L_v[W_a^{h,v}, W_b^{i,v}, W_c^{j,v}] \text{ is } (d_3, \delta_3^{1/20}, r')\text{-regular with respect to } \mathcal{H}^{hij} \right\}$$

and set $\mathbf{H}_{\text{bad}}^{hij}(v) = [t_v]^3 \setminus \mathbf{H}_{\text{good}}^{hij}(v)$.

It suffices to prove Parts A and B from Claim 2.2 in the following form.

Proposition 3.2 (Part A). All but $4k^2\delta_3^{1/2}n$ vertices $v \in V_1$ satisfy that for all $1 < i < j \leq k$,

$$\left| \mathbf{L}_{\text{bad}}^{ij}(v) \right| \leq 3\delta_3^{1/4}t_v^2.$$

induction hypothesis on Theorem 1.5. We stress that what is crucial to apply the induction hypothesis is that the hierarchy of constants in upcoming (6) (cf. (2)) is satisfied. As for the slight lack of uniformity of densities or vertex class sizes, we could easily alter our induction hypothesis to accommodate the small fluctuations we may incur. For simplicity, we suppress handling these minor details.

Proposition 3.3 (Part B). *All but $k^3\delta_2^{1/2}n$ vertices $v \in V_1$ satisfy that for all $1 < i < j \leq k$,*

$$\left| \mathbf{H}_{\text{bad}}^{hij}(v) \cap \mathbf{L}_{\text{good}}^{hij}(v) \right| < 2\delta_3^{1/20}t_v^3.$$

Indeed, Propositions 3.2 and 3.3 together imply that all but

$$4k^2\delta_3^{1/2}n + k^3\delta_2^{1/2}n \leq \delta_3^{1/4}n$$

vertices $v \in V_1$ satisfy

$$\begin{aligned} & t_v^{k-3} \sum_{1 < i < j \leq k} \left| \mathbf{L}_{\text{bad}}^{ij}(v) \right| + t_v^{k-4} \sum_{1 < h < i < j \leq k} \left| \mathbf{H}_{\text{bad}}^{hij}(v) \right| \\ &= t_v^{k-3} \sum_{1 < i < j \leq k} \left| \mathbf{L}_{\text{bad}}^{ij}(v) \right| + t_v^{k-4} \sum_{1 < h < i < j \leq k} \left| \mathbf{H}_{\text{bad}}^{hij}(v) \cap \mathbf{L}_{\text{good}}^{hij}(v) \right| + t_v^{k-4} \sum_{1 < h < i < j \leq k} \left| \mathbf{H}_{\text{bad}}^{hij}(v) \cap \mathbf{L}_{\text{bad}}^{hij}(v) \right| \\ &\leq 2t_v^{k-3} \sum_{1 < i < j \leq k} \left| \mathbf{L}_{\text{bad}}^{ij}(v) \right| + t_v^{k-4} \sum_{1 < h < i < j \leq k} \left| \mathbf{H}_{\text{bad}}^{hij}(v) \cap \mathbf{L}_{\text{good}}^{hij}(v) \right| \\ &\leq 6\delta_3^{1/4} \binom{k}{2} t_v^{k-1} + 2\delta_3^{1/20} \binom{k}{3} t_v^{k-1} \leq \delta_3^{1/20} k^3 t_v^{k-1}, \end{aligned}$$

as promised by Claim 2.2.

Proposition 3.2 is easy to prove and we give this proof below. Proposition 3.3 is more challenging to prove and we devote Section 4 to its proof.

3.1. Proof of Proposition 3.2. With the constants $k, d_3, \delta_3, d_2, \varepsilon_0, T_0, r', \delta_2$ and r given in (2), recall \mathcal{H} and P are as in Setup 1.4 with associated family of ε_0 -regular partitions and $\{\mathbf{P}_v\}_{v \in V_1}$. We prove all but $4k^2\delta_3^{1/2}n$ vertices $v \in V_1$ satisfy that for all $1 < i < j \leq k$, $|\mathbf{L}_{\text{bad}}^{ij}(v)| \leq 3\delta_3^{1/4}t_v^2$. Let us clarify this goal. Fix $1 < i < j \leq k$. Since \mathbf{P}_v is ε_0 -regular for every $v \in V_1$, at most $\varepsilon_0 t_v^2 \leq \delta_3^{1/4}t_v^2$ pairs $W_a^{v,i}, W_b^{v,j}$, $1 \leq a, b \leq t_v$, can be irregular. Thus, the burden of proving Proposition 3.2 concerns establishing the density assertion. This assertion is easily handled using the following (easy) material in [3] and [11].

3.1.1. Good vertices. We begin with the following definition.

Definition 3.4. *Let $L \subseteq P$ be bipartite graphs with bipartition $U_1 \cup U_2$ and let $\alpha, \gamma > 0$ and integer r be given. We say L is (α, γ, r) -regular with respect to P if every r -tuple $\mathbf{B} = (B_1, \dots, B_r)$ of induced subgraphs $B_i \subseteq P$ satisfying $|\bigcup_{s=1}^r B_s| > \gamma|P|$ also satisfies $|L \cap \bigcup_{s=1}^r B_s| = (\alpha \pm \gamma)|\bigcup_{s=1}^r B_s|$.*

The following Fact 3.5 appeared (in slightly different language) as Fact 4.3.2 of [11], pg 281, and asserts that for \mathcal{H} and P as in Setup 1.4, most vertices $v \in V_1$ satisfy that their links L_v^{ij} , $1 < i < j \leq k$, are $(\alpha, 2\delta_3^{1/2}, r)$ -regular.

Fact 3.5 (most links are regular). *Let k, d_3, δ_3, d_2 and r be given³ as in (2). With $\delta_2 = \delta_2(d_2) > 0$ sufficiently small as in (2), whenever \mathcal{H} and P are as in Setup 1.4 with constants $k, d_3, \delta_3, d_2, \delta_2$ and r , then all but $2k^2\delta_3^{1/2}n$ vertices $v \in V_1$ satisfy that for all $1 < i < j \leq k$, L_v^{ij} is $(d_3, 2\delta_3^{1/2}, r)$ -regular with respect to P_v^{ij} .*

As can be seen in [11], Fact 3.5 is quite easy to prove and follows easily from the condition in Setup 1.4 that all 3-graphs \mathcal{H}^{1ij} , $1 < i < j \leq k$, are (d_3, δ_3, r) -regular.

As in Fact 3.5, we say that a vertex $v \in V_1$ is a *good vertex* if for all $1 < i < j \leq k$, L_v^{ij} is $(d_3, 2\delta_3^{1/2}, r)$ -regular with respect to P_v^{ij} and if it is typical in the sense of Fact 2.1 (see previous section). Let $V_1^{\text{good}} = V_1^{\text{good}}(k, d_3, \delta_3, d_2, \delta_2, r)$ be the set of all good vertices $v \in V_1$. Facts 2.1 and 3.5 assert that with $\delta_2 = \delta_2(d_2)$ sufficiently small, $|V_1^{\text{good}}| > (1 - 3k^2\delta_3^{1/2})n$.

³Fact 3.5, as it appears in [11], has a very loose quantification of constants. All that is required is that $\delta_2 \ll d_3, \delta_3, d_2$ and integer r can be arbitrary.

3.1.2. *Fact 3.5* \implies *Proposition 3.2*. The proof of Proposition 3.2 is now immediate. In the context of Setup 1.4, almost every vertex $v \in V_1$ is a good vertex. Indeed, from (2), δ_2 is sufficiently small. Now, fix $1 < i < j \leq k$. The key observation is that every good vertex $v \in V_1^{\text{good}}$ satisfies that all but $2\delta_3^{1/4}t_v^2$ pairs $W_a^{v,i}, W_b^{v,j}$, $1 \leq a, b \leq t_v$, have density $d_2d_3(1 \pm \delta_3^{1/4})$.

Indeed, let $v \in V_1^{\text{good}}$ but suppose $\{W_a^{v,i}, W_b^{v,j}\}_{\{a,b\} \in I}$ is a collection of pairs, each with density, say, smaller than $d_2d_3(1 - \delta_3^{1/4})$, such that $|I| \geq \delta_3^{1/4}t_v^2$. We claim the vector $\mathbf{B} = (P_v^{ij}[W_a^{v,i}, W_b^{v,j}] : \{a, b\} \in I)$ contradicts the $(d_3, 2\delta_3^{1/2}, r)$ -regularity of L_v^{ij} with respect to P_v^{ij} .

Note that (2) gives that $r \geq T_0^2 \geq t_v^2 \geq |I|$. The vector \mathbf{B} is therefore an r -tuple of *induced* subgraphs of P_v^{ij} . We claim \mathbf{B} is an r -tuple of induced subgraphs of P_v^{ij} satisfying

$$\left| \bigcup_{\{a,b\} \in I} P_v^{ij} [W_a^{v,i}, W_b^{v,j}] \right| > 2\delta_3^{1/2} |P_v^{ij}| \quad (8)$$

and

$$\left| L_v^{ij} \cap \bigcup_{\{a,b\} \in I} P_v^{ij} [W_a^{v,i}, W_b^{v,j}] \right| < (d_3 - 2\delta_3^{1/2}) \left| \bigcup_{\{a,b\} \in I} P_v^{ij} [W_a^{v,i}, W_b^{v,j}] \right|. \quad (9)$$

Once (8) and (9) are established, we see the vector \mathbf{B} contradicts the $(d_3, 2\delta_3^{1/2}, r)$ -regularity of L_v^{ij} with respect to P_v^{ij} . This will prove Proposition 3.2.

Both of (8) and (9) are easily established and we sketch their proofs, beginning with the former. Observe that

$$\left| \bigcup_{\{a,b\} \in I} P_v^{ij} [W_a^{v,i}, W_b^{v,j}] \right| = \sum_{\{a,b\} \in I} |P_v^{ij} [W_a^{v,i}, W_b^{v,j}]|. \quad (10)$$

Fix $\{a, b\} \in I$. Recall that $\delta_2 \ll 1/T_0 \leq 1/t_v$ in (2) and that by Fact 2.1

$$|W_a^{v,i}|, |W_b^{v,j}| = (d_2 \pm 2\delta_2) \frac{n}{t_v}.$$

Consequently, the (d_2, δ_2) -regularity of P^{ij} implies that

$$|P_v^{ij} [W_a^{v,i}, W_b^{v,j}]| = (d_2 \pm \delta_2) |W_a^{v,i}| |W_b^{v,j}| = (d_2 \pm \delta_2) \left((d_2 \pm 2\delta_2) \frac{n}{t_v} \right)^2 = (d_2 \pm 2\delta_2)^3 \frac{n^2}{t_v^2}. \quad (11)$$

The (d_2, δ_2) -regularity of P^{ij} also implies

$$|P_v^{ij}| = (d_2 \pm \delta_2) \left((d_2 \pm \delta_2) n \right)^2 = (d_2 \pm \delta_2)^3 n^2. \quad (12)$$

Consequently, with $|I| \geq \delta_3^{1/4}t_v^2$, (10), (11) and (12) easily establish (8).

Observe that (9) is equivalent to

$$\sum_{\{a,b\} \in I} |L_v^{ij} [W_a^{v,i}, W_b^{v,j}]| < (d_3 - 2\delta_3^{1/2}) \sum_{\{a,b\} \in I} |P_v^{ij} [W_a^{v,i}, W_b^{v,j}]|. \quad (13)$$

Fix $\{a, b\} \in I$. Our assumption is that

$$|L_v^{ij} [W_a^{v,i}, W_b^{v,j}]| < d_2d_3 \left(1 - \delta_3^{1/4} \right) |W_a^{v,i}| |W_b^{v,j}|$$

which, with (11), implies

$$|L_v^{ij} [W_a^{v,i}, W_b^{v,j}]| < d_3 \frac{1 - \delta_3^{1/4}}{1 - \delta_2 d_2^{-1}} |P_v^{ij} [W_a^{v,i}, W_b^{v,j}]| < (d_3 - 2\delta_3^{1/2}) |P_v^{ij} [W_a^{v,i}, W_b^{v,j}]| \quad (14)$$

where the last inequality follows from $\delta_2 \ll d_2$ and $\delta_3 \ll d_3$ in (2). As (14) holds for each $\{a, b\} \in I$, (13) (cf. (9)) follows.

4. PROOF OF PART B

With the constants $k, d_3, \delta_3, d_2, \varepsilon_0, T_0, r', \delta_2$ and r given in (2), recall \mathcal{H} and P are as in Setup 1.4 with the associated family of ε_0 -regular partitions $\{\mathbf{P}_v\}_{v \in V_1}$. We show all but $k^3 \delta_2^{1/2} n$ vertices $v \in V_1$ satisfy $|\mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v)| < 2\delta_3^{1/20} t_v^3$ for all $1 < h < i < j \leq k$. In the remainder of this paper, we fix $1 < h < i < j \leq k$. It suffices to prove that all but $\delta_2^{1/2} n$ vertices $v \in V_1$ satisfy $|\mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v)| < 2\delta_3^{1/20} t_v^3$ for the fixed indices $1 < h < i < j \leq k$.

Assume, on the contrary, there exists a set $A_{hij} \subseteq V_1$ of size

$$|A_{hij}| > \delta_2^{1/2} n \quad (15)$$

consisting of vertices for which

$$\left| \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right| \geq 2\delta_3^{1/20} t_v^3. \quad (16)$$

We show that (15) contradicts our hypothesis of Setup 1.4 that the 3-graph \mathcal{H}^{hij} , $1 \leq h < i < j \leq k$ fixed above, is (d_3, δ_3, r) -regular with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$. Our approach, while technical in appearance, is straightforward. We outline our approach in the following remark.

Remark 4.1. Fix $v \in A_{hij}$ and fix $(a, b, c) \in \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v)$. Since $(a, b, c) \in \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \subseteq \mathbb{H}_{\text{bad}}^{hij}(v)$, we appeal to Notation 3.1 and Definition 1.3 to infer that there exists a vector (witness) $\mathbf{Q}_{vabc}^{hij} = (Q_{vabc}^{hij}(p) : 1 \leq p \leq r')$, $Q_{vabc}^{hij}(p) \subseteq L_v[W_a^{h,v}, W_b^{i,v}, W_c^{j,v}]$, satisfying

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \right| > 2\delta_3^{1/20} \left| \mathcal{K}_3^{(2)} \left(L_v \left[W_a^{h,v}, W_b^{i,v}, W_c^{j,v} \right] \right) \right|, \quad (17)$$

but

$$\left| d_{\mathcal{H}} \left(\mathbf{Q}_{vabc}^{hij} \right) - d_3 \right| \geq 2\delta_3^{1/20}. \quad (18)$$

In upcoming Construction 4.3, we collect the witnesses \mathbf{Q}_{vabc}^{hij} over $(a, b, c) \in \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v)$ and $v \in A_{hij}$ to create a ‘‘big witness’’ \mathbf{Q}_{hij} that will contradict the (d_3, δ_3, r) -regularity of \mathcal{H}^{hij} with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$.

In the process of collecting the witnesses \mathbf{Q}_{vabc}^{hij} over $(a, b, c) \in \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v)$ and $v \in A_{hij}$, we do not need the entire set A_{hij} , and in fact, we need only a small subset thereof. Over two separate rounds, we refine the set A_{hij} into two nested subsets $A'_{hij} \subseteq A''_{hij} \subseteq A_{hij}$ where the final subset A''_{hij} produces the big witness \mathbf{Q}_{hij} promised.

4.1. Refining the set A_{hij} . We begin with the first round.

4.1.1. Round 1. We obtain the intermediate subset $A'_{hij} \subseteq A_{hij}$ using the following fact. The easy fact below states that from A_{hij} we may find a fairly large subset of vertices A'_{hij} , every pair of which has the ‘‘right’’ shared P^{1q} -neighborhood, $q \in \{h, i, j\}$.

Fact 4.2. Set

$$f = 256 \frac{\delta_3^{2/5}}{d_3^3 d_2^2}. \quad (19)$$

There exists $A'_{hij} = \{v_1, \dots, v_{2f}\} \subseteq A_{hij}$ satisfying that for each $q \in \{h, i, j\}$, for each $1 \leq s < s' \leq 2f$,

$$\left| N_{P^{1q}}(v_s) \cap N_{P^{1q}}(v_{s'}) \right| = (d_2 \pm \delta_2)^2 n. \quad (20)$$

Observe that Fact 4.2 is only an assertion on the behaviour of the graphs P^{1h} , P^{1i} and P^{1j} . Due to the (d_2, δ_2) -regularity of these 3 graphs where $\delta_2 \ll d_2$ in (2), Fact 4.2 is easy to prove from the hypothesis in (15) that $|A_{hij}| \geq \delta_2^{1/2}n$. In particular, Fact 4.2 is essentially proved in [3] on pp. 155.

4.1.2. *Round 2.* To identify the subset $A''_{hij} \subset A'_{hij} = \{v_1, \dots, v_{2f}\}$, we use the following considerations. Fix $v \in \{v_1, \dots, v_{2f}\}$. Let

$$\left(\mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right)^- \subseteq \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v)$$

denote the set of triples (a, b, c) for which the density in (18) is too small, i.e., $d_{\mathcal{H}}(\mathbf{Q}_{vabc}^{hij}) < d_3 - 2\delta_3^{1/20}$, which is to say

$$\left| \mathcal{H}^{hij} \cap \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \right| < \left(d_3 - 2\delta_3^{1/20} \right) \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{vabc}^{hij}(p)) \right|. \quad (21)$$

Let

$$\left(\mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right)^+ \subseteq \mathbb{H}_{\text{bad}}^{hij}(v)$$

denote the set of triples (a, b, c) for which the density in (18) is too large, i.e. $d_{\mathcal{H}}(\mathbf{Q}_{vabc}^{hij}) > d_3 - 2\delta_3^{1/20}$. Now, let

$$\{v_1, \dots, v_{2f}\}^- \subseteq \{v_1, \dots, v_{2f}\}$$

denote the set of vertices v for which

$$\left| \left(\mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right)^- \right| \geq \frac{1}{2} \left| \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right| > \delta_3^{1/20} t_v^3 \quad (22)$$

where the last equality holds from (16). Let

$$\{v_1, \dots, v_{2f}\}^+ \subseteq \{v_1, \dots, v_{2f}\}$$

denote the set of vertices v for which

$$\left| \left(\mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right)^+ \right| \geq \frac{1}{2} \left| \mathbb{L}_{\text{good}}^{hij}(v) \cap \mathbb{H}_{\text{bad}}^{hij}(v) \right| > \delta_3^{1/20} t_v^3.$$

Clearly, one of

$$\left| \{v_1, \dots, v_{2f}\}^- \right| \geq f, \quad \left| \{v_1, \dots, v_{2f}\}^+ \right| \geq f$$

holds. In our proof, it does not matter which holds as the cases are symmetric. We assume, without loss of generality, that the former holds. By reordering the vertices, we take

$$A''_{hij} = \{v_1, \dots, v_f\} \subseteq \{v_1, \dots, v_{2f}\}^- \quad (23)$$

and construct the witness \mathbf{Q}_{hij} based on $A''_{hij} = \{v_1, \dots, v_f\}$.

4.2. The promised witness. With the set $A''_{hij} = \{v_1, \dots, v_f\}$ above, we construct the promised witness \mathbf{Q}_{hij} .

Construction 4.3. Fix $1 \leq p \leq r'$ and $1 \leq s \leq f$. Define the graph

$$Q_{v_s}^{hij}(p) = \bigcup_{(a,b,c) \in (\mathbb{L}_{\text{good}}^{hij}(v_s) \cap \mathbb{H}_{\text{bad}}^{hij}(v_s))^-} Q_{v_s abc}^{hij}(p). \quad (24)$$

Define the $r'f$ -tuple

$$\mathbf{Q}_{hij} = (Q_{v_s}^{hij}(p) : 1 \leq p \leq r', 1 \leq s \leq f). \quad (25)$$

We assert \mathbf{Q}_{hij} is the promised r -tuple witnessing the (d_3, δ_3, r) -irregularity of \mathcal{H}^{hij} with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$. We first observe \mathbf{Q}_{hij} is a vector of appropriate length. Indeed, by construction \mathbf{Q}_{hij} is an $r'f$ -tuple of subgraphs from $P^{hi} \cup P^{ij} \cup P^{hj}$. Recall $f = 256\delta_3^{2/5}d_3^{-3}d_2^{-3}$ given in (19). Since $r \gg r'$ and $r \gg d_2^{-1}$ from (2), we see $r \geq r'f$, making \mathbf{Q}_{hij} an r -tuple of subgraphs.

Since \mathbf{Q}_{hij} is an r -tuple of subgraphs from $P^{hi} \cup P^{ij} \cup P^{hj}$, the following Claims 4.4 and 4.5 provide a direct contradiction to the (d_3, δ_3, r) -regularity of \mathcal{H}^{hij} with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$.

Claim 4.4. $\left| \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right| > \delta_3 \left| \mathcal{K}_3^{(2)}(P^{hi} \cup P^{ij} \cup P^{hj}) \right|$.

Claim 4.5.

$$\frac{\left| \mathcal{H} \cap \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right|}{\left| \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right|} < d_3 - \delta_3.$$

Since Claims 4.4 and 4.5 provide a direct contradiction to the (d_3, δ_3, r) -regularity of \mathcal{H}^{hij} with respect to $P^{hi} \cup P^{ij} \cup P^{hj}$, our proof of Proposition 3.3 will be complete upon proving these two claims.

4.3. Proof of Claim 4.4. Observe that

$$\begin{aligned} \left| \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right| &\geq \sum_{s=1}^f \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| \\ &\quad - \sum_{1 \leq s < s' \leq f} \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \cap \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_{s'}}^{hij}(p)) \right|. \end{aligned} \tag{26}$$

We bound the two terms on the right hand side of (26) in the following two facts.

Fact 4.6. For $1 \leq s \leq f$ fixed,

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| \geq \frac{\delta_3^{1/10}}{128} d_3^3 d_2^6 n^3.$$

Fact 4.7. For $1 \leq s < s' \leq f$ fixed,

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \cap \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_{s'}}^{hij}(p)) \right| \leq 16d_2^9 n^3.$$

Facts 4.6 and Fact 4.7 conclude the proof of Claim 4.4. Indeed, applying Facts 4.6 and 4.7 to (26), we see

$$\left| \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right| \geq f \frac{\delta_3^{1/10}}{128} d_3^3 d_2^6 n^3 - 8f^2 d_2^9 n^3 = d_2^3 n^3 \left(\frac{f d_3^3 d_2^3 \delta_3^{1/10}}{128} - 8f^2 d_2^6 \right).$$

Inserting the value $f = 256\delta_3^{2/5}d_3^{-3}d_2^{-3}$ from (19), we infer the further lower bound

$$d_2^3 n^3 \left(\delta_3^{1/2} - \frac{2^{19}}{d_3^6} \delta_3^{4/5} \right) = \delta_3^{1/2} d_2^3 n^3 \left(1 - \frac{2^{19}}{d_3^6} \delta_3^{3/10} \right) \geq \frac{1}{2} \delta_3^{1/2} d_2^3 n^3, \tag{27}$$

where the last inequality follows from the fact that $\delta_3 \ll d_3$ from (2). On the other hand, we conclude from Fact 1.2 that

$$\left| \mathcal{K}_3^{(2)}(P^{hi} \cup P^{ij} \cup P^{hj}) \right| \leq (d_2 + \theta(\delta_2))^3 n^3 \leq 2d_2^3 n^3$$

where $\theta(\delta_2) \rightarrow 0$ as $\delta_2 \rightarrow 0$. Comparing this inequality against (27) proves Claim 4.4. Thus, it remains to verify Facts 4.6 and 4.7.

4.3.1. *Proof of Fact 4.6.* Fix $1 \leq s \leq f$. Observe from (24) that

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| = \sum_{(a,b,c) \in (\mathbb{L}_{\text{good}}^{hij}(v_s) \cap \mathbb{H}_{\text{bad}}^{hij}(v_s))^-} \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s abc}^{hij}(p)) \right|. \quad (28)$$

Using (17), we further conclude

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| \geq \sum_{(a,b,c) \in (\mathbb{L}_{\text{good}}^{hij}(v_s) \cap \mathbb{H}_{\text{bad}}^{hij}(v_s))^-} \delta_3^{1/20} \left| \mathcal{K}_3^{(2)} \left(L_v \left[W_a^{h,v}, W_b^{i,v}, W_c^{j,v} \right] \right) \right|. \quad (29)$$

We further estimate (29) by appealing to Fact 1.2.

Fix $(a, b, c) \in (\mathbb{L}_{\text{good}}^{hij}(v_s) \cap \mathbb{H}_{\text{bad}}^{hij}(v_s))^- \subseteq \mathbb{L}_{\text{good}}^{hij}(v_s)$. By the definition of $\mathbb{L}_{\text{good}}^{hij}(v_s)$, each of the bipartite graphs $L_{v_s}^{hi}[W_a^{h,v_s}, W_b^{i,v_s}]$, $L_{v_s}^{ij}[W_b^{i,v_s}, W_c^{j,v_s}]$, $L_{v_s}^{hj}[W_a^{h,v_s}, W_c^{j,v_s}]$, is ε_0 -regular with density $d_3 d_2 (1 \pm \delta_3^{1/4})$. Applying Fact 1.2 to $L_v[W_a^{h,v}, W_b^{i,v}, W_c^{j,v}]$, we therefore conclude

$$\begin{aligned} \left| \mathcal{K}_3^{(2)} \left(L_v \left[W_a^{h,v}, W_b^{i,v}, W_c^{j,v} \right] \right) \right| &\geq \left(d_3 d_2 \left(1 - \delta_3^{1/4} \right) - \theta_{d_3 d_2}(\varepsilon_0) \right)^3 |W_a^{h,v_s}| |W_b^{i,v_s}| |W_c^{j,v_s}|, \\ &\geq \frac{(d_3 d_2)^3}{4} |W_a^{h,v_s}| |W_b^{i,v_s}| |W_c^{j,v_s}| \geq \frac{(d_3 d_2)^3}{128} \left(\frac{d_2 n}{t_v} \right)^3 = \frac{d_3^3 d_2^6 n^3}{128 t_v^3} \end{aligned} \quad (30)$$

where the last inequality follows from Fact 2.1 in Section 2.2. Applying (30) to (29), we conclude

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| \geq \frac{\delta_3^{1/20} d_3^3 d_2^6}{128 t_v^3} n^3 \left| (\mathbb{L}_{\text{good}}^{hij}(v_s) \cap \mathbb{H}_{\text{bad}}^{hij}(v_s))^- \right|.$$

We use (22) to conclude the proof of Fact 4.6.

4.3.2. *Proof of Fact 4.7.* Fix $1 \leq s < s' \leq f$. We use the notation $P_{v_s v_{s'}}^{hi} = P_{v_s}^{hi} \cap P_{v_{s'}}^{hi}$ where we recall $P_{v_s}^{hi} = P^{hi}[N_{P^{1h}}(v_s), N_{P^{1i}}(v_s)]$. Define $P_{v_s v_{s'}}^{ij}$ and $P_{v_s v_{s'}}^{hj}$ similarly. Then,

$$\left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \cap \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_{s'}}^{hij}(p)) \right| \leq \left| \mathcal{K}_3^{(2)} \left(P_{v_s v_{s'}}^{hi} \cup P_{v_s v_{s'}}^{ij} \cup P_{v_s v_{s'}}^{hj} \right) \right|. \quad (31)$$

To bound the right hand side, we apply Fact 1.2 to the graph $P_{v_s v_{s'}}^{hi} \cup P_{v_s v_{s'}}^{ij} \cup P_{v_s v_{s'}}^{hj}$ and may do so as (via (20)) each of $P_{v_s v_{s'}}^{hi}$, $P_{v_s v_{s'}}^{ij}$ and $P_{v_s v_{s'}}^{hj}$ is $(d_2, \delta_2^{1/2})$ -regular⁴. Applying Fact 1.2,

$$\frac{\left| \mathcal{K}_3^{(2)} \left(P_{v_s v_{s'}}^{hi} \cup P_{v_s v_{s'}}^{ij} \cup P_{v_s v_{s'}}^{hj} \right) \right|}{|N_{P^{1h}}(v_s, v_{s'})| |N_{P^{1i}}(v_s, v_{s'})| |N_{P^{1j}}(v_s, v_{s'})|} \leq \left(d_2 + \theta_{d_2}(\sqrt{\delta_2}) \right)^3 \leq 2d_2^3$$

from which it follows (via (20)) that

$$\left| \mathcal{K}_3^{(2)} \left(P_{v_s v_{s'}}^{hi} \cup P_{v_s v_{s'}}^{ij} \cup P_{v_s v_{s'}}^{hj} \right) \right| \leq 16d_2^3 n^3. \quad (32)$$

Combining (31) and (32) proves Fact 4.7.

⁴It is easy to show that each of $P_{v_s v_{s'}}^{hi}$, $P_{v_s v_{s'}}^{ij}$ and $P_{v_s v_{s'}}^{hj}$ is $(d_2, \delta_2^{1/2})$ -regular from the hypothesis that each of P^{hi} , P^{ij} and P^{hj} is (d_2, δ_2) -regular.

4.4. **Proof of Claim 4.5.** The proof of Claim 4.5 is easy and follows largely from work in Claim 4.4. We bound

$$\frac{\left| \mathcal{H} \cap \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right|}{\left| \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right|}$$

and begin with the numerator.

Clearly

$$\left| \mathcal{H} \cap \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right| \leq \sum_{1 \leq s \leq f} \left| \bigcup_{p=1}^{r'} \mathcal{H} \cap \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right|.$$

We infer from (24) and (25) that for fixed $1 \leq s \leq f$,

$$\left| \bigcup_{p=1}^{r'} \mathcal{H} \cap \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| = \sum_{(a,b,c) \in (L_{\text{good}}^{hij}(v_s) \cap H_{\text{bad}}^{hij}(v_s))^-} \left| \mathcal{H} \cap \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s abc}^{hij}(p)) \right|$$

and hence, from (22),

$$\begin{aligned} \left| \bigcup_{p=1}^{r'} \mathcal{H} \cap \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| &\leq (d_3 - \delta_3^{1/20}) \sum_{(a,b,c) \in (L_{\text{good}}^{hij}(v_s) \cap H_{\text{bad}}^{hij}(v_s))^-} \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s abc}^{hij}(p)) \right| \\ &= (d_3 - \delta_3^{1/20}) \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right| \end{aligned}$$

where the last equality follows from (24). Consequently,

$$\left| \mathcal{H} \cap \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right| \leq (d_3 - \delta_3^{1/20}) \sum_{1 \leq s \leq f} \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right|. \quad (33)$$

Using (26) and (33), we see

$$\frac{\left| \mathcal{H} \cap \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right|}{\left| \bigcup \left\{ \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) : 1 \leq p \leq r', 1 \leq s \leq f \right\} \right|} < (d_3 - \delta_3^{1/20}) \frac{1}{1 - \frac{A}{B}},$$

where $A = \sum_{1 \leq s < s' \leq f} \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \cap \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_{s'}}^{hij}(p)) \right|$ and $B = \sum_{s=1}^f \left| \bigcup_{p=1}^{r'} \mathcal{K}_3^{(2)}(Q_{v_s}^{hij}(p)) \right|$. Using Facts 4.6 and 4.7, we obtain further upper bound

$$(d_3 - \delta_3^{1/20}) \left(1 - \frac{8f^2 d_2^9 n^3}{(f \delta_3^{1/10} d_3^3 d_2^6 n^3)/(128)} \right)^{-1} = (d_3 - \delta_3^{1/20}) \left(1 - \frac{2^{18} f d_2^3}{\delta_3^{1/10} d_3^3} \right)^{-1}.$$

Using the value of f in (19), we obtain further upper bound $(d_3 - \delta_3^{1/20}) \left(1 - \frac{2^{18} \delta_3^{3/10}}{d_3^6} \right)^{-1}$. Claim 4.5 then follows the fact that $\delta_3 \ll d_3$ in (2). \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEVADA, RENO, RENO, NV, 89557, USA
E-mail address: `nagle@unr.edu`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY, ATLANTA, GA, 30032, USA
E-mail address: `{rod1|mschach}@mathcs.emory.edu`