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Cantor's Diagonalization Process and the Baire-Moore Category Theorems

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1. INTRODUCTION. Cantor's diagonalization argument is well known and is frequently used to show that the real numbers are uncountable. It is also well known that the uncountability of the reals is an immediate consequence of the following theorem that we call the Baire theorem for short.

Theorem 1.1 (Baire Category Theorem). *The intersection of a countable collection of dense open sets in a space X is dense in X.*

The Baire theorem is valid in a complete metric space or in a locally compact Hausdorff space (see [**3,** p. 200]).

It seems to be less well known that the following theorem from R. L. Moore's Colloquium Publication (Theorem 53 in [**4,** p. 21]) is stronger than the Baire theorem but can be proved using an argument much like Cantor's simple diagonalization method. We refer to Theorem 1.2 as the Moore theorem.

Theorem 1.2 *No closed point set M is the sum of countably many closed point sets such that if H is one of them every point of H is a limit point of $M - H$.*

This theorem is also valid in a complete metric space or a compact Hausdorff space. We do not know if R. L. Moore is responsible for this theorem or not. He does have a similar theorem in [**5**].

In this article we provide an interpretation of Cantor's proof that is conceptually simpler in that the representation of the real numbers as decimals is not explicitly used. Then we show modifications of that proof to provide relatively simple proofs of several theorems including the Moore and Baire theorems. We have found this approach helpful in getting students to understand or even to discover a proof for the Baire theorem.

2. CANTOR'S PROOF. We begin by briefly recalling one version of the Cantor diagonalization proof (see [**2,** p. 43]). For simplicity we show that the interval $[0,1]$ is not countable. Assume to the contrary that there is a sequence $x_1, x_2, x_3, \ldots$ that contains all numbers in $[0,1]$ and express x_i as the decimal:

$$x_i = 0.a_{i1}a_{i2}a_{i3}\ldots$$

where a_{ij} is one of the integers $0, 1, 2, \cdots, 9$.

Let $x = .a_1a_2a_3\ldots$ where for each n a_n is any integer from 1 through 8 except a_{nn}. The number x is unambiguously represented as a decimal that is not in the sequence $x_1, x_2, x_3, \ldots$ and this contradiction completes the proof.

In this article an *interval* is a closed interval and an *open interval* is an interval except for its endpoints. Our interpretation of Cantor's argument is based on the observation that if $y = 0.y_1y_2y_3\ldots$, then y_i determines a unique interval of length $1/10^i$ that contains y and contains the interval determined by y_{i+1}. In other words, we consider the decimal representation of y as determining in a natural way a unique nested sequence of intervals having y as the only point common to all of them.

Interpreted in this manner Cantor's procedure produces a nested sequence of intervals such that the i^{th} interval in the sequence does not contain x_i. Hence any point in the intersection of these interval is not included in the sequence $x_1, x_2, x_3, \ldots$.

Such a sequence of intervals is easily constructed as follows. Select a subinterval I_1 of $[0,1]$ that does not contain x_1. For example one might divide $[0,1]$ into three non-overlapping intervals. One of them must fail to contain x_1. One might think of this as Cantor's proof in a ternary system. Next select a subinterval I_2 of I_1 that does not contain x_2. This process can be continued to obtain a nested sequence of intervals that have a non-empty intersection each point of which lies in $[0,1]$ and is not in the sequence $x_1, x_2, x_3, \ldots$.

3. PERFECT SETS. As a first step in modifying Cantor's method to prove other theorems we modify the proof in the last paragraph of section 2 to show that no perfect set is countable. Recall that a set M is *perfect* if it is closed and each point of M is a limit point of M. This time we will need that a nested sequence of non-empty closed and bounded sets of numbers has a non-empty intersection.

Assume that there is a perfect set M and a sequence $x_1, x_2, x_3, \ldots$ of points of M such that each point of M is in the sequence. Let S_0 be an open interval about x_1. There is a smallest integer n_1 such that x_{n_1} is in S_0 and is not x_1. Let S_1 be an open interval containing x_{n_1} whose closure I_1 does not contain x_1 and is a subset of S_0. Next we want a subinterval I_2 of I_1 that does not contain x_2. Again there is a smallest integer n_2 such that x_{n_2} is in S_1 and is not x_{n_1}. Select an open interval S_2 containing x_{n_2} whose closure I_2 does not contain x_{n_1} and is a subset of S_1. Clearly I_2 does not contain x_2 since it contains no x_j with $j \leq n_1$. This process can be continued to define an infinite sequence $I_1, I_2, I_3, \ldots$ such that for each i x_i is not in I_i. There is a point common to all the sets in the sequence $I_1 \cap M, I_2 \cap M, I_3 \cap M, \ldots$ that is in M but not in the sequence $x_1, x_2, x_3, \ldots$.

For the Moore theorem our proof differs from the preceding one in that the points $x_1, x_2, x_3, \ldots$ are replaced by a sequence of closed sets. We assume that the closed set M is the union of the sets $M_1, M_2, M_3, \ldots$ and that each point of each M_i is a limit point of $M - M_i$.

We begin as before by selecting an open interval S_0 containing a point p_0 of M_1. As p_0 is a limit point of $M - M_1$, there is a least integer $n_1 > 1$ such that S_0 contains a point p_1 of M_{n_1} that is not in M_1. Let S_1 be an open interval containing p_1 such that its closure I_1 lies in S_0 and contains no point of M_1. Note that I_1 contains no point of M_i for $i < n_1$. From the hypothesis S_1 contains a point of $M - M_{n_1}$, so there is a least positive integer n_2 such that S_1 contains a point p_2 of M_{n_2} that is not in M_{n_1}. As before, let S_2 be an open interval that contains p_2 and whose closure I_2 lies in S_1 and contains no point of M_{n_1}. There is no point of M_1 or M_2 in I_2. In fact, there is no point of M_j in I_2 for $j < n_2$ We can continue this process to determine a nested sequence $M \cap I_1, M \cap I_2, M \cap I_3, \ldots$ of closed and bounded sets that have a non-empty intersection each point of which is in M but not in any of the sets of the sequence $M_1, M_2, M_3, \ldots$. This contradiction completes the proof.

While we have described this proof for subsets of the real numbers, it is equally valid in a locally compact and therefore regular Hausdorff space.

In this case, the open intervals are replaced with open sets and the intervals with closures of open sets. It is also valid in a complete metric space, and the only additional modification required is to select the diameters of the open sets converging to zero fast enough to ensure that the closures of the open sets have a non-empty intersection.

4. COMPARING THE TWO THEOREMS. In a nice M. S. thesis at Emory University in 1965 Janice Thomas Astin showed that in an arbitrary topological space the Moore theorem implies the Baire theorem and gave an example of a topological space in which the Baire theorem holds but the Moore theorem does not. This is not surprising since the Baire theorem refers only to the space whereas the Moore theorem states a property that holds for all closed sets in the space. It is not difficult to show that in a topological space, if every closed subset considered as a subspace satisfies the Baire theorem, then the Moore theorem is valid in the space. What follows is essentially from Ms. Astin's thesis.

Theorem 4.3 (Astin). *Suppose that S is a topological space and no closed set M is the union of a countable number of closed sets such that if g is one of them, each point of g is a limit point of $M - g$. Then every countable collection of dense open sets has a dense intersection.*

Proof. Let $O_1, O_2, O_3, \ldots$ be a countable collection of dense open sets and define $M_i = S - O_i$ for each $i > 0$. Since M_i is closed and each point of M_i is a limit point of $S - M_i = O_i$ then $\bigcup_{i>0} M_i$ is not closed. Thus there is a point that is not in $\bigcup_{i>0} M_i$ It follows that $K = \bigcap_{i>0} O_i \neq \emptyset$.

It remains to show that K is dense in S. Assume to the contrary that there is an open set O that contains no point of K. Then each point of O is in $H_i = O - O_i$ for some $i > 0$ and we have that $O = \bigcup_{i>0} H_i$. It is not difficult to check that $\overline{O} = \overline{\bigcup_{i>0} H_i} = (\bigcup_{i>0} \overline{H_i}) \cup (\overline{O} - O)$. So we have the closed set $\overline{O}$ is the union of a countable collection of closed sets. We show that contrary to our hypothesis each point of each of these closed sets is a limit point of its complement in $\overline{O}$. Clearly each point of the closed set $\overline{O} - O$ is a limit point of $\overline{O} - (\overline{O} - O) = O$. We show next that each point of $\overline{H_i}$ is a limit point of $\overline{O} - \overline{H_i}$. We first note that if x is a point of H_i, then x is in O and a limit point of $O - H_i$ and thus a limit point of $\overline{O} - \overline{H_i}$. Now let p be a point of $\overline{H_i}$. If p is in H_i then as was just shown p is a limit point of

$\overline{O} - \overline{H_i}$. On the other hand if p is in $\overline{H_i} - H_i$ and U is an open set containing p, then U contains a point q of H_i, and again we have that q is a limit point of $\overline{O} - \overline{H_i}$ so U contains a point of $\overline{O} - \overline{H_i}$ whence p is a limit point of $\overline{O} - \overline{H_i}$. This involves a contradiction that completes our proof. ■

Finally we provide the example from Ms. Astin's thesis in which Theorem 1.1 holds but Theorem 1.2 does not. Let S denote the subspace of the Euclidean plane obtained by deleting the points J on the x-axis with an irrational coordinate. The set of points M on the x-axis with rational first coordinates is a closed and countable set in S, and each point of M is a limit point of M so Theorem 1.2 is not valid in S. On the other hand if $D_1, D_2, D_3, \ldots$ is a countable collection of dense open sets in the space S, then each D_i is $O_i \cap S$ for some open dense set O_i in E^2. Now E^2 is a complete metric space so Theorem 1.1 is valid and $\bigcap_{i>0} O_i$ is dense in E^2. Thus $\bigcap_{i>0} D_i = \bigcap_{i>0} O_i - J$ is dense in S and Theorem 1.1 is valid in S.

References

1. Janice Thomas Astin, A study of Baire-Moore Theorems, *M. S. Thesis, Emory University*, 1966.

2. Colin W. Clark, *Elementary Mathematical Analysis*, 2nd ed., Wadsworth Publishers of Canada, Ltd., Belmont, California, 1982.

3. John L. Kelley, *General Topology*, D. Van Nostrand Company, Inc., New York, 1955.

4. R. L. Moore, *Foundations of Point Set Theory*, Revised ed., American Mathematical Society, Providence, RI 1962.

5. ______________, An Extension of the Theorem that no Countable Point Set is Perfect, *Proceedings of the National Academy of Sciences*, **10** No. 5 (1924) 168–170.