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Interesting Dynamics and Inverse Limits in a Family of One-Dimensional Maps

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1. INTRODUCTION. In the past thirty or forty years inverse limits have been used extensively in dynamical systems as well as in continuum theory as a means to attack a myriad of unsolved problems. A study of one-dimensional branched manifolds by R. F. Williams [**16**] provided an early demonstration of the utility of the inverse limit construction in dynamical systems. In continuum theory, many surprising and complicated examples have been constructed using inverse limits. For example, a continuum admitting only constant maps and the identity as continuous transformations of the continuum into itself was constructed by Howard Cook [**4**] in the 1960s via inverse limits.

The inverse limit technique is particularly useful in that it allows one to build complicated structures out of simpler ones. For instance, in Example 2 of this paper (see Figure 2) we demonstrate how a continuum containing a $\sin(1/x)$-curve results from an inverse limit on the interval $[0, 1]$ using a single simple piecewise-linear continuous function on the interval. Example 3 (see Figure 6) shows that even more complicated continua (indecomposable continua) may show up in inverse limits on intervals with equally simple looking bonding maps. This particular example actually consists of two copies of a continuum naturally associated with the "Smale horseshoe" glued together at a common end-point.

Interest in inverse limits is increased by the realization that through the inverse limit construction one is able to turn the study of a dynamical system consisting of a space and a continuous function on that space into the study of another space with a *homeomorphism* on that space (Theorem 3.9).

After reading with interest a paper by Susan Bassein that appeared in this MONTHLY in 1998 [**2**], we decided that the simple two-parameter family of unimodal maps on the interval $[0,1]$ whose dynamics she discusses would be a nearly perfect family to use for the purpose of introducing the inverse limit construction to a more general audience. We noted a remarkable correlation between the maps in this family that have complicated dynamics and the pathology of the continua that are inverse limits of these maps. Indeed, even the hypotheses of Bassein's theorems that imply an absence of chaos also imply a simple inverse limit. In addition, when the hypotheses of her theorems imply chaos, they often imply that the inverse limit contains an indecomposable continuum. If the chaos is extensive, the inverse limit is likely to be indecomposable.

The well-known Smale horseshoe is described by starting with a topological disk, typically a square disk with two semicircular disks attached at opposite sides (see Figure 3.2 in [**6,** p. 182]), and then "folding" this topological disk inside itself. If this process is continued in a natural way, then the intersection of the resulting disks is an indecomposable continuum. It is arguably the simplest example of an indecomposable continuum. A similar, but plane-separating, continuum was described by Brouwer in 1910. Brouwer's example was simplified to a nonseparating plane continuum by Janiszewski in 1911, and Janiszewski's example was given a nice geometric description by Knaster in 1922 [13, p. 204]. It is not easy to show that the continuum that arises from the Smale horseshoe (or any other continuum) is an indecomposable continuum. A careful look at the history of this continuum presented in the Kuratowski book has prompted some to refer to it as the Brouwer-Jansizewski-Knaster (or BJK) continuum, [**5,** p. 15]. Because it is also important in dynamics we refer to it as a *BJK-horseshoe*.

Our goal in this paper is to provide an elementary introduction to inverse limits. We begin this in section 2 by presenting a sequence of theorems basic to the study of inverse limits, supplying elementary proofs wherever possible. In the remainder of the paper we concentrate on the family of maps investigated by Bassein. In particular, we indicate correlations between the chaotic nature of those maps (as discussed by Bassein) and the complexity of their inverse limits. Each member of this family is a map of $[0,1]$ onto itself whose graph consists of two straight line segments, one having end-points $(0,b)$ and $(a,1)$ and the other joining $(a,1)$ to $(1,0)$, where $0 < a < 1$ and $0 \leq b \leq 1$. The member of the family determined by the two parameters a and b is denoted f_{ab}, while we use $\mathcal{F}$ to denote the entire family. To be more explicit, the members f_{ab} of the family $\mathcal{F}$ are defined for a in $(0,1)$ and b in

$[0,1]$ as follows:

$$f_{ab}(x) = \begin{cases} b + \dfrac{1-b}{a}\,x & \text{if } 0 \le x < a, \\ \dfrac{1-x}{1-a} & \text{if } a \le x \le 1. \end{cases}$$

The continuum produced by the inverse limit on $[0,1]$ using f_{ab} with $a = 1/2$ and $b = 0$ is the BJK-horseshoe.

The family of tent maps has been of particular interest to dynamicists and continuum theorists alike, and inverse limits arising in the tent family continue to attract the attention of researchers to this day. The *tent family* is a one-parameter family $\{T_\lambda \mid 0 < \lambda \le 1\}$ of mappings of $[0,1]$ into itself given by $T_\lambda(x) = 2\lambda x$ for x in $[0,1/2]$ and $T_\lambda(x) = 2\lambda(1-x)$ for x in $[1/2,1]$. For $\lambda > 1/2$, T_λ maps $[T_\lambda(\lambda), \lambda]$ onto itself and $T_\lambda \,|\, [T_\lambda(\lambda), \lambda]$ is called the *core* of T_λ. The cores of the tent maps are conjugate to maps in the two-parameter family $\mathcal{F}$ appearing along the curve $b = (2a-1)/(a-1)$, where $0 < a \le 1/2$.

We have restricted our references to items accessible to a beginner or directly relevant to the paper. The interested reader wishing to delve deeper into the subject will find an extensive bibliography available in the citations listed at the end of this paper. In particular, **[9]** contains 220 references, together with an annotated bibliography of early papers and books dealing with inverse limits that was compiled by Ralph Bennett in the late 1960s.

2. DEFINITIONS, BASIC THEOREMS, EXAMPLES. We hope that this paper will be accessible to an advanced undergraduate. Familiarity with the concepts of continuity, connectedness, and compactness in metric spaces should suffice as background. By a *mapping* (or, for short, a *map*) we mean a continuous function. If $I_1, I_2, I_3, \dots$ is a sequence of subintervals of $[0,1]$ and $\mathbf{f} = (f_1, f_2, f_3, \dots)$ is a sequence of mappings such that $f_i : I_{i+1} \to I_i$, the *inverse limit* of this sequence is the set of points of the Hilbert cube $\mathcal{Q} = [0,1]^\infty$ that contains the point $p = (p_1, p_2, p_3, \cdots)$ if and only if $f_i(p_{i+1}) = p_i$. We endow $\mathcal{Q}$ with a topology by defining a metric on it: the distance between sequences $x = (x_1, x_2, x_3, \cdots)$ and $y = (y_1, y_2, y_3, \cdots)$ is given by

$$d(x,y) = \sum_{i>0} \frac{|x_i - y_i|}{2^i}.$$

(The undergraduate reader may have to assume without proof that $\mathcal{Q}$ is compact and connected in the metric topology.) We denote the inverse limit by $\varprojlim \mathbf{f}$, and we refer to the mappings f_i as *bonding maps.* In most instances of interest here the sequence of functions under consideration is the constant sequence generated by some member of the family $\mathcal{F}$. If g is a mapping from a subinterval $[a, b]$ of the interval $[0, 1]$ into $[a, b]$, then $\varprojlim \mathbf{g}$ is the inverse limit of the constant sequence each term of which is g.

Theorems 2.1 and 2.3 are well known and are readily accessible in many sources on inverse limits, including **[7]**, **[8]**, and **[9]**. We furnish simple proofs both for the sake of completeness and in the interest of providing an introduction to inverse limits.

Theorem 2.1. *If $I_1, I_2, I_3, \ldots$ is a sequence of subintervals of $[0, 1]$ and $\mathbf{f} = (f_1, f_2, f_3, \ldots)$, where f_n is a mapping from I_{n+1} into I_n, then $\varprojlim \mathbf{f}$ is a compact and connected subset of the Hilbert cube $\mathcal{Q}$.*

Proof. For each $n > 1$, let M_n be the set of all points $x = (x_1, x_2, x_3, \ldots)$ in $\mathcal{Q}$ such that $f_i(x_{i+1}) = x_i$ for each $i < n$. It is clear that $\varprojlim \mathbf{f}$ is the intersection of the nested sequence $M_2, M_3, M_4, \ldots$ and is thus nonempty, compact, and connected, provided that each M_n is compact and connected. To see that M_n is closed, let $x^1, x^2, x^3, \ldots$ be a sequence of points of M_n that converges to x in $\mathcal{Q}$. For each i let $x^i = (x^i_1, x^i_2, x^i_3, \ldots)$. If $i < n$, then convergence in $\mathcal{Q}$ implies that the sequence $x^1_{i+1}, x^2_{i+1}, x^3_{i+1}, \ldots$ converges to x_{i+1} and the sequence $x^1_i, x^2_i, x^3_i, \ldots$ to x_i. Because f_i is continuous and $f_i(x^j_{i+1}) = x^j_i$ for each j, $f_i(x_{i+1}) = x_i$ whenever $i < n$, so we see that x belongs to M_n. As a closed subset of $\mathcal{Q}$, M_n is compact. For convenience of notation, denote by f^n_i the composition $f_i \circ f_{i+1} \circ \cdots \circ f_{n-1}$. Since M_n is the image of $\mathcal{Q}$ under the mapping that takes the point $(x_1, x_2, x_3, \ldots)$ of $\mathcal{Q}$ to $(f^n_1(x_n), f^n_2(x_n), \cdots, f^n_{n-1}(x_n), x_n, x_{n+1}, \ldots)$, it follows that M_n is connected. ■

A *continuum* is a nondegenerate (i.e., it contains at least two points), compact, and connected subset of a metric space. The reader should be aware that the definition of continuum in most topology books does not require that continua be nondegenerate; we adopted this terminology here to avoid having to make extensive use of the word nondegenerate throughout this paper. Theorem 2.1 describes conditions under which the inverse limit of a sequence of mappings is either a continuum or a point. Throughout this paper we make frequent use of the following standard theorem **[15,** Theorem

26.6, p. 167]: *if $f : X \to Y$ is a one-to-one and continuous transformation of a compact space X onto a Hausdorff space Y, then f is a homeomorphism.*

The next theorem provides a simplification of the description of the topology of an inverse limit space and is frequently used in the study of inverse limits. In the remainder of this paper we denote by π_i the projection of $\mathcal{Q}$ onto its ith factor space, i.e., π_i is given by $\pi_i(x_1, x_2, x_3, \dots) = x_i$.

Theorem 2.2. *Let $\mathbf{f}$ be a sequence of mappings of a subinterval $[a, b]$ of $[0, 1]$ into itself, and let $\mathcal{B}$ denote the collection of subsets of $\mathcal{Q}$ defined as follows: the set O belongs to $\mathcal{B}$ if and only if there exist a positive integer i and an open subset U of $[0, 1]$ such that $O = (\varprojlim \mathbf{f}) \cap \pi_i^{-1}(U)$. Then $\mathcal{B}$ is a basis for the relative topology of $\varprojlim \mathbf{f}$ as a subspace of $\mathcal{Q}$.*

Proof. Let $x = (x_1, x_2, x_3, \dots)$ be a point of $M = \varprojlim \mathbf{f}$, and let O be a basic open set in $\mathcal{Q}$ containing x. There is a positive integer n and a finite sequence $U_1, U_2, U_3, \cdots, U_n$ of open sets in $[0, 1]$ such that x_i lies in U_i for $1 \le i \le n$ and $O = U_1 \times U_2 \times \cdots U_n \times \mathcal{Q}$. To see that $\mathcal{B}$ is a basis for the subspace M, we need to note only that for $1 \le i < n$ there is an open subset V_i of $[0, 1]$ such that x_n lies in V_i and $f_i^n(V_i)$ is contained in U_i. It follows that

$$x \in \pi_n^{-1}(V_1 \cap V_2 \cap \cdots \cap V_{n-1} \cap U_n) \cap M \subseteq O \cap M. \quad \blacksquare$$

If f is a mapping of $[0, 1]$ into itself, then the composition $f \circ f = f^2$ also maps $[0, 1]$ into itself. The reader should recall that $\varprojlim \mathbf{f^2}$ is simply the inverse limit of the constant sequence each term of which is f^2. The next theorem allows us to investigate $\varprojlim \mathbf{f^2}$ instead of $\varprojlim \mathbf{f}$, since the two are homeomorphic.

Theorem 2.3. *If f is a mapping of $[0, 1]$ into itself, then $M_1 = \varprojlim \mathbf{f}$ is homeomorphic to $M_2 = \varprojlim \mathbf{f^2}$.*

Proof. It is easy to see that $x' = (x_1, x_3, x_5, \dots)$ is a point of M_2 whenever $x = (x_1, x_2, x_3, \dots)$ is a point of M_1. Define $h : M_1 \to M_2$ by $h(x) = x'$. If $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ are different points of M_1, there is a positive integer k such that $x_k \neq y_k$. If k is odd, then plainly $h(x) \neq h(y)$. If k is even, then by the definition of M_1 $f(x_{k+1}) = x_k$ and $f(y_{k+1}) = y_k$. Since $x_k \neq y_k$, $x_{k+1} \neq y_{k+1}$. Accordingly, $h(x) \neq h(y)$ in this case as well, so h is one-to-one. To show that h is continuous, let

$x = (x_1, x_2, x_3, \dots)$ be a point of M_1, and let B denote a basic open set in M_2 containing $x' = (x_1, x_3, x_5, \dots)$. There exist a positive integer n and an open set V in $[0,1]$ containing $x'_n = x_{2n-1}$ such that $B = \pi_n^{-1}(V) \cap M_2$. Since f is continuous and $f(x_{2n}) = x_{2n-1}$, there is an open set U in $[0,1]$ containing x_{2n} such that $f(U)$ is a subset of V. If $R = \pi_{2n}^{-1}(U) \cap M$, then R is a basic open set in M_1 and $h(R)$ is a subset of B. Thus h is continuous and therefore a homeomorphism of M_1 onto its range. It is easy to see from the definition of M_2 that if $x' = (x'_1, x'_2, x'_3, \dots)$ is in M_2, and $x = (x'_1, f(x'_2), x'_2, f(x'_3), x'_3, \dots)$, then x is in M_1 and $h(x)$ is x'. We conclude that h maps M_1 onto M_2, making these spaces homeomorphic. ■

Next we present three examples. Each of these is an inverse limit on $[0,1]$ using a single bonding map $f = f_{ab}$ chosen from the family $\mathcal{F}$. We fix $a = 1/2$ and describe $\varprojlim \mathbf{f}$ for different values of b.

Example 1. For our first example we choose $a = 1/2$ and $b = 1$. In this case $\varprojlim \mathbf{f}$ is homeomorphic to the interval $[0,1]$ (i.e., the inverse limit is an arc). In Figure 1 we provide side-by-side graphs of f and f^2 for the convenience of the reader.

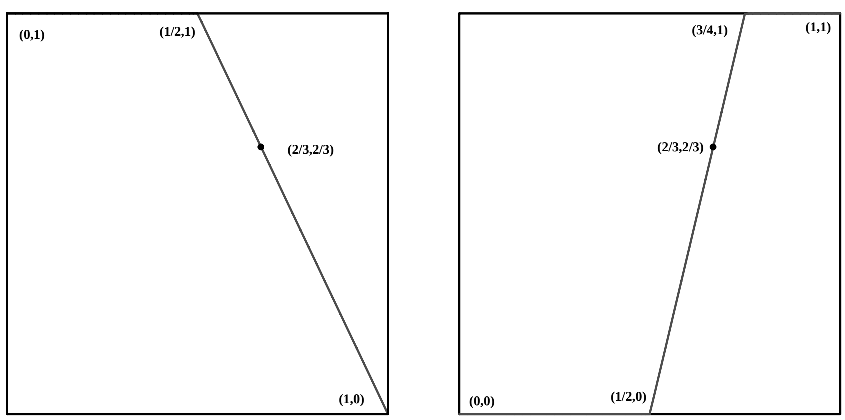

Figure 1.

To see this, we actually show that $M = \varprojlim \mathbf{f^2}$ is homeomorphic to $[0,1]$ and then appeal to Theorem 2.3. To this end, let x be a point of M and define $h(x) = d(x, (0,0,0,\dots))$. Thus h is a function from M into $[0,1]$. The distance function on a metric space is continuous, so h is continuous. If x and y are points of M and $x \neq y$, there is a positive integer i such that $\pi_i(x) \neq \pi_i(y)$. We may assume that $\pi_i(x) < \pi_i(y)$. Since f is nonincreasing,

f^2 is nondecreasing. It follows from the definition of M that $\pi_j(x) \le \pi_j(y)$ for all positive integers j. Thus, $d(x,(0,0,0,\cdots)) < d(y,(0,0,0,\cdots))$. Therefore h is one-to-one and so is a homeomorphism. The points $x = (0,0,0,\dots)$ and $y = (1,1,1,\dots)$ are points of M since $f^2(0) = 0$ and $f^2(1) = 1$. It follows that $h(x) = 0$ and $h(y) = 1$, so $h(M) = [0,1]$.

Next we provide a second proof that $\varprojlim \mathbf{f}$ is an arc, illustrating a technique that is frequently used in the study of inverse limits. The proof relies on the characterization of an arc in Theorem 2.4, which follows from [**8,** Theorem 1-18, p. 49] and [**8,** Theorem 2-27, p. 54]. We omit its proof because it bears little relevance to our study of chaotic continua. A point P of a continuum M is a *separating point* if $M - P$ is not connected, while P is a *nonseparating point* if $M - P$ is connected. Separating and nonseparating points are sometimes called *cut points* and *noncut points*, respectively.

Theorem 2.4. *If M is a continuum that contains at most two nonseparating points, then M is an arc.*

Returning to Example 1, let $p = (p_1, p_2, p_3, \dots)$ denote a point of $\varprojlim \mathbf{f}$ different from $x = (0,1,0,1,0,1,\dots)$ and $y = (1,0,1,0,1,0,\dots)$, each of which is a point of M. There is a positive integer n such that p_n is not in $\{0,1\}$. Since $f^{-1}(p_n)$ consists of a single point of $(1/2, 1)$, we have $\pi_n^{-1}(p_n) \cap M = \{p\}$. If $H = \pi_n^{-1}([0, p_n))$ and $K = \pi_n^{-1}((p_n, 1])$, then H and K are mutually exclusive basic open sets in $\mathcal{Q}$ whose union contains $M - \{p\}$. It follows that $M - \{p\}$ is not connected, so p is a separating point of M. Theorem 2.4 shows that M is an arc.

By a *topological ray* we mean a nondegenerate, locally compact, connected point set containing a single nonseparating point. The nonseparating point is referred to as the *end-point* of the ray. If R is a topological ray such that $\overline{R}$ is compact, $\overline{R} - R$ is its *remainder.*

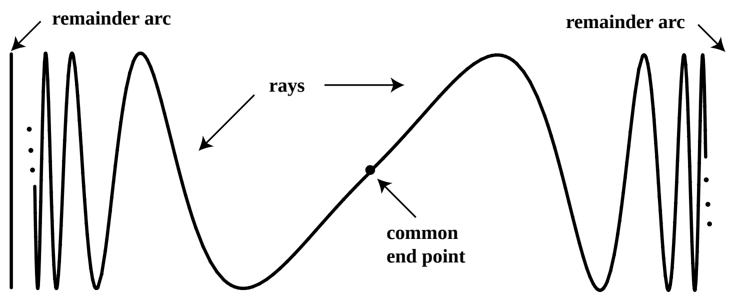

Figure 2.

Example 2. For this example we choose $a = 1/2$ and $b = 3/4$. In this case the inverse limit $\varprojlim \mathbf{f}$ is the union of the closures of two topological rays that intersect only at a common end-point and that have arcs as remainders. Figure 2 depicts this inverse limit. For graphs of f and f^2 see Figure 3.

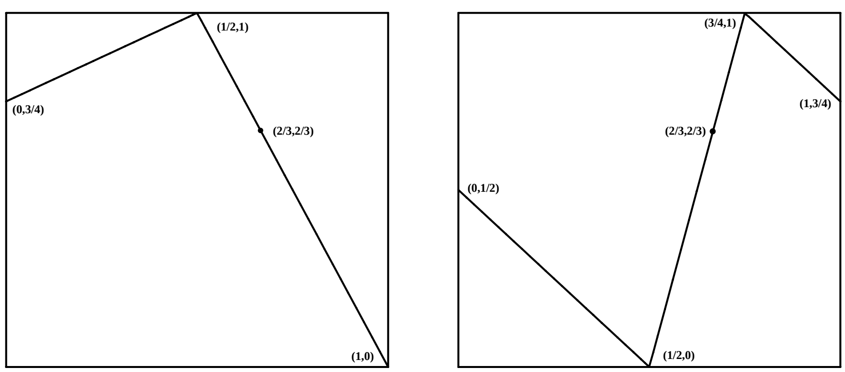

Figure 3.

We consider $M = \varprojlim \mathbf{f^2}$ and begin by identifying two subcontinua and H and K of M. Let $H = \varprojlim \mathbf{h}$ for $h = f^2 \,|\, [0, 2/3]$ and $K = \varprojlim \mathbf{k}$ for $k = f^2 \,|\, [2/3, 1]$. Note that $M = H \cup K$ and, since $2/3$ is a fixed point for f^2, that $H \cap K = \{(2/3, 2/3, 2/3, \dots)\}$. We show that there exist a topological ray R and an arc A such that A is the remainder arc of R and $H = R \cup A$. We then complete our analysis of M by showing that H and K are homeomorphic.

For each positive integer n, let A_n denote the set of all points $(x_1, x_2, x_3, \dots)$ of H such that x_i belongs to $[1/2, 2/3]$ if $i \geq n$. The set A_n is an arc, for π_n is a homeomorphism from A_n onto $[1/2, 2/3]$. Let $R = A_1 \cup A_2 \cup A_3 \cdots$. Since A_i is a proper subset of A_{i+1} for each i, R is connected and locally compact and, thus, a topological ray (whose end-point is $(2/3, 2/3, 2/3, \dots)$).

Note that $g = f^2 \mid [0, 1/2]$ is a homeomorphism. We show that $A = \varprojlim \mathbf{g}$ is an arc by observing that g^2 is the identity and applying Theorem 2.3. It is easy to see that $\pi((x, x, x, \cdots)) = x$ defines a one-to-one continuous function from $A' = \varprojlim \mathbf{g^2}$ onto $[0, 1/2]$, whence A' is an arc. Theorem 2.3 ensures that A is homeomorphic to A' so A is an arc. Moreover, every point of A is a limit point of $H - A$. To see this, suppose that $x = (x_1, x_2, x_3, \dots)$ is a point of A and that B is a basic open set in H containing x. There exist an index n and an open set U in $[0, 2/3)$ such that $B = \pi_n^{-1}(U) \cap H$ and x_n belongs to U. Note that there is a point w in $[1/2, 2/3)$ such that $f^2(w) = x_n$. Then, because $f^2 \mid [1/2, 2/3]$ is a homeomorphism, there is a point $y = (y_1, y_2, y_3, \dots)$ in H such that $y_{n+1} = w$ and y_i lies in $(1/2, 2/3]$ if $i > n + 1$. Since $y_n = x_n$, y is a point of B but y is not in A. Thus, A is a subset of $\overline{R}$. Since the points of H that do not belong to R are points of A and $H = R \cup A$, $\overline{R} - R = A$.

Furthermore, H and K are homeomorphic. To see this, define $\hat{f} : H \to K$ by $\hat{f}((x_1, x_2, x_3, \dots)) = (f(x_1), f(x_2), f(x_3), \dots)$. If x and y are points of H and $x \neq y$, then there is a positive integer i such that $x_j \neq y_j$ for each $j \geq i$. If there is an integer $j > i$ such that $f(x_j) = f(y_j)$, then $x_i = y_i$, so it follows that $\hat{f}$ is one-to-one. That $\hat{f}$ is continuous follows from the fact that f is continuous. Therefore $\hat{f}$ is a homeomorphism. Note that f maps $[0, 2/3]$ onto $[2/3, 1]$ (see Figure 3). If $x' = (x'_1, x'_2, x'_3, \dots) = (f^2(x'_2), f^2(x'_3), f^2(x'_4), \dots)$ is in K, then $x = (f(x'_2), f(x'_3), f(x'_4), \dots)$ is in H and $\hat{f}(x) = x'$. We conclude that $\hat{f}$ is surjective and that H and K are homeomorphic.

Included in Bassein's paper are some "web" diagrams. These are similar to the graphical analysis pictures included by Devaney in his book *An Introduction to Chaotic Dynamical Systems* [**6,** p. 21]. These same pictures can be viewed "in reverse" to see points in the inverse limit. We have included such a picture in Figure 4. Following Devaney's convention, we identify the interval $[0, 1]$ with the diagonal Δ in the obvious way. To get a point of the inverse limit with a specific first coordinate x start from the point (x, x) on Δ and move *horizontally* to a point on the graph of the bonding map f. In most cases, there are several points of the graph on this horizontal line, each of which will produce a point of the inverse limit if f is surjective. Choose one of these points on the graph. Its first coordinate is the second coordinate of the point of the inverse limit under construction. Next, travel vertically to the diagonal and then horizontally to a point of the graph, if possible. If f is surjective, this is always possible; if f is not surjective, the process may terminate at this stage. If the process does not terminate, the first coordinate of this point will be the third coordinate of the point of the inverse limit under construction. Continuing this process will generate a point of the desired inverse limit, unless it terminates at some stage. In Figure 4, we show how three "typical" points p, q, and r of A_3 from Example 2 are obtained in this manner. (The graph shown in Figure 4 is that of $f = f_{ab}^2$ for $a = 1/2$ and $b = 3/4$.) The first coordinates of these points are x, y, and z, respectively. For a point to belong to A_3, it is important that its *third* coordinate be in $[1/2, 2/3]$. The third coordinates of p and q and the second coordinate of r are in $[1/2, 2/3]$. Of course, A_3 contains points *all* of whose coordinates are in $[1/2, 2/3]$. Notice, that once a coordinate of a point is in $(1/2, 2/3]$, all of the remaining coordinates are completely determined and are also in this interval.

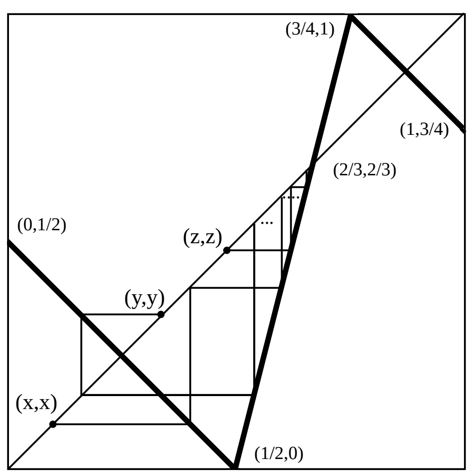

Figure 4.

An *indecomposable continuum* is a continuum that is not the union of two proper subcontinua. To the reader encountering this concept for the first time: it is not at all evident that indecomposable continua exist nor, as indicated in the introduction, is it a simple matter to establish their existence. However, as we see in Example 3, indecomposable continua arise in inverse limits on $[0,1]$ using a single bonding map chosen from the family $\mathcal{F}$.

Example 3. Here we consider $f = f_{ab}$ with $a = 1/2$ and $b = 2/3$. In this case, the inverse limit $\varprojlim \mathbf{f}$ is the union of two indecomposable continua $M = \varprojlim \mathbf{g}$ and $N = \varprojlim \mathbf{h}$, where $g = f^2 \,|\, [0, 2/3]$ and $h = f^2 \,|\, [2/3, 1]$ (see Figure 5 for the graphs of f and f^2 and Figure 6 for the inverse limit).

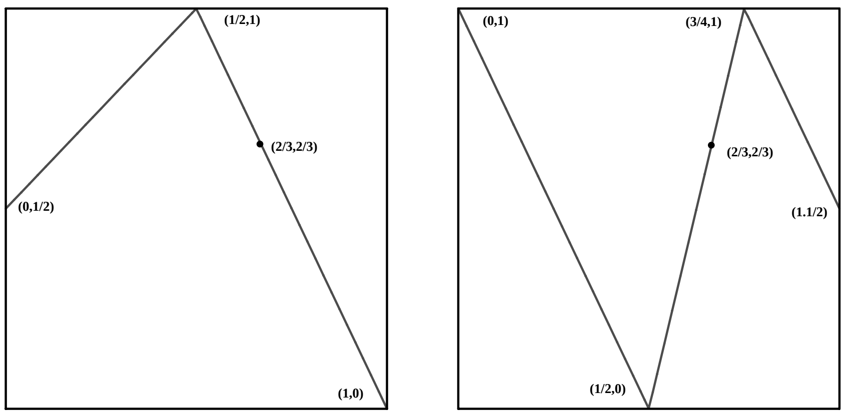

Figure 5.

We proceed as in the previous example and consider only $M = \varprojlim \mathbf{g}$. A homeomorphism from M onto N is given as in Example 2 by $\hat{f}((x_1, x_2, x_3, \dots)) = (f(x_1), f(x_2), f(x_3), \dots)$. Thus N is indecomposable if M is. The continuum M is not the union of two proper subcontinua. For if $M = H \cup K$, where H and K are proper subcontinua of M, then there is a positive integer j such that $\pi_n(H) \neq [0, 2/3]$ and $\pi_n(K) \neq [0, 2/3]$ if $n \geq j$. If, for instance, $\pi_n(H) = [0, 2/3]$ for every n, then H is M. However, $\pi_{j+1}(H)$ and $\pi_{j+1}(K)$ are intervals whose union is $[0, 2/3]$. Assume that 0 is in $\pi_{j+1}(H)$ and $2/3$ is in $\pi_{j+1}(K)$. The point $1/2$ must lie in either $\pi_{j+1}(H)$ or $\pi_{j+1}(K)$. Now $g(1/2) = 0$ and $g(2/3) = g(0) = 2/3$. Thus if $1/2$ is in $\pi_{j+1}(H)$, then $\pi_j(H) = g[\pi_{j+1}(H)] = [0, 2/3]$; if $1/2$ is in $\pi_{j+1}(K)$, then $\pi_j(K) = g[\pi_{j+1}(K)] = [0, 2/3]$. Either possibility is contrary to the choice of j.

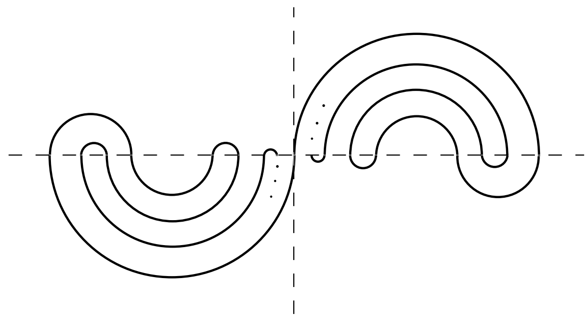

Figure 6.

Remark. In Figure 6 we have begun representing two one-to-one continuous images of a ray having a common endpoint such that the closure of each of them is a BJK-horseshoe. In constructing the image in the right half-plane, the points on the dashed horizontal line form the endpoints of the Cantor ternary set if the rightmost crossing is at the point $(1,0)$. The BJK-horseshoe on the right would thus include the Cantor set. It does not include any other points on the dashed line. The continuum in this figure results from any function from the family $\mathcal{F}$ with $b=0$.

3. AN INVERSE LIMIT TOOLKIT. In this section, we introduce the theorems we need in section 4 to address the connections between the topology of the inverse limits and the dynamics of the family $\mathcal{F}$. We have already seen special instances of these theorems in developing Examples 1, 2, and 3. Theorem 2.3 is a special case of the following theorem. A proof of Theorem 3.1 is left to the reader, since it is similar to the one that we gave for Theorem 2.3.

Theorem 3.1. *Suppose that* $M = \varprojlim \mathbf{f}$, *where* $\mathbf{f} = (f_1, f_2, f_3, \dots)$ *is a sequence of mappings of a subinterval* $[a,b]$ *of* $[0,1]$ *onto itself, and that for* $j > i$ f_i^j *denotes the composition* $f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$. *If* $n_1, n_2, n_3, \dots$ *is an increasing sequence of positive integers, then* M *is homeomorphic to* $\varprojlim \mathbf{g}$, *where* $\mathbf{g} = (f_{n_1}^{n_2}, f_{n_2}^{n_3}, f_{n_3}^{n_4}, \dots)$.

Of particular interest is the direct analogue of Theorem 2.3.

Corollary 3.2. *If* $M = \varprojlim \mathbf{f}$, *where* f *is a mapping of a subinterval* $[a,b]$ *of* $[0,1]$ *onto itself and* n *is a positive integer, then* $\varprojlim \mathbf{f^n}$ *is homeomorphic to* $\varprojlim \mathbf{f}$.

If $f : X \to Y$ is a mapping between topological spaces X and Y and A is a subset of X, we denote by $f(A)$ the image of A under f, i.e., $f(A) = \{f(x) \mid x \in A\}$. Thus a mapping $f : X \to Y$ is surjective if $f(X) = Y$. When $f : X \to Y$ is surjective, we often write $f : X \twoheadrightarrow Y$.

Theorem 3.3. *If I is an interval and $f : I \to I$ is not surjective, then there is a subinterval J of I, possibly degenerate, such that $g = f \mid J$ maps J onto itself and has the property that $\varprojlim \mathbf{f} = \varprojlim \mathbf{g}$.*

Proof. The conclusion holds for $J = \bigcap_{i>0} f^i(I)$, where f^i denotes the i-fold composition of f with itself. ■

The next result can be found in [**9,** Theorem 2.16, p. 20]. We include a proof for the sake of completeness.

Theorem 3.4 (Bennett). *Suppose that f is a mapping of the interval $[a, b]$ onto itself and d is a number between a and b (i.e., $a < d < b$) such that (1) $f([d, b])$ is a subset of $[d, b]$, (2) $f \mid [a, d]$ is increasing, and (3) there is a positive integer j such that $f^j[a, d] = [a, b]$. Then, if $g = f \mid [d, b]$, $\varprojlim \mathbf{f}$ is the union of the continuum $K = \varprojlim \mathbf{g}$ and a topological ray R such that $\overline{R} - R = K$.*

Proof. Because f is increasing on $[a, d]$ and $f^j([a, d]) = [a, b]$, it follows that $f(d) > d$. Hence there is only one number c in $[a, d)$ such that $f(c) = d$. For each $n > 0$, let $A_n = \pi_{n+1}^{-1}([a, c]) \cap \varprojlim \mathbf{f}$. The projection π_{n+1} is continuous and one-to-one on A_n since f is increasing on $[a, d]$. Thus π_{n+1} is a homeomorphism of A_n onto $[a, c]$, so A_n is an arc having $\pi_{n+1}^{-1}(a) \cap A_n = (a, a, a, \cdots)$ as one end-point. Also A_n is a proper subset of A_{n+1}, for $\pi_{n+2}^{-1}(c) \cap A_{n+1}$ is a point of A_{n+1} but not of A_n since $\pi_{n+1}(\pi_{n+2}^{-1}(c)) = d$. It follows that $\cup_{n=1}^{\infty} A_n$ is a topological ray. Now if $K = \varprojlim \mathbf{g}$ and x belongs to $R = \varprojlim \mathbf{f} - K$, then there is an integer n such that $\pi_n(x) < d$. This implies that $\pi_{n+1}(x)$ lies in $[a, c]$, whence x is a point of A_n. Thus $R = \bigcup_{n=1}^{\infty} A_n$.

It remains to show that each point of K is a limit point of R. To this end, we consider $\varprojlim \mathbf{f^j}$, which by Theorem 3.1 is homeomorphic to $\varprojlim \mathbf{f}$. The natural homeomorphism between these inverse limits takes K onto $K' = \varprojlim \mathbf{g^j}$, so it suffices to show that each point of K' is a limit point of $\varprojlim \mathbf{f^j} - K'$. Let x belong to K' and let $O = \pi_n^{-1}(u, v)$ be a basic open set containing

x. Because $f^j([a,d]) = [a,b]$ there is a point y in O such that $\pi_n(y) = x_n(x)$ and $\pi_{n+1}(y) < d$. It follows that y is not a member of K'. We infer that x is a limit point of $\varprojlim \mathbf{f^j} - K'$. ■

It is a significant technical fact that closed sets in an inverse limit are the inverse limits of their projections. For inverse limits on intervals, we have:

Theorem 3.5. *If $[a,b]$ is a subinterval of $[0,1]$, $\mathbf{f} = (f_1, f_2, f_3, \dots)$ is a sequence of mappings of $[a,b]$ onto $[a,b]$, and K is a closed subset of $\varprojlim \mathbf{f}$, then $K = \varprojlim \mathbf{g}$, where $\mathbf{g} = (f_1 \,|\, \pi_2(K), f_2 \,|\, \pi_3(K), f_3 \,|\, \pi_4(K), \dots)$.*

Proof. Let $L = \varprojlim \mathbf{g}$. Clearly K is a subset of L. Assume that there is a point $x = (x_1, x_2, x_3, \dots)$ in L that is not in K. As K is closed there exist an integer n and an open set U in $[a,b]$ containing x_n such that the basic open set $(\varprojlim \mathbf{f}) \cap \pi_n^{-1}(U)$ contains no point of K, contradicting the fact that x_n is in $\pi_n(K)$. ■

Corollary 3.6 . *Let $\mathbf{f} = (f_1, f_2, f_3 \dots)$ be a sequence of mappings of a subinterval $[a,b]$ of $[0,1]$ onto itself. If $M = \varprojlim \mathbf{f}$ and H is a subcontinuum of M with the property that $\pi_n(H) = [a,b]$ for each positive integer n, then $H = M$.*

The next theorem provides a simple criterion for determining whether an inverse limit is indecomposable.

Theorem 3.7. *Let f be a mapping from $[0,1]$ onto itself. If there exist numbers x, y, and z satisfying $0 \le x < y < z \le 1$ such that either $f(x) = f(z) = 0$ and $f(y) = 1$ or $f(x) = f(z) = 1$ and $f(y) = 0$, then $\varprojlim \mathbf{f}$ is an indecomposable continuum.*

$f(y) = 0$, *Proof.* The proof is similar to that given in Example 3. Suppose that $M = H \cup K$, where H and K are proper subcontinua of M. It follows from Theorem 3.5 that there is a positive integer N such that $\pi_i(H) \neq [0,1]$ and $\pi_i(K) \neq [0,1]$ whenever $i \ge N$. Since $\pi_i(H) \cup \pi_i(K) = [0,1]$ for each i, one of $[x,y]$ and $[y,z]$ is a subset of either $\pi_{N+1}(H)$ or $\pi_{N+1}(K)$. However, this leads to a contradiction, for $f[x,y] = f[y,z] = [0,1]$. ■

The concept of conjugacy is important in both dynamics and the study of inverse limits. Devaney says **[6,** p. 47]: "Mappings which are topologically conjugate are completely equivalent in terms of their dynamics." If f is a mapping from A into A and g is a mapping from B into B, then f and g are said to be *topologically conjugate* if there is a homeomorphism h from A into B such that $h \circ f = g \circ h$. The relevance of this concept to inverse limits in the case where A and B are subintervals of $[0,1]$ is shown by:

Theorem 3.8. *Let $[a,b]$ and $[c,d]$ be subintervals of $[0,1]$. If $f : [a,b] \twoheadrightarrow [a,b]$ and $g : [c,d] \twoheadrightarrow [c,d]$ are topologically conjugate mappings, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$.*

For a proof we refer the reader to Corollary 4a on page 159 of the first volume of Kuratowski's book *Topology* [**13**].

We close this section with a word about the *shift homeomorphism*. We actually made use of the shift homeomorphism in Examples 2 and 3 without explicitly defining it there.

Theorem 3.9. *If $f : [0,1] \twoheadrightarrow [0,1]$ is a mapping, then the function $\hat{f}$ given by $\hat{f}(x_1, x_2, x_3, \dots) = (f(x_1), x_1, x_2, \dots) = (f(x_1), f(x_2), f(x_3), \dots)$ is a homeomorphism of $\varprojlim \mathbf{f}$ onto $\varprojlim \mathbf{f}$.*

Proof. It is straightforward to see that $\hat{f}$ is one-to-one and surjective and that the preimage of a basic open set under $\hat{f}$ is a basic open set, so $\hat{f}$ is continuous. Accordingly, $\hat{f}$ is a homeomorphism. ■

The statement of Theorem 3.9 remains true if the interval $[0,1]$ is replaced by an arbitrary compact metric space. One of the reasons the inverse limit construction is of such interest to dynamicists is the fact that passing to an inverse limit changes the study of a dynamical system (a space and a self-mapping of it) based on a compactum into the study of a homeomorphism.

4. DYNAMICAL PROPERTIES AND INVERSE LIMITS. Suppose f is a mapping of an interval I into itself. A point x of I is called *periodic* provided $f^n(x) = x$ for some positive integer n. If S is an infinite subset of I and $f(S) \subset S$, f is *chaotic on S* provided

(1) f is *topologically transitive on S*; i.e., for each two open intervals U and V intersecting S there is a positive integer n such that $f^n(U \cap S)$ intersects V and

(2) if U is an open interval intersecting S then $U \cap S$ contains a periodic point for f.

We now turn our attention to the two-parameter family $\mathcal{F}$. In order that the reader be able to see that we have addressed all of parameter space, we provide Figure 7, which depicts this space and the two curves $C_1 : b = 1 - a + a^2$ and $C_2 : b = 1/(2 - a)$ that arise naturally to subdivide it.

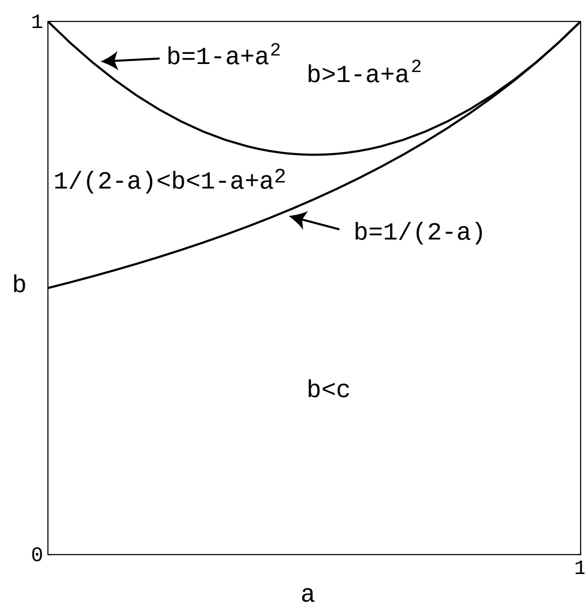

Figure 7.

Non-chaotic dynamics: $\mathbf{b > 1 - a + a^2}$. Bassein begins her discussion of the dynamics of maps in $\mathcal{F}$ with the case $b > 1 - a + a^2$ by showing that in this case $f = f_{ab}$ is not chaotic on any infinite set. The next theorem shows that under the same hypothesis $\varprojlim \mathbf{f}$ is the simplest continuum, an arc.

Theorem 4.1. *If $f = f_{ab}$ with $b > 1 - a + a^2$, then $\varprojlim \mathbf{f}$ is an arc.*

Proof. The fact that $\varprojlim \mathbf{f}$ is an arc is Theorem 1 of [**10**], but we provide a complete and different proof. Bassein shows that f is not chaotic on any infinite set by making use of 2-cycles. We use the composite function $g = f^2$ and show that $M = \varprojlim \mathbf{g}$ is an arc. Theorem 2.3 then ensures that $\varprojlim \mathbf{f}$ is an arc.

To see that M is an arc, let c denote the fixed point of f. There are two fixed points of g different from c, c_1 in $[0, a)$ and c_2 in $(1-a+a^2, 1]$. We demonstrate that each point of M other than $(c_1, c_1, c_1, \dots)$ and $(c_2, c_2, c_2, \dots)$ separates M. Let $x = (x_1, x_2, x_3, \dots)$ be a point of M, and assume that x_1 is less than c and x_1 is not c_1. Bassein observes that c_1 attracts every point in $[0, c)$ under f^2 (i.e., $\lim_{n\to\infty} g^n(t)$ is c_1 for every number t in $[0, c)$). For similar reasons there is a positive integer n such that $x_n > a_1$, where $a_1 = g(0)$. There is only one point p of M (namely, $p = x$) such that $\pi_n(p) = x_n$. It follows that $M - \{p\}$ is the union of the two mutually exclusive basic open sets $\pi_n^{-1}[0, x_n) \cap M$ and $\pi_n^{-1}(x_n, 1] \cap M$. Therefore x is a separating point of M. In view of Theorem 2.4, M is an arc. ■

In the region of parameter space where $b > 1 - a + a^2$, Bassein shows that the orbit of every point except the fixed point of f is attracted to the 2-cycle of f. This implies that f has only one fixed point, a unique period 2 orbit, and no other periodic orbits. Except in the case $b = 1$ (where one can show directly that $\varprojlim \mathbf{f}$ is an arc, as was done in Example 1), the following theorem of Block and Schumann **[3]** provides a third proof that $\varprojlim \mathbf{f}$ is an arc when $b > 1 - a + a^2$. By a *unimodal map* is meant a map on $[0, 1]$ such that for some number c satisfying $0 < c < 1$ either f is increasing on $[0, c]$ and decreasing on $[c, 1]$ or f is decreasing on $[0, c]$ and increasing on $[c, 1]$.

Theorem 4.2 (Block and Schumann). *Let f be a unimodal map of $[0, 1]$ into $[0, 1]$. If f has only one fixed point c, has a period 2 orbit, and has no other periodic points, then $\varprojlim \mathbf{f}$ is an arc.*

Theorem 4.2 is much more general than is appropriate to discuss in this paper. But for maps of the type considered by Bassein this theorem can be established by an argument similar to our proof of Theorem 4.1. In that proof $f^2 = g$ and there is a point c_1 that attracts each point of $[0, c)$ under iteration of g. To prove that $\varprojlim \mathbf{g}$ is an arc, one needs to verify the following: if c is the fixed point of f and x is a point of $\varprojlim \mathbf{g}$ such that $\pi_1(x) < c$, then either $\pi_1(x) = c_1$ or $\pi_n(x) > g(0)$ for some n, in which event the sequence $\{\pi_i(x)\}_{i=1}^{\infty}$ converges to c_1. This follows easily from the fact that f is piecewise linear.

The boundary between chaotic and nonchaotic dynamics: $\mathbf{b = 1 - a + a^2}$. Here, as in the case $b > 1 - a + a^2$, f is not chaotic on any infinite set but in this case, the 2-cycle is not attracting. In fact, Bassein points out that each point in $[0, a]$ except the 2-cycle has period 4. The points in parameter space where $b = 1 - a + a^2$ form the boundary separating the parameters for which f is chaotic from those for which it is not. Similarly, it is the boundary between the regions where the inverse limit contains an indecomposable continuum and where it does not. For (a, b) on this boundary the inverse limit is no longer locally connected, but it is still a relatively simple continuum. It is the union of two continua each of which is homeomorphic to the closure of the graph of $y = \sin(1/x)$ for $0 < x \leq 1$, and it is also homeomorphic to the continuum described in Example 2.

Theorem 4.3. *If $f = f_{ab}$ with $b = 1 - a + a^2$, then $\varprojlim \mathbf{f}$ is the closure of the union of two topological rays R_1 and R_2 with a common end-point, and with remainders $\overline{R}_1 - R_1$ and $\overline{R}_2 - R_2$ that are arcs.*

Proof. Let $M = \varprojlim \mathbf{f^2}$. Since $f(0) = b = 1 - a + a^2 = f^{-1}(a)$, we have $f^2(0) = a$ and $f^2(a) = 0$. Now $g = f^2 \,|\, [0, a]$ is a homeomorphism of $[0, a]$, and $f^2 \,|\, [b, 1]$ is a homeomorphism of $[b, 1]$. We note that g is linear on $[0, a]$ with slope -1, so g^2 is the identity and $\varprojlim \mathbf{g}$ is an arc by Theorem 2.3. The proof continues along the lines of Example 2 by showing that $\varprojlim \mathbf{g}$ is the remainder arc of a ray lying in M. In fact, it is the remainder of the ray that is the union of the sequence of arcs $A_1, A_2, A_3, \ldots$, where for each positive integer n A_n is the set of all points $(x_1, x_2, x_3, \ldots)$ of M such that x_i belongs to $[0, 1/(2-a)]$ when $i < n$ and to $[a, 1/(2-a)]$ when $i \geq n$. Again, as in Example 2, if $h = f^2 \,|\, [0, c]$ and $k = f^2 \,|\, [c, 1]$ for $c = 1/(2-a)$, then $H = \varprojlim \mathbf{h}$ is homeomorphic to $K = \varprojlim \mathbf{k}$, $M = H \cup K$, and $H \cap K = \{(c, c, c, \ldots)\}$. ∎

Chaotic dynamics: b < c. Recall that $c = 1/(2-a)$ is the fixed point of f. If $b < c$, Bassein shows that f is chaotic on the closure of $S_0 = \{x \mid f^n(x) = c \text{ } for\ some\ n > 0\}$. In terms of inverse limits, S_0 is the set of all coordinates of points in $\varprojlim \mathbf{f}$ whose first coordinate is c. In her proof she uses the following theorem [**2,** p. 122]:

Theorem 4.4 (Bassein). *If $f = f_{ab}$ with $b < c = 1/(2-a)$. and I is a subinterval of $[0, 1]$ containing c, then there is a positive integer n such that $f^n(I) = [0, 1]$.*

We show that the conclusion of Theorem 4.4 implies indecomposability in the inverse limit.

Theorem 4.5. *Let $f = f_{ab}$ with $b < c = 1/(2-a)$. If for each subinterval I of $[0,1]$ containing c there is a positive integer n such that $f^n(I) = [0,1]$, then $\varprojlim \mathbf{f}$ is an indecomposable continuum.*

Proof: Let x and y be numbers with $x < c < y$. Then there is an integer n such that $f^n([x,c]) = f^n([c,y]) = [0,1]$. It follows from Theorem 3.7 that $\varprojlim \mathbf{f^n}$ is an indecomposable continuum and from Theorem 3.2 that $\varprojlim \mathbf{f}$ is an indecomposable continuum. ■

Chaos on a subinterval: $\mathbf{c \le b < 1 - a + a^2}$. Let

$$W = \{(a,b) \mid 0 < a < 1,\ c \le b < 1 - a + a^2\},$$

where as usual $c = 1/(2-a)$. We show for $f = f_{ab}$ with (a,b) in W that the inverse limit $\varprojlim \mathbf{f}$ is not indecomposable but that it *contains* an indecomposable continuum. First, we state and prove a couple of theorems that give general information about the nature of the inverse limits when $b \ge c$.

Theorem 4.6. *If $f = f_{ab}$ with $b \ge c = 1/(2-a)$ and $M = \varprojlim \mathbf{f}$, then M is the union of two homeomorphic continua that intersect in a single point.*

Proof. Let $I_j = [0,c]$ if j is odd and $I_j = [c,1]$ if j is even. For each positive integer i, let $h_i = f \,|\, I_{j+1}$. Let $\mathbf{h} = (h_1, h_2, h_3, \ldots)$, $\mathbf{k} = (h_2, h_3, h_4, \ldots)$, $H = \varprojlim \mathbf{h}$, and $K = \varprojlim \mathbf{k}$. Note that $f([0,c]) = [c,1]$ and $f([c,1]) = [0,c]$. That H and K are subcontinua of M follows from Theorem 2.1. Moreover, the shift homeomorphism $\hat{f}$ transforms H onto K. To see that $M = H \cup K$, observe that if x is a point of M different from $(c,c,c,\cdots)$, then there is a positive integer N such that $x_i \ne c$ when $i \ge N$. Then x_i is in $[0,c]$ if and only if x_{i+1} is in $[c,1]$, so x is either in H or K but not in both. ■

Theorem 4.7. *Suppose that $f = f_{ab}$ with $b > c = 1/(2-a)$. Let $I_j = [c,1]$ if j is odd and $[0,c]$ if j is even, and let $k_j = f \,|\, I_{j+1}$. If $\mathbf{k} = (k_1, k_2, k_3, \ldots)$ and $K = \varprojlim \mathbf{k}$, then K is the union of a topological ray R and a continuum J such that $J = \overline{R} - R$.*

Proof. Theorems 4.1 and 4.3 cover the cases where $b > 1 - a + a^2$ and $b = 1 - a + a^2$, so we may assume that $b < 1 - a + a^2$. It is simpler to identify J and R in a homeomorphic copy of K. To that end, observe that $k_i \circ k_{i+1}$

is $f^2 \,|\, [c, 1]$ when i is odd. Invoking Theorem 3.1 with $g = f^2 \,|\, [c, 1]$, we see that K is homeomorphic to $\varprojlim \mathbf{g}$. We may apply Theorem 3.4 to $\varprojlim \mathbf{g}$ since $g^2([c, b]) = [c, 1]$, $g([b, 1]) = [b, 1]$, and g is increasing on $[c, b]$. We thus obtain a topological ray and its remainder, which we then transform to R and J via the homeomorphism between $\varprojlim \mathbf{g}$ and K. ■

The points where $b = c$ form the boundary between the parameter regions where the inverse limit is an indecomposable continuum and where the inverse limit is decomposable but contains an indecomposable continuum. This case provides an interesting inverse limit.

Theorem 4.8. *If $f = f_{ab}$ with $b = c = 1/(2 - a)$, then $\varprojlim \mathbf{f}$ is the union of two indecomposable continua.*

We omit the proof because it is essentially identical to one given in Example 3. As was pointed out in the remark following this example, these two continua can be shown to be BJK-horseshoes (see [**10**]).

In [**2,** pp. 125–126], Bassein treats chaos on a subinterval through renormalization. We introduce notation that allows us to formulate her results in such a way that we can give a general description of the inverse limit when the parameter pair is in W. Let

$$T = \{(a, b) \mid 0 < a < 1,\, 0 \leq b < 1 - a\},$$

and define $\phi : W \to T$ by

$$\phi(a, b) = (1 - \frac{a(1 - a)}{1 - b}, 1 - \frac{1}{(1 - a)^2} + \frac{a}{(1 - a)(1 - b)}) = (a_1, b_1).$$

(In the third paragraph of [**2,** sec. 5, p. 125] Bassein shows that, if (a, b) lies in W, then $0 \leq b_1 < 1 - a_1$, i.e., $\phi(a, b)$ belongs to T.) Most of [**2,** sec. 5] is devoted to proving the following theorem, which we state in terms of the function ϕ.

Theorem 4.9 (Bassein). *If (a, b) belongs to W, then there is a positive integer n such that $\phi^n(a, b)$ belongs to* $\{(a, b) \mid 0 < a < 1 \text{ and } 0 \leq b < c = 1/(2 - a)\}$.

In order to give the reader a feeling for the general result of the present subsection, we include the following theorem. It reflects the nature of the inverse limit in the situation where Bassein's first renormalization results in a map that exhibits chaotic dynamics on $[0,1]$ but has no attracting periodic orbits.

Theorem 4.10. *If $f = f_{ab}$ where (a,b) belongs to W and $\phi(a,b) = (a_1, b_1)$ belongs to $\{(a,b) \mid 0 \leq b < c\}$, then $\varprojlim \mathbf{f}$ is the union of two homeomorphic continua intersecting at $(c,c,c,\cdots)$, each of which is the union of a topological ray R with end-point $(c,c,c,\cdots)$ and an indecomposable continuum K such that $\overline{R} - R = K$.*

Proof. That $\varprojlim \mathbf{f}$ is the union of two homeomorphic continua is a consequence of Theorem 4.6. Next, appealing to Theorem 4.7, we conclude that each of these is the closure of a topological ray. Finally, that the remainder $\overline{R} - R$ is indecomposable may be seen as follows. It is tedious but straightforward to show that $f^2 \,|\, [0, f(b)]$, is conjugate to $f_{a_1 b_1}$, in which $(a_1, b_1) = \phi(a,b)$. One inverts $f^2 \,|\, [0, f(b)]$ with the conjugacy $h^{-1} f^2 h$, where $h(x) = f(b) - x$ for x in $[0, f(b)]$, and then scales the resulting function linearly so that it maps $[0,1]$ onto $[0,1]$. It then follows from Theorem 4.5 that $\varprojlim \mathbf{g}$, where $g = f_{a_1 b_1}$ is indecomposable and from Theorem 3.8 that K is indecomposable. ∎

In order to state the general theorem of this section, we adopt the following language: we say that a continuum is of *type* R_1 if it is the union of two topological rays intersecting at a common end-point such that each ray has an indecomposable continuum as its remainder. If n is a positive integer, we say that a continuum is of *type* R_{n+1} if it is the union of two topological rays intersecting at a common end-point such that each topological ray has as remainder a continuum of type R_n. Note that the inverse limit in Theorem 4.10 is of type R_1.

Theorem 4.11. *If $f = f_{ab}$ where (a,b) belongs to W, $\phi^n(a,b)$ belongs to W, and $\phi^{n+1}(a,b)$ belongs to $\{(a,b) \mid 0 < a < 1 \text{ and } 0 \leq b < c\}$, then $\varprojlim \mathbf{f}$ is a continuum of type R_n. Moreover, $\varprojlim \mathbf{f}$ contains 2^n homeomorphic indecomposable continua.*

Proof. This theorem can be established by induction using Theorem 4.10. ■

More chaos: $\mathbf{1 - a \le b \le a}$. This subsection is concerned primarily with the region of parameter space where $1 - a \le b \le a$, although Theorem 4.12 holds more generally. We saw earlier that, if $f = f_{ab}$ for (a, b) is in this region, then $\varprojlim \mathbf{f}$ is an indecomposable continuum. However, certain phenomena occur here that allow us to say more about the nature of these inverse limits.

Let $f = f_{ab}$ with $1 - a \le b \le a$. Following Bassein [**2**], we denote by m the least positive integer i such that $f^i(0) > a$ and remark that $m \ge 2$, since $f(0) = b \le a$. In [**2,** p. 127], Bassein observes that there is a number x, with $0 \le x < b$, such that $f^{m+1}(x) = x$. By choosing x in $[0, b)$, it follows from the choice of m that $x < f(x) < \ldots < f^m(x)$. Adopting Bassein's terminology, we call the set $\{x, f(x), \ldots, f^m(x)\}$ a *fundamental* $(m+1)$*-cycle.* (In [**11**], x is called a *stair-step periodic point.*) Although Bassein's emphasis is on the fundamental $(m + 1)$-cycles, there are other periodic orbits present within the region under consideration and these orbits persist outside the region. For instance, with $a = 0.75$ and $b \approx 0.47972$, 0 is in a period five orbit that is not stair-step. A third period five orbit occurs when $a < b < c = 1/(2 - a)$. It is interesting to note that, although where these three period five orbits occur similar inverse limits surface (see Theorem 4.12), they are topologically distinct [**1**]. When a is in a periodic orbit, we can say more about the nature of the inverse limit. Recall that a point p of a continuum M is an *end-point* of M under the following condition: if H and K are subcontinua of M containing p, then H is a subset of K or K is a subset of H. If a continuum has n end-points but does not have $n + 1$ end-points, we say it has *only* n *end-points.*

Theorem 4.12. *Let $f = f_{ab}$. Suppose that a belongs to a periodic orbit of period $n \ge 3$ and that $f^j(0)$ is the first positive member* (*in the usual order on* $[0, 1]$) *in the orbit of* 0 *under f. If n and j are relatively prime, then $\varprojlim \mathbf{f}$ is an indecomposable continuum having only n end-points, and every proper subcontinuum of $\varprojlim \mathbf{f}$ is an arc.*

The proof of this theorem is technical and does not rely on relationships between a and b. The interested reader should consult [**12**].

Corollary 4.13. *If $f = f_{ab}$ where $1 - a \le b \le a$ and a belongs to the fundamental* $(m+1)$*-cycle, then $\varprojlim \mathbf{f}$ is an indecomposable continuum having only $m + 1$ end-points, and every proper subcontinuum of $\varprojlim \mathbf{f}$ is an arc.*

Proof. This follows from Theorem 4.12, since $f(0) = b$ is the first positive member of the orbit of 0 under f. ■

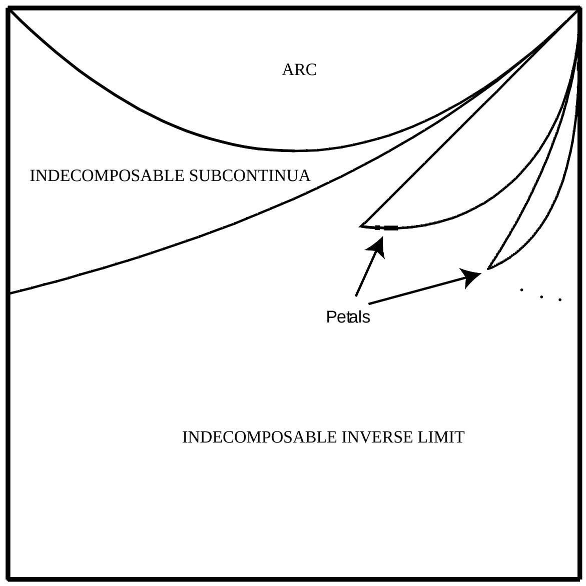

Figure 8.

Bassein identifies the region of parameter space where the fundamental $(m+1)$-cycle is attracting. This is the set of parameter values (a, b) for which

$$\frac{(1-a)(1-b)}{ab} \leq \left(\frac{1-b}{a}\right)^m < 1-a \ .$$

Bassein refers to this region as a "petal." Accompanying the region of attraction is a region she calls a "sliver" where the stable periodicity has ceased but in which f_{ab}^{m+1} has $m+1$ mutually exclusive invariant intervals. The inequalities determining the sliver in parameter space are found on page 129 just below Figure 12. These inequalities are

$$1-a < \left(\frac{1-b}{a}\right)^m \leq \frac{(1-b+ab)(1-a)}{a} \ .$$

The petals for $m = 2$ and $m = 3$ are shown in Figure 8. The slivers abut the petals on their lower sides but are too thin to depict in a small figure. The following theorems address the nature of the inverse limit $\varprojlim \mathbf{f}$ within the petal, along the lower boundary of the petal, and within the sliver, respectively.

Theorem 4.14. *If $f = f_{ab}$ for (a, b) in the region where the fundamental $(m+1)$-cycle is attracting, i.e., where*

$$\frac{(1-a)(1-b)}{ab} \leq (\frac{1-b}{a})^m < 1-a \ ,$$

then $\varprojlim \mathbf{f}$ is an indecomposable continuum having only $m+1$ end-points, and every proper subcontinuum of $\varprojlim \mathbf{f}$ is an arc.

For a proof of this theorem, the interested reader is referred to [**11,** Theorem 13]. The proof is technical and does not follow readily from the relationships between a and b, as do most of the proofs in the present paper. The following theorems are not found explicitly in [**11**], but the theorems and their proofs are part of the remarks at the end of that paper.

Theorem 4.15. *If $f = f_{ab}$ for (a, b) on the curve described by*

$$(\frac{1-b}{a})^m = 1-a \ for \ 1-a \leq b \leq a \ ,$$

then $\varprojlim \mathbf{f}$ is an indecomposable continuum having only $2(m+1)$ end-points. Furthermore each nondegenerate proper subcontinuum of $\varprojlim \mathbf{f}$ is either an arc or the closure of a topological ray with an arc as its remainder.

Theorem 4.16. *If $f = f_{ab}$ for (a, b) in the region described by the conditions*

$$1-a < (\frac{1-b}{a})^m \leq \frac{(1-b+ab)(1-a)}{a} \ for \ 1-a \leq b \leq a \ ,$$

then $\varprojlim \mathbf{f}$ is an indecomposable continuum having $(m+1)$ homeomorphic subcontinua, each of which contains an indecomposable continuum. Furthermore, when

$$(\frac{1-b}{a})^m = \frac{(1-b+ab)(1-a)}{a} \ ,$$

these $m+1$ subcontinua of the inverse limit are BJK-horseshoes.

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