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Inverse limits with subsets of [0, 1] cross [0, 1]

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Inverse limits with subsets of $[0,1] \times [0,1]$

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Abstract

The characterization of chainable continua as inverse limits on $[0,1]$ has resulted in many papers which have contributed to our knowledge of both inverse limits and chainable continua. Quite simple maps on $[0,1]$ give rise to quite complicated inverse limits. In this paper we begin an investigation of continua that can be represented by inverse limits of closed subsets of the unit square $I^2 = [0,1] \times [0,1]$. We show that some of the usual properties of inverse limits are valid in this situation, give numerous examples, and provide conditions under which the inverse limit is a continuum.

1 Introduction

The study of continua as inverse limits has a long history. Much of the research involved inverse limits of maps on the interval $[0,1]$ and much of that with a single bonding map. The reasons are quite clear to anyone who has worked with inverse limits. Very simple maps on $[0,1]$ can give rise to extremely complicated continua even in this special case. See for example Ingram [4]. This has limited the study to that of chainable continua. A few authors have studied those continua that are inverse limits of maps on simple triods or circles. For example see Davis and Ingram [3] or Anderson and Choquet [1]. In this article we continue to concentrate on the interval $[0,1]$, but instead of maps on $[0,1]$, we consider inverse limits of closed subsets of the unit square. In this setting we find that many of the basic theorems about inverse limits of maps on $[0,1]$ still apply but the inverse limits need not be chainable.

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2 Basic definitions and notation

In the following we shall use bold characters to represent sequences and italic characters to represent the terms of the sequence. For example, $\mathbf{M}$ will denote the sequence whose terms are $(M_1, M_2, M_3, ...)$. The interval $[0, 1]$ will be denoted by I, I^2 will denote the unit square $[0, 1] \times [0, 1]$, and Z^+ is the set of positive integers. If X is a topological space, then 2^X is the set of all compact subsets of X. By a *map or a mapping* we mean a continuous function.

The Hilbert cube $\mathcal{Q}$ is the set of all sequences $(x_1, x_2, x_2, \dots)$ such that $x_i \in I$ for $i \in Z^+$ with the topology determined by defining a metric on it: the distance from the point $\mathbf{x} = (x_1, x_2, x_3, \dots)$ to the point $\mathbf{y} = (y_1, y_2, y_3, \dots)$ is given by

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i>0} \frac{|x_i - y_i|}{2^i}.$$

If $\mathbf{M} = (M_1, M_2, M_3, \dots)$ is a sequence of closed subsets of I^2, and for each $i \in Z^+$ the projection of M_i on the x-axis contains the projection of M_{i+1} on the y-axis, then $\underleftarrow{lim}\, \mathbf{M}$ will denote the subspace of $\mathcal{Q}$ such that $\mathbf{x} \in \underleftarrow{\mathbf{lim}}\, \mathbf{M}$ if and only if $(x_{i+1}, x_i) \in M_i$ for each $i \in Z^+$. This definition is more general than is needed for this paper since we will restrict our attention to the case where $M_i = M_1$ for each i and the x-projection of M_1 is I. However this definition is closer to the one normally used for inverse limits of mappings and will allow for sub inverse limits, e.g., inverse limits using subsets of the M_i. We note that in this case there may be points (x_2, x_1) of M_1 for which there is no point in $\underleftarrow{lim}\, \mathbf{M}$ whose first two terms are x_1 and x_2. This is the case in Example 1 in section 3. On the other hand if we require that the x-projection of M_i be a subset of the y-projection of M_{i+1}, then for each point (x_2, x_1) of M_1 there is a corresponding point $(x_1, x_2, x_3, \dots)$ in $\underleftarrow{lim}\, \mathbf{M}$. This is analogous to the case for inverse limits of surjective mappings. In the special case where M is a closed subset of I^2 and $M_i = M$ for each $i \in Z^+$, then we will abuse our notation and use $\underleftarrow{lim}\, \mathbf{M}$ for the inverse limit.

We use the usual notation where π_i is the projection of $\mathcal{Q}$ onto its ith factor space, i.e., if $\mathbf{x} = (x_1, x_2, x_3, \dots)$, π_i is given by $\pi_i(\mathbf{x}) = \mathbf{x_i}$.

Let f be a function from I into the set 2^I of closed subsets of I. The *graph of* f, $G(f)$, is the set of all points $(x, y) \in I^2$ such that $y \in f(x)$. The statement that f is *upper semicontinuous* at the point $x \in I$ means that if V is an open set in I containing $f(x)$, then there is an open set U in I containing x such that if $y \in U$, $f(y) \subseteq V$.

If $M \subseteq I^2$ then M_x is the projection of M on the x-axis and M_y is the projection on the y-axis.

3 Basic theorems and examples

Our first theorem shows that if M is a closed subset of I^2 and $M_x = I$, then it is the graph of an upper semicontinuous function from I into 2^I. The following theorem is probably well known but is included here for completeness as the proof is short.

Theorem 1. *If M is a closed subset of I^2 such that $M_x = I$, then there is an upper semicontinuous function f from I into 2^I such that $M = G(f)$.*

Proof. For each $x \in I$, let $f(x)$ be the set of all points t of I such that (x,t) is in M. Clearly $M = G(f)$. Let V be an open set in I containing $f(x)$. If f is not upper semicontinuous at x, then there is a sequence of points $\{x_i\}_{i=1}^{i=\infty}$ in I converging to x and a sequence $\{y_i\}_{i=1}^{i=\infty}$ in I such that if $i \in Z^+$, then $y_i \in f(x_i) - V$. For each i, (x_i, y_i) is in M and some subsequence of $\{(x_i, y_i)\}_{i=1}^{i=\infty}$ converges to a point (x,y). Since M is closed, $(x,y) \in M$. But $y \notin V$, so y is not in $f(x)$ and thus (x,y) is not in M. This contradiction implies that f is upper semicontinuous. ■

Next we observe that our inverse limits are compact.

Theorem 2. *If M is a closed subset of I^2, then $\varprojlim M$ is compact.*

Proof. Since $\mathcal{Q}$ is compact it suffices to show that $\varprojlim M$ is closed. Let $\mathbf{p} = (p_1, p_2, p_3, \dots)$ be a point of $\mathcal{Q} - \varprojlim \mathbf{M}$ and let $n \in Z^+$ such that $(p_{n+1}, p_n) \notin M$. Since M is closed, there are open sets S and T in I such that $p_{n+1} \in S$, $p_n \in T$ and $S \times T$ contains no point of M. Now $\pi_{n+1}^{-1}(S) \cap \pi_n^{-1}(T)$ is an open set in $\mathcal{Q}$ containing $\mathbf{p}$ but no point of $\varprojlim M$, so $\mathbf{p}$ is not a limit point of $\varprojlim M$. It follows that $\varprojlim \mathbf{M}$ is compact. ■

Unlike inverse limits with continuous functions from I into I, inverse limits on subsets of I^2 need not be connected as the following simple example shows.

Example 1. *Let M consist of the point $(1,1)$ together with all points $(x, x/2)$ for $x \in I$.*

If $\mathbf{p} = (p_1, p_2, p_3, \dots)$ is in $\mathcal{Q}$ and $1/2^{n+1} < p_1 < 1/2^n$ for some $n \in Z^+$, then $1/2 < p_{n+1} < 1$ and there is no point (x, p_{n+1}) in M. Thus there is no point in $\varprojlim \mathbf{M}$ whose first coordinate is p_1. It follows that if $\mathbf{p}$ is a point in the inverse limit, then p_1 is 0, 1 or $1/2^n$ for some positive integer n. Clearly $(0,0,0,\dots)$ and $(1,1,1,\dots)$ are in $\varprojlim \mathbf{M}$. If $\mathbf{p}$ is in $\varprojlim M$ and for some $n \in Z^+$, $p_1 = 1/2^n$, then $p_n = 1/2$ and for $j > n$, $p_j = 1$. We conclude that the inverse limit is a sequence of points that converges to the point $(0,0,0,\dots)$.

We next consider conditions under which $\varprojlim \mathbf{M}$ is connected. First we give an example to show that in order for $\varprojlim M$ to be connected it is not sufficient for M to be connected.

Example 2. *Let M be the union of the straight line intervals joining the following pairs of points:* $(0, 3/4)$ *and* $(1/2, 1)$*,* $(1/2, 1)$ *and* $(1, 1)$*,* $(0, 3/4)$ *and* $(1/2, 1/2)$*, and* $(1/2, 1/2)$ *and* $(1, 1/2)$

Let $\mathbf{p} \in \varprojlim \mathbf{M}$. Since there is no point of M whose 2^{nd} coordinate is less than $1/2$, then $p_n \geq 1/2$ for any $n \in Z^+$. However if $1/2 < p_n < 1$, then $p_{n+1} < 1/2$. Thus we have that for each n, $p_n = 1/2$ or $p_n = 1$. It follows that $\varprojlim M$ is not only not connected, it is a totally disconnected perfect set, thus homeomorphic to the ternary Cantor set. See [**7,** p. 217].

We were unable to find an example of a closed and connected subset M of I^2 with $M_x = M_y = I$ and with $\varprojlim \mathbf{M}$ not connected.

In view of the preceding example, some additional condition on M is needed to insure that $\varprojlim \mathbf{M}$ is connected. If M is a subset of I^2, then by a *vertical section of M* we mean the intersection of M with some vertical line. We will show that a sufficient condition that $\varprojlim \mathbf{M}$ be connected is that each vertical section of M be connected. This condition is illustrated in the next example.

Example 3 *Let M be the same as that of Example 1 but with the vertical interval from* $(1, 1)$ *to* $(1/2, 1)$ *added.*

Theorem 2 implies that $\varprojlim \mathbf{M}$ is compact and we will show later that it is a continuum since each vertical section of M is connected. But it is easy to see directly that $\varprojlim \mathbf{M}$ is an arc. Note that if y is in $[0, 1]$, then there is only one point in M whose 2^{nd} coordinate is y. This implies that π_1 is a 1-1 mapping from the compact set $\varprojlim \mathbf{M}$ onto $[0, 1]$ and is thus a homeomorphism. See [**7,** Theorem 17.14, p. 123]. ■

For the next three theorems, we assume that M is a closed subset of I^2, $M_x = I$, and f is the upper semi-continuous function given by Theorem 1. For each $n \geq 1$, let $G_n(f)$ be the set of all points $(x_1, x_2, x_3 \dots)$ in $\mathcal{Q}$ such that $x_i \in f(x_{i+1})$ for $i \leq n$. Since $\varprojlim \mathbf{M} = \bigcap_{n=1}^{\infty} G_n(f)$ then $\varprojlim \mathbf{M}$ is compact if each $G_n(f)$ is compact, which we now demonstrate.

Theorem 3. *$G_n(f)$ is compact for each $n > 1$.*

Proof. Our proof is essentially the same as was used in Theorem 2. If $\mathbf{p} \in \mathcal{Q} - G_n(f)$, then (p_{i+1}, p_i) is not in M for some i with $1 \leq i \leq n$. So there are open sets U and V in I containing p_{i+1} and p_i respectively such that $U \times V$ contains no point of M. But $\pi_i^{-1}(V) \cap \pi_{i+1}^{-1}(U)$ is an open set in $\mathcal{Q}$ that contains no point of $G_n(f)$ and thus $\mathbf{p}$ is not a limit point of G_f. This implies that $G_n(f)$ is closed in $\mathcal{Q}$ and thus is compact.■

The next theorem is a first step in determining a sufficient condition that $\varprojlim \mathbf{M}$ is connected.

Theorem 4. *M is connected if each vertical section of M is connected.*

Proof. If M is not connected, it is the union of two mutually exclusive compact sets H and K. Since the projection of M on the x-axis is $[0, 1]$, there is a number x in the x-projections of both H and K. But the vertical section of M that is a subset of the vertical line containing the point $(x, 0)$ is connected and must be a subset of one of H or K. ■

We are now ready to provide a sufficient condition that $\varprojlim \mathbf{M}$ be connected. A *continuum* is a compact and connected set.

Theorem 5. *$\varprojlim \mathbf{M}$ is a continuum if each vertical section of M is connected.*

Proof. Since $\varprojlim \mathbf{M} = \bigcap_{n=1}^{\infty} G_n(f)$, then M is a continuum if each $G_n(f)$ is a continuum. $G_n(f)$ is compact by Theorem 3 so we only need to show that $G_n(f)$ is connected which we do by induction. It follows from the previous theorem that $G_1(f)$ is connected. Assume that there is an $n > 1$ such that $G_n(f)$ is not connected but $G_{n-1}(f)$ is connected. Let H and K be two mutually exclusive compact sets with union $G_n(f)$. Let h be the shift map from $\mathcal{Q}$ onto $\mathcal{Q}$ defined by $h(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. Since $M_x = I$, $h(G_n(f)) = G_{n-1}(f)$ and h is continuous, it follows that $h(H)$ and $h(K)$ are compact subsets of $G_{n-1}(f)$ with union $G_{n-1}(f)$. Since $G_{n-1}(f)$ is connected, there is a point $\mathbf{y} = (y_2, y_3, y_4, \dots)$ in $h(H) \cap h(K)$. Now $f(y_2)$ is connected so the set A of all points $(y_1, y_2, y_3, \dots) \in \mathcal{Q}$ such that $y_1 \in f(y_2)$ is a connected subset of $G_n(f)$ and thus a subset of only one of H or K. Since $f(y_2)$ is a closed and connected subset of $\{y_2\} \times I$, then A is an arc or a point. In either case this involves a contradiction since if A is degenerate this implies it is common to H and K, and if it is an arc there is a point of $H \cap A$ that maps to $\mathbf{y}$ and a point of $K \cap A$ that maps to $\mathbf{y}$. ■

One might think that if $G_1(f)$ were connected, then $G_n(f)$ would be connected for all n, but it is not difficult to show that if M is as in Example 2 then $G_2(f)$ is not connected.

Our next example shows that the assumption that each vertical section of M is connected is not necessary in order that $\varprojlim \mathbf{M}$ be connected. In our discussion of this example we will consider inverse limit sequences. By an *inverse limit sequence* is meant a sequence of pairs $(X_1, f_1), (X_2, f_2), (X_3, f_3), \dots$ such that for each $i \in Z^+$, X_i is a topological space and f_i is a map from X_{i+1} into X_i. If $(X_1, f_1), (X_2, f_2), (X_3, f_3), \dots$ is an inverse limit sequence then by the *inverse limit* of the sequence, that we denote by $\varprojlim \mathbf{f}$ where as usual $\mathbf{f} = (f_1, f_2, f_3, \dots)$ is meant the subset of $\Pi_{i=1}^{\infty} X_i$ to which the point $\mathbf{x} = (x_1, x_2, x_3, \dots)$ belongs only if $f(x_{i+1}) = x_i$ for each i. In our example each X_i is a subset of $[0, 1]$ and our inverse limit is a subset of $\mathcal{Q}$.

Example 4. *Let M be the union of the graphs of the two functions f and g where $f(x) = x$ and $g(x) = 1 - x$ for each number x in $[0, 1]$.*

We show that $\varprojlim \mathbf{M}$ is connected by showing it is the union of a collection of arcs each having the point $\mathbf{A} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ as one endpoint. Let $\mathcal{C}$ denote the set of all sequences each term of which is is 0 or 1. Clearly $\mathcal{C} \subset \varprojlim M$. M is the union of the 4 intervals from the corners of I^2 to the point $(1/2, 1/2)$. Label these intervals as $I_{(x,y)}$ where $(x, y) \in \{0, 1\}^2$ is one of the corners of I^2. If $\mathbf{s} \in \mathcal{C}$, then let $I_{\mathbf{s}}$ denote the set of all points $\mathbf{p}$ in $\varprojlim \mathbf{M}$ such that if $i \in Z^+$, $(p_{i+1}, p_i) \in I_{(s_{i+1}, s_i)}$.

Note that if I^2 is partitioned into 4 non-overlapping squares, each having one corner at $(\frac{1}{2}, \frac{1}{2})$ then the intersection of M with each of these squares is the graph of a 1-1 map whose domain is either the interval $[0, 1/2]$ or $[1/2, 1]$ and whose range is also one of those intervals. It follows that for $\mathbf{s} \in \mathcal{C}$, $I_{\mathbf{s}}$ is the inverse limit of the sequence $(X_1, f_1), (X_2, f_2), (X_3, f_3), \dots$ where for each $i \in Z^+$, X_i is $[0, 1/2]$ or $[1/2, 1]$, f_i is a homeomorphism, and $f_i(1/2) = 1/2$. Thus we have that if $x_1 \in X_1$ then there is only one point $\mathbf{x}$ in $\varprojlim \mathbf{f}$ such that $\pi_1(\mathbf{x}) = x_1$. It follows that $I_{\mathbf{s}}$ is an arc since π_1 is a homeomorphism of $I_{\mathbf{s}}$ onto X_1. Moreover $(1/2, 1/2, 1/2, \dots)$ and $\mathbf{s}$ are the endpoints of $I_{\mathbf{s}}$. Being the union of a collection of arcs having a common endpoint $\varprojlim \mathbf{M}$ is connected. Actually since $\mathcal{C}$ is a Cantor set, $\varprojlim \mathbf{M}$ is the cone over a Cantor set.

The following theorem is a simplification of one due to Ralph Bennett [2]. A slightly strengthened version of Bennett's theorem can be found in [**5,** Theorem 2.16, p. 20]. A *topological ray* is a locally compact connected set having only one nonseparating point, and a *nonseparating point* of a connected set M is a point p of M such that $M - \{p\}$ is connected.

Theorem 6. *(***Bennett***) Assume that $0 < a < b < 1$ and that f is a map whose graph contains the straight line interval from $(0, 0)$ to $(a, 1)$ and the one from $(a, 1)$*

to $(b, f(b))$ *and* $f([b,1]) = [b,1]$. *Then* $\varprojlim \mathbf{f}$ *is a compactification of a topological ray* R *such that* $\overline{R} - R = \varprojlim (\mathbf{f} \mid [\mathbf{b}, \mathbf{1}])$.

In an earlier version of this paper, with the same hypothesis as that of the next theorem, the author attempted to show that $\varprojlim \mathbf{M}$ was a compactification of a topological ray as is the case in Bennett's theorem. Tom Ingram discovered an error in our argument that resulted in our proving the following weaker theorem in which R need not be a topological ray.

Theorem 7. *Assume that* $0 < a < b < 1$ *and that* M *is the union of a closed subset* K *of* $[b,1] \times [b,1]$ *and two straight line intervals, one from* $(0,0)$ *to* $(a,1)$ *and the other from* $(a,1)$ *to the point* (b,b). *Assume also that* $M_x = I$ *and each vertical section of* M *is connected. Then* $\varprojlim M$ *is the disjoint union of a connected set* R *and a continuum* H *such that each point of* H *is a limit point of* R.

Proof. For each $n > 0$, let R_n be the set of all points $\mathbf{p} = (p_1, p_2, p_3, \dots)$ of $\varprojlim \mathbf{M}$ such that p_n is in $[0,a]$. Note that when p_n is in $[0,a]$, then p_j is uniquely determined for $j > n$. This implies that $\pi_1 \,|\, R_1$ is a homeomorphism of R_1 onto $[0,a]$ and that R_1 is an arc. Let f be the upper semicontinuous function given by Theorem 1 whose graph is M. If p_2 is in $[0,a]$, then $f(p_2) = \{p_1\}$ is degenerate so $\pi_2 \,|\, R_2$ is a homeomorphism of R_2 onto $[0,a]$ and R_2 is an arc. For $n > 2$, p_j is uniquely determined for $j > n$ but for $j < n-2$, p_j may not be uniquely determined since it may depend on the set K. But for $n > 2$, R_n must be connected and this can be seen as follows: If $\mathbf{p} = (p_1, p_2, p_3, \dots) \in G_{n-2}(f)$, then let $h(p) = (p_1, p_2, p_3, \cdots, p_{n-1}, ap_{n-1}, a^2 p_{n-1}, \dots)$. Since $p_{n-1} \in [0,1]$, then $\pi_n(h(\mathbf{p})) \in [0,a]$ and we have that $h(\mathbf{p}) \in R_n$. Appealing to the proof of Theorem 5, we see that $G_{n-2}(f)$ is connected. Moreover h is continuous so R_n is connected. Finally, we have that $R = \bigcup_{i>0} R_i$ is connected since for each n R_n contains the point $(0,0,0,\dots)$.

Let H denote the set of all points $\mathbf{x} = (x_1, x_2, x_3, \dots)$ of $\varprojlim \mathbf{M}$ such that $x_i \in [b,1]$ for each i. We show that M is the union of R and H. To this end, let $\mathbf{p}$ be a point of $\varprojlim \mathbf{M}$ that is not in H. Then p_n is in $[0,b)$ for some n and thus p_{n+1} is in $[0,a]$ so $\mathbf{p}$ is in $R_{n+1} \subseteq R$. It follows that $\varprojlim \mathbf{M} = H \cup R$, and H and R are clearly mutually exclusive. Finally to see that each point of H is a limit point of R, let $\mathbf{p}$ be a point of H and let n be a positive integer. Since p_n is in $[b,1]$ and $f([0,a]) = [0,1]$, there is a point $x_{n+1} \in [0,a]$ such that (x_{n+1}, p_n) is in M. Thus we have a point $(p_1, p_2, \cdots, p_n, x_{n+1}, ax_{n+1}, a^2 x_{n+1}, \dots)$ in R_{n+1} that has the same first n coordinates as $\mathbf{p}$. It follows that $\mathbf{p}$ is a limit point of R. ■

4 Comments and questions

All of our examples yield inverse limits that are either infinite dimensional or 1 dimensional continua. We suspect that this is true in general and that $\varprojlim M$ must either contain a Hilbert cube or be 1-dimensional. While working on this paper we noted that much of what we have done for closed subsets of I^2 can be done in a more general setting where the inverse limit is for a sequence of spaces and functions that are upper semi-continuous functions from a space to the compact subsets of the space. We are working on this generalization.

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