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Chromatic numbers and products

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Abstract

Let $\Lambda(n)$ be the smallest number so that there are two n chromatic graphs whose product has chromatic number $\Lambda(n)$. Under the assumption that a certain sharper result than one obtained by Duffus, Sands and Woodrow [1] holds we will prove that $\Lambda(n) \geq n/2$.

MR Subject Classifications [2000]: 05C15
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1 Introduction

We denote by $\mathbf{G}$ the class of all finite symmetric graphs, with loops allowed. If $\mathrm{G} \in \mathbf{G}$ then $V(\mathrm{G})$ is the set of *vertices* of G and $E(\mathrm{G})$, a subset of the set of one-element and two-element subsets of $V(\mathrm{G})$, is the set of *edges* of G. We denote by K_n the complete graph with $V(\mathrm{K}_n) = \underline{n} = \{0, 1, 2, \dots, n-1\}$ and $E(\mathrm{K}_n)$, the set of all two-element subsets of $\underline{n}$. We denote by $\mathring{1}$ the graph with $V(\mathring{1}) = \{1\}$ and $E(\mathring{1}) = \{\{1\}\}$, that is, the graph with one vertex and with a loop on that vertex.

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The product, $\mathrm{G}\times\mathrm{H}$, of two graphs G and H is the graph with $V(\mathrm{G}\times\mathrm{H}) = V(\mathrm{G})\times V(\mathrm{H})$ and edges all $\{(g_0,h_0),(g_1,h_1)\}$ such that $\{g_0,g_1\}\in E(\mathrm{G})$ and $\{h_0,h_1\}\in E(\mathrm{H})$. It is easily seen that, with χ denoting the usual vertex chromatic number of graphs, $\chi(\mathrm{G}\times\mathrm{H})\leq\min\{\chi(\mathrm{G}),\chi(\mathrm{H})\}$ ([7] equation (2)). Hedetniemi's conjecture, stated in [4], is that the chromatic number of the product of two n-chromatic graphs is n. This holds true for $n=1$ trivially and is not too difficult to prove for $n=2$ and $n=3$, (see [4] or [7] Theorem 1). It follows from [3] that the product of two 4-chromatic graphs has 4-chromatic.

Let $\Lambda(n)$ be the smallest number so that there are two n-chromatic graphs whose product has chromatic number $\Lambda(n)$. It is not known if this function tends to infinity. Indeed, the following curious statement is the best result to date: $\Lambda(n)$ either tends to infinity with n or is bounded by 9. (See [5] and [8] for this strengthening of the original result of Poljak and Rödl [6].)

It is proven in [1] that if $\chi(\mathrm{G})\geq n+1$ and $\chi(\mathrm{H})\geq n+1$ for two connected graphs G and H which both contain a complete graph on n vertices then $\chi(\mathrm{G}\times\mathrm{H})\geq n+1$. We will show that a stronger version of that result implies that $\Lambda(n)$ is unbounded. Let us say that the number n is *semi-stable* if $\chi(\mathrm{G})=n+1$ and $\chi(\mathrm{H})=n+1$ for two connected graphs G and H, at least one containing a complete graph on n vertices, then $\chi(\mathrm{G}\times\mathrm{H})=n+1$. We will prove, Theorem 4.1, that if the number n has the weak Hedetniemi property then $\Lambda(n)\geq n/2$.

Using terminology introduced in [2], a graph G with chromatic number $n+1$ is *stable with respect to* n (or just *stable*) if $\chi(\mathrm{G}\times\mathrm{H})=n+1$ for every graph H with $\chi(\mathrm{H})=n+1$. In the terminology just introduced, a number n is semi-stable if and only if every graph G with $\chi(\mathrm{G})=n+1$ which contains a complete subgraph on n vertices is stable.

2 Notation

If G and H are two graphs with $V(\mathrm{G})\cap V(\mathrm{H})=\emptyset$ then and only then we define $\mathrm{G}+\mathrm{H}$ to be the graph with $V(\mathrm{G}+\mathrm{H})=V(\mathrm{G})\cup V(\mathrm{H})$ and $E(\mathrm{G}+\mathrm{H})=E(\mathrm{G})\cup E(\mathrm{H})$.

Let $\mathrm{G},\mathrm{H}\in\mathbf{G}$ be two graphs and $f:V(\mathrm{G})\mapsto V(\mathrm{H})$ a function of $V(\mathrm{G})$ to $V(\mathrm{H})$. The function f is a *homomorphism* of G to H if $\{f(x),f(y)\}$ is an edge of H for every edge $\{x,y\}$ of G.

For two graphs $\mathrm{G},\mathrm{H}\in\mathbf{G}$ we write $\mathrm{G}\to\mathrm{H}$ if there is a homomorphism of the graph G to the graph H. The graphs G and H are *equivalent*, $\mathrm{G}\sim\mathrm{H}$, if $\mathrm{G} \to \mathrm{H}$ and $\mathrm{H} \to \mathrm{G}$. Observe that $\mathrm{G} \to \mathring{1}$ for every graph G.

A *core* of a graph $\mathrm{G} \in \mathbf{G}$ is a graph C with a minimum number of vertices under the condition that $\mathrm{C} \sim \mathrm{G}$.

Note that every finite graph G has a core, that any two cores of G are isomorphic, and that every homomorphism of a core of a graph into itself is an automorphism. If $\mathrm{G} \sim \mathrm{H}$ then every core of G is isomorphic to every core of H. Hence, we can speak of a core of a $\sim$-equivalence class. If $\mathrm{G} = \mathrm{H} + \mathrm{M}$ and no connected component of H has a homomorphism to M and no connected component of M has a homomorphism to H then every core of G is of the form $\mathrm{H}' + \mathrm{M}'$ where H' is a core of H and M' is a core of M.

We will select one core, core(G), for every graph $\mathrm{G} \in \mathbf{G}$ subject to the following conditions:

- core(G) is isomorphic to a core of G.
- If $\mathrm{G} \sim \mathrm{H}$ then $\mathrm{core(G)} = \mathrm{core(H)}$.
- If G contains a loop then $\mathrm{core(G)} = \mathring{1}$.
- If $\chi(\mathrm{G}) = n$ and G contains a complete subgraph on n vertices then $\mathrm{core(G)} = \mathrm{K}_n$.
- If $\mathrm{G} = \mathrm{L} + \mathrm{H}$, $\mathrm{core(L)} = \mathrm{K}_n$, $\chi(\mathrm{H}') > n$ for every connected component H' of H, and $\mathrm{K}_n \not\to \mathrm{H}$ then $\mathrm{core(G)} = \mathrm{K}_n + \mathrm{core(H)}$. (That is, we select core(H) so that $V(\mathrm{core(H)}) \cap \underline{n} = \emptyset$.)

Given a graph G be a graph and a positive integer n, following [3], let $\mathcal{C}_n(\mathrm{G})$ be the graph whose vertex set is the set of all functions of $V(G)$ into $\underline{n}$, with two such functions f and g adjacent if for all adjacent vertices a and b of G, $f(a) \neq g(b)$. The vertex f of $\mathcal{C}_n(\mathrm{G})$ is a good (vertex) coloring of G, in the usual sense, if $f(a) \neq f(b)$ for all adjacent vertices a and b of G. Note that f is a good coloring of G if and only if f is a loop of $\mathcal{C}_n(\mathrm{G})$. Also, the chromatic number $\chi(\mathrm{G})$ is the smallest n so that $V(\mathcal{C}_n(\mathrm{G}))$ contains a good coloring of G. Observe that the chromatic number of G is n if and only if $\mathrm{G} \to \mathrm{K}_n$ and $\mathrm{G} \not\to \mathrm{K}_{n-1}$.

The graph $\mathcal{C}_n(\mathrm{G})$ is the "exponential graph" $\mathrm{K}_n^{\mathrm{G}}$ studied in [2] and [7]. Here, we use exponentiation only for the base K_n, so we write $\mathcal{C}_n(\mathrm{G})$ instead. The collection of all these *n-colouring graphs*, for fixed n has an interesting structure. Let

$$\mathfrak{B}_n := \{\mathrm{core}(\mathcal{C}_n(\mathrm{G})) \mid \mathrm{G} \in \mathbf{G}\}.$$

It follows from Theorem 8 in [7] that $\mathfrak{B}_n$ is a Boolean lattice. While we will not need this result,we will make use of several of the properties involving this exponentiation operation, other operations and homomorphisms, obtained in [2], [3], and [7]. In particular, we will use the following construction: for $\mathrm{C} \in \mathfrak{B}_n$, let $\overline{\mathrm{C}} := \mathrm{core}(\mathcal{C}_n(\mathrm{C}))$. For convenience, the results on $\mathfrak{B}_n$ are itemized below.

For two graphs G and H the following hold.

1. If G is connected and has chromatic number larger than n then the graph $\mathcal{C}_n(\mathrm{G})$ contains exactly one complete subgraph on n vertices. ([7], Lemma 4)
2. $\chi(\mathrm{G}) \leq n$ if and only if $\mathrm{G} \to \mathrm{K}_n$.
3. $\mathcal{C}_n(\mathrm{G} + \mathrm{H}) = \mathcal{C}_n(\mathrm{G}) \times \mathcal{C}_n(\mathrm{H})$. ([7], Theorem 8 (8))
4. $\mathrm{G} \to \mathcal{C}_n(\mathcal{C}_n(\mathrm{G}))$. ([7], Theorem 8 (11))
5. $\mathrm{core}(\mathcal{C}_n(\mathrm{G})) = \overset{\circ}{1}$ if and only if $\mathrm{G} \to \mathrm{K}_n$. ([7], Theorem 8 (5))
6. $\mathrm{G} \times \mathrm{H} \to \mathrm{K}_n$ if and only if $\mathrm{H} \to \mathcal{C}_n(\mathrm{G})$. ([7], Theorem 8 (6))
7. If $\mathrm{C} \in \mathfrak{B}_n$ then $\overline{\overline{\mathrm{C}}} = \mathrm{C}$. ([7], Theorem 8 (7))
8. $\mathcal{C}_n(\mathrm{K}_n) \sim \overset{\circ}{1}$ and $\overset{\circ}{1} \times \mathrm{G} \sim \mathrm{G}$. ([7], Theorem 8 (5))

3 The role of K_n in the Boolean lattice $\mathfrak{B}_n$

A graph G is *n-full* if it contains a complete subgraph on n vertices and any two complete subgraphs of G on n vertices are in the same connected component of G. Note that if $\mathrm{G} \not\sim \overset{\circ}{1}$ is n-full then the core of G is also n-full.

Lemma 3.1. *Let* G *be a graph in which every connected component has chromatic number greater than n, where $n \geq 3$. Then $\mathcal{C}_n(\mathrm{G})$ is n-full.*

Proof. The set of constant functions in $\mathcal{C}_n(\mathrm{G})$ induces a complete subgraph of $\mathcal{C}_n(\mathrm{G})$.

Let $\{\mathrm{H}_0, \mathrm{H}_1, \ldots, \mathrm{H}_{r-1}\}$ be the set of connected components of G. Then $\mathrm{G} = \mathrm{H}_0 + \mathrm{H}_1 + \cdots + \mathrm{H}_{r-1}$ which, according to §2.3 (item (3) in §2), implies that $\mathcal{C}_n(\mathrm{G}) = \mathcal{C}_n(\mathrm{H}_0) \times \mathcal{C}_n(\mathrm{H}_1) \times \cdots \times \mathcal{C}_n(\mathrm{H}_{r-1})$. For all $i \in \underline{r}$, the graph $\mathcal{C}_n(\mathrm{H}_i)$ contains exactly one induced subgraph isomorphic to K_n, say L_i, according to §2.1.

Every complete subgraph on n vertices of $\mathcal{C}_n(\mathrm{G})$ has to be a subgraph of $\mathrm{L} := \mathrm{L}_0 \times \mathrm{L}_1 \times \cdots \times \mathrm{L}_{r-1}$. The result follows because L is connected – since $n \geq 3$, it actually has diameter two. $\square$

Theorem 3.1. *Let* $\mathrm{C} \in \mathfrak{B}_n$ *with* $n \geq 3$ *and* $\overset{\circ}{1} \neq \mathrm{C} \neq \mathrm{K}_n$*. Then* C *is* n*-full.*

Proof. For any graph G, the set of constant functions in $\mathcal{C}_n(\mathrm{G})$ induces a complete n-vertex subgraph of $\mathcal{C}_n(\mathrm{G})$. Note that $\overset{\circ}{1} \neq \overline{\mathrm{C}} \neq \mathrm{K}_n$ according to item §2.7. So, the cores C and $\overline{\mathrm{C}}$ each have all connected components of chromatic number at least n, at most one with chromatic number equal to n and, if so, that component must contain K_n. We distinguish two cases.

Case 1: There is a connected component N of $\overline{\mathrm{C}}$ which contains a complete subgraph of n vertices and has chromatic number equal to n. Then, because $\overline{\mathrm{C}}$ is a core, the connected component N is equal to K_n and every other connected component of $\overline{\mathrm{C}}$ has chromatic number larger than n. Hence $\overline{\mathrm{C}} = \mathrm{K}_n + \mathrm{H}$ where every connected component of H has chromatic number larger than n. We obtain, according to §2.7, §2.3 and §2.5, that:

$$\mathrm{C} \sim \mathcal{C}_n(\overline{\mathrm{C}}) = \mathcal{C}_n(\mathrm{K}_n + \mathrm{H}) \sim \mathcal{C}_n(\mathrm{K}_n) \times \mathcal{C}_n(\mathrm{H}) \sim \overset{\circ}{1} \times \mathcal{C}_n(\mathrm{H}) \sim \mathcal{C}_n(\mathrm{H}).$$

Lemma 3.1 yields that C is n-full.

Case 2: Every connected component of $\overline{\mathrm{C}}$ has chromatic number larger than n. In this case, Lemma 3.1 immediately shows that C is n-full and the theorem follows. $\square$

The elements of $\mathfrak{B}_n \setminus \{\overset{\circ}{1}, \mathrm{K}_n\}$ of the form $\mathrm{K}_n + \mathrm{H}$ are called the *non-joined* elements of $\mathfrak{B}_n$ and all the other elements of $\mathfrak{B}_n \setminus \{\overset{\circ}{1}, \mathrm{K}_n\}$ are the *joined* elements of $\mathfrak{B}_n$.

Lemma 3.2. *The number* n *has is semi-stable if and only if no element in* $\mathfrak{B}_n \setminus \{\overset{\circ}{1}, \mathrm{K}_n\}$ *is joined.*

Proof. Let n be semi-stable and assume for a contradiction that the element $\mathrm{C} \in \mathfrak{B}_n$ with $\overset{\circ}{1} \neq \mathrm{C} \neq \mathrm{K}_n$ is joined. Then C has a connected component N containing a complete subgraph of n vertices and with $\chi(\mathrm{N}) > n$. Since $\overline{\mathrm{C}} \neq \mathrm{K}_n$, $\chi(\overline{\mathrm{C}}) > n$and C has a connected component H with $\chi(\mathrm{H}) > n$. Then $\chi(\mathrm{N} \times \mathrm{H}) \geq n + 1$ because n is semistable. But there is always a good n-coloring of $\mathrm{C} \times \overline{\mathrm{C}}$, according to §2.6, so

$$n \geq \chi(\mathrm{C} \times \overline{\mathrm{C}}) \geq \chi(\mathrm{N} \times \mathrm{H}) \geq n + 1,$$

a contradiction.

Suppose that no element in $\mathfrak{B}_n \setminus \{\mathring{1}, \mathrm{K}_n\}$ is joined. Let G be a connected graph with $\chi(\mathrm{G}) = n+1$ and which contains a complete subgraph on n vertices. Then $\mathrm{G} \to \mathrm{core}(\mathcal{C}_n(\mathcal{C}_n(\mathrm{G}))) := \mathrm{C} \in \mathfrak{B}_n$ according to §2.4. Hence $\mathrm{C} = \mathring{1}$ because otherwise C would be joined. It follows from §2.5 that $\chi(\mathcal{C}_n(\mathrm{G})) = n$.

In order to show that G is stable, let H be a graph with $\chi(\mathrm{H}) = n+1$. If $\chi(\mathrm{G} \times \mathrm{H}) \to n$ then $\mathrm{H} \to \mathcal{C}_n(\mathrm{G})$, according to §2.6, which contradicts $\chi(\mathcal{C}_n(\mathrm{G})) = n$.

□

Corollary 3.1. *The number n is semi-stable if and only if every element $\mathrm{C} \in \mathfrak{B}_n \setminus \{\mathring{1}, \mathrm{K}_n\}$ is of the form $\mathrm{C} = \mathrm{K}_n + \mathrm{H}$ where $\chi(\mathrm{H}) \geq n+1$ and H does not contain a complete subgraph on n vertices.*

Corollary 3.2. *The number n is semi-stable if and only if $\chi(\mathcal{C}_n(\mathrm{G})) = n$ for every connected graph G of chromatic number at least $n+1$ which contains a complete subgraph on n vertices.*

Proof. Let n be semi-stable. It follows from §2.4 that $\mathrm{G} \to \mathrm{core}\big(\mathcal{C}_n(\mathcal{C}_n(\mathrm{G}))\big) :=$ C. This implies that $\mathrm{C} \neq \mathrm{K}_n$ and that C is joined unless $\mathrm{C} = \mathring{1}$. Hence it follows from Lemma 3.2 that $\mathrm{C} = \mathring{1}$. According to §2.5 $\mathrm{C} = \mathring{1}$ implies that $\chi(\mathcal{C}_n(\mathrm{G})) = n$.

To prove the converse, let G be a connected graph of chromatic number at least $n+1$ and assume G contains a complete subgraph on n vertices. So, $\chi(\mathcal{C}_n(\mathrm{G})) = n$. Let H be a graph with $\chi(\mathrm{H}) = n+1$. If $\chi(\mathrm{G} \times \mathrm{H}) \to n$ then $\mathrm{H} \to \mathcal{C}_n(\mathrm{G})$, according to §2.6, in contradiction to $\chi(\mathcal{C}_n(\mathrm{G})) = n$.

□

4 The main results

Lemma 4.1. *Let n be semi-stable and let $\mathrm{C} = \mathrm{K}_n + \mathrm{H} \in \mathfrak{B}_n \setminus \{\mathring{1}, \mathrm{K}_n\}$. Then $\chi(\overline{\mathrm{C}}) \leq 2n$. If H contains a triangle then $\chi(\overline{\mathrm{C}}) \leq n+1$.*

Proof. If H does not contain a triangle assume, for a contradiction, that $\chi(\overline{\mathrm{C}}) \geq 2n+1$. Then $\chi(\mathcal{C}_n(\mathrm{C})) \geq 2n+1$ because $\overline{\mathrm{C}}$ is the core of $\mathcal{C}_n(\mathrm{C})$. If H contains a triangle assume for a contradiction that $\chi(\overline{\mathrm{C}}) \geq n+2$ in which case $\chi(\mathcal{C}_n(\mathrm{C})) \geq n+2$.

Let $\{\pi_i \mid i \in \underline{n!}\}$ be the set of good colorings of K_n and for every $i \in \underline{n!}$ let M_i be the induced subgraph of $\mathcal{C}_n(\mathrm{C})$ where $V(\mathrm{M}_i)$ is the set of functions

$f \in V(\mathcal{C}_n(\mathrm{C}))$ whose restriction to $V(\mathrm{K}_n)$ is π_i. We may assume without loss of generality that $\pi_0(0) = 0$. Let N be the induced subgraph of $\mathcal{C}_n(\mathrm{C})$ where $V(\mathrm{N})$ is the set of functions in $\mathcal{C}_n(\mathrm{C})$ whose restriction to $V(\mathrm{K}_n)$ is not a good coloring of K_n.

Note that if $f \in V(\mathrm{M}_i)$ and $g \in V(\mathcal{C}_n(\mathrm{C})) \setminus V(\mathrm{M}_i)$ then there are vertices a and b in $V(\mathrm{K}_n)$ so that $a \neq b$ and $f(a) = g(b)$. Hence f and g are not adjacent in $\mathcal{C}_n(\mathrm{C})$ and therefore:

$$\mathcal{C}_n(\mathrm{C}) = \mathrm{N} + \sum_{i \in \underline{n!}} \mathrm{M}_i.$$

We claim that $\chi(\mathrm{N}) \leq n$. For every $f \in V(\mathrm{N})$ there are distinct vertices x_f and y_f in $V(\mathrm{K})$ with $f(x_f) = f(y_f)$. Let $\alpha(f) = f(x_f)$. Then α is a good n-coloring of N. Any two graphs M_i and M_j for $i, j \in \underline{n!}$ are isomorphic under the isomorphism which maps $f \in \mathrm{M}_i$ to $\pi_j \circ \pi_i^{-1} \circ f \in \mathrm{M}_j$. It follows that $\chi(\mathrm{M}_0) \geq 2n+1$ if H does not contain a triangle and $\chi(\mathrm{M}_0) \geq n+2$ if H does contain a triangle.

For every $f \in V(\mathrm{M}_0)$ let f' be the restriction of f to $V(\mathrm{H})$. Note that the functions $f, g \in V(\mathrm{M}_0)$ are adjacent if and only if the functions $f', g' \in V(\mathcal{C}_n(\mathrm{H}))$ are adjacent and that $V(\mathcal{C}_n(\mathrm{H})) = \{f' \mid f \in V(\mathrm{M}_0)\}$. It follows that $\chi(\mathcal{C}_n(\mathrm{H})) \geq 2n+1$ if H does not contain a triangle and $\chi(\mathcal{C}_n(\mathrm{H})) \geq n+2$ if H does contain a triangle.

Select a vertex $b \in V(\mathrm{H})$ and let S be the set of those vertices of H adjacent to b. Note that since C is a core, S is not empty. Also, if H contains a triangle, select b as one of its vertices and let c and d be the others.

For every $i \in \underline{n}$ let B_i be the set of functions f in $V(\mathrm{M}_0)$ for which $f(s) = i$ for all $s \in S$. Let $B_i' = \{f' \mid f \in B_i\}$, let B_i be the subgraph of M_0 induced by B_i and B_i' the subgraph of $\mathcal{C}_n(\mathrm{H})$ induced by B_i'. Let $A_i = V(\mathrm{M}_0) \setminus B_i$, let $A_i' = V(\mathcal{C}_n(\mathrm{H})) \setminus B_i'$, let A_i be the subgraph of M_0 induced by A_i and A_i' the subgraph of $\mathcal{C}_n(\mathrm{H})$ induced by A_i'.

Then $\chi(\mathrm{A}_i) = \chi(\mathrm{A}_i')$ and $\chi(\mathrm{B}_i') = \chi(\mathrm{B}_j')$ for all $i, j \in \underline{n}$. The second assertion follows because B_i' is isomorphic to B_j' under the transposition of $\underline{n}$ which switches i with j. It follows from this and the fact that B_1' is an induced subgraph of A_0' that $\mathrm{B}_0' \to \mathrm{A}_0'$. This implies that if $\chi(\mathrm{A}_0') \leq n$ then $\chi(\mathcal{C}_n(\mathrm{H})) \leq 2n$. Hence, we may assume that $\chi(\mathrm{A}_0) = \chi(\mathrm{A}_0') \geq n+1$ in the case that H does not contain a triangle.

If H contains a triangle then $f(c) = f(d) = 0$ for all $f \in B_0$. Hence no two functions $f, g \in B_0$ are adjacent. It follows that $\chi(\mathrm{M}_0) \leq \chi(\mathrm{A}_0) + 1$. Thus, also in the case that H does contain a triangle, $\chi(\mathrm{A}_0) \geq n+1$.

It follows from the definition of A_0 that for every $f \in A_0$ there is an element $s \in S$ so that $f(s) \neq 0$. Select such a vertex s and let $\beta(f) = s$. Note that $f(\beta(f)) \neq 0$ for all $f \in A_0$. Let a be a vertex with $a \notin V(\mathrm{C})$. Let D be the graph with $V(\mathrm{D}) = V(\mathrm{C}) \cup \{a\}$ and $E(\mathrm{D}) = E(\mathrm{C}) \cup \{\{0, a\}, \{a, b\}\}$.

We associate with every function $f \in A_0 = V(\mathrm{A}_0)$ a function $f^* \in \mathcal{C}_n(\mathrm{D})$ so that

$$f^*(x) = \begin{cases} f(x), & \text{if } x \in V(\mathrm{C}); \\ f(\beta(f)), & \text{if } x = a. \end{cases}$$

Let $A_0^* = \{f^* \mid f \in A_0\}$ and A_0^* the subgraph of $\mathcal{C}_n(\mathrm{D})$ induced by A_0^*. Let f and g be two adjacent functions in A_0. Then f^* and g^* are adjacent as well. The only two edges to check are $\{a, b\}$ and $\{0, a\}$. If $f^*(a) = g^*(b)$ then $f(\beta(f)) = g(b)$ which can not be the case because f and g are adjacent and b and $\beta(f)$ are are adjacent. If $f^*(0) = g^*(a)$ then $0 = g(\beta(g))$. But $g \in A_0$ and hence $g(\beta(g)) \neq 0$.

It follows that $n + 1 \leq \chi(\mathrm{A}_0) \leq \chi(\mathrm{A}_0^*) \leq \chi(\mathcal{C}_n(\mathrm{D}))$. According to §2.5, $\mathcal{C}_n(\mathcal{C}_n(\mathrm{D})) \not\sim \mathring{1}$. Let $\mathrm{R} = \mathrm{core}\big(\mathcal{C}_n(\mathcal{C}_n(\mathrm{D}))\big)$. There is a homomorphism of D into $\mathcal{C}_n(\mathcal{C}_n(\mathrm{D}))$ according to §2.4 and hence there is a homomorphism of D into $\mathrm{R} \in \mathfrak{B}_n$. But this homomorphism and the construction of D show that R is joined. Lemma 3.2 shows that this contradicts the assumption that n is semi-stable.

□

Corollary 4.1. *If n is semi-stable and $\mathrm{C} \in \mathfrak{B}_n$ and $\mathrm{C} \neq \mathring{1}$ then $\chi(\mathrm{C}) \leq 2n$.*

Proof. If $\mathrm{C} = \mathrm{K}_n$ the result follows trivially. If $\mathring{1} \neq \mathrm{C} \neq \mathrm{K}_n$ then $\mathring{1} \neq \overline{\mathrm{C}} \neq \mathrm{K}_n$ according to §2.7. Hence $\overline{\mathrm{C}} = \mathrm{K}_n + \mathrm{H}$ for some graph H because of the assumption that n has the weak Hedetniemi property. Because $\mathrm{C} = \overline{\overline{\mathrm{C}}}$ the corollary follows from Lemma 4.1. □

Theorem 4.1. *Let n be semi-stable and let G be a graph with $\chi(\mathrm{G}) \geq n+1$. Then $\chi(\mathrm{G} \times \mathrm{H}) \geq n + 1$ for every graph H with $\chi(\mathrm{H}) \geq 2n + 1$.*

Proof. Assume for a contradiction that $\chi(\mathrm{G} \times \mathrm{H}) \leq n$. Then there is a homomorphism of H into $\mathcal{C}_n(\mathrm{G})$ according to §2.6 and hence a homomorphism of H into $\mathrm{core}(\mathcal{C}_n(\mathrm{G})) \in \mathfrak{B}_n$ which implies that $\chi(\mathrm{core}(\mathcal{C}_n(\mathrm{G}))) \geq 2n + 1$. According to §2.5, it follows from $\chi(\mathrm{G}) > n$ that $\mathrm{core}(\mathcal{C}_n(\mathrm{G})) \neq \mathring{1}$ and hence we arrived at a contradiction to Corollary 4.1. □

Theorem 4.2. *Let n be semi-stable and let G be a connected graph which contains a triangle. If $\chi(\mathrm{G}) \geq n + 1$ then $\chi(\mathrm{G} \times \mathrm{H}) \geq n + 1$ for every graph H with $\chi(\mathrm{H}) \geq n + 2$.*

Proof. Assume for a contradiction that $\chi(\mathrm{G} \times \mathrm{H}) \leq n$. Then, according to §2.6, there is a homomorphism of H into $\mathcal{C}_n(\mathrm{G})$ and hence a homomorphism of H into $\mathrm{core}(\mathcal{C}_n(\mathrm{G})) := \mathrm{C} \in \mathfrak{B}_n$. Hence $\chi(\mathrm{C}) \geq n+2$.

There is a homomorphism of G into $\mathrm{core}(\mathcal{C}_n(\mathcal{C}_n(\mathrm{G})) = \overline{\mathrm{C}}$ according to §2.4. Hence $\overline{\mathrm{C}}$ contains a triangle in a connected component which has chromatic number larger than n. Also $\overline{\mathrm{C}} \neq \overset{\circ}{1}$ because the chromatic number of C is larger than n and $\overline{\mathrm{C}} \neq \mathrm{K}_n$ because its chromatic number is larger than n. Hence $\overline{\mathrm{C}} = \mathrm{K}_n + \mathrm{L}$ for some graph L because n is semi-stable. Also, since $\overline{\mathrm{C}}$ contains a triangle in a connected component of chromatic number larger than n, the graph L contains a triangle. Hence we can apply Lemma 4.1 to obtain that $\chi(\overline{\overline{\mathrm{C}}}) \leq n+1$. But this is a contradiction because $\overline{\overline{\mathrm{C}}} = \mathrm{C}$ according to §2.7. □

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