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Spectral Properties of the Hermitian and Skew-Hermitian Splitting Preconditioner for Saddle Point Problems

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SPECTRAL PROPERTIES OF THE HERMITIAN AND SKEW-HERMITIAN SPLITTING PRECONDITIONER FOR SADDLE POINT PROBLEMS*

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Abstract. In this paper we derive bounds on the eigenvalues of the preconditioned matrix that arises in the solution of saddle point problems when the Hermitian and Skew-Hermitian splitting preconditioner is employed. We also give sufficient conditions for the eigenvalues to be real. A few numerical experiments are used to illustrate the quality of the bounds.

Key words. saddle point problems, iterative methods, preconditioning, eigenvalues

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1. Introduction.

We are given the saddle point problem

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \qquad \Leftrightarrow \qquad \mathcal{A}x = b. \tag{1.1}$$

with $A \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite and $B \in \mathbb{R}^{m\times n}$ with $\text{rank}(B) = m \leq n$. We assume that the null spaces of A and B have trivial intersection, which implies that $\mathcal{A}$ is nonsingular. We set

$$\mathcal{H} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{S} = \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix},$$

so that $\mathcal{A} = \mathcal{H}+\mathcal{S}$. We consider the preconditioner $\mathcal{P} = (2\alpha)^{-1}(\mathcal{H}+\alpha I)(\mathcal{S}+\alpha I)$, with real $\alpha > 0$, and we study the eigenvalue problem associated with the preconditioned matrix, that is

$$(\mathcal{H}+\mathcal{S})x = \eta(2\alpha)^{-1}(\mathcal{H}+\alpha I)(\mathcal{S}+\alpha I)x. \tag{1.2}$$

This preconditioner has been studied in a somewhat more general setting in [4], motivated by the paper [1]. Letting $D(1,1) := \{z \in \mathbb{C};\ |z-1| < 1\}$, it was shown in [4] that the spectrum of the preconditioned matrix satisfies $\sigma(\mathcal{P}^{-1}\mathcal{A}) \subset \overline{D(1,1)}\backslash\{0\}$. Furthermore, $\sigma(\mathcal{P}^{-1}\mathcal{A}) \subset D(1,1)$ if A is positive definite. Some rather special cases (including the case $A = I$) have been studied in [2, 3]. The purpose of this paper is to provide more refined inclusion regions for the spectrum of $\mathcal{P}^{-1}\mathcal{A}$ for saddle point problems of the form (1.1). Most of our bounds are in terms of the extreme eigenvalues and singular values of the blocks A and B, respectively. Although these quantities may be difficult to estimate, our results can still be used to guide the choice of the parameter α. For instance, we show that sufficiently small values of α always result in preconditioned matrices having a real spectrum consisting of two tight clusters.

Throughout the paper, we write M^T for the transpose of a matrix M and u^* for the conjugate transpose of a complex vector u. Also, $A > 0$ ($A \geq 0$) means that matrix A is symmetric positive definite (respectively, semidefinite).

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2. Spectral bounds. In this section we provide bounds for the eigenvalues of the preconditioned matrix.

In the following we shall use the fact that A is symmetric positive semidefinite, so that

$$0 \le \lambda_n \le \frac{u^* A u}{u^* u} \le \lambda_1, \qquad \forall\, u \in \mathbb{C}^n,\, u \neq 0, \tag{2.1}$$

where λ_n, λ_1 are the smallest and largest eigenvalues of A, respectively. Moreover, we denote by $\sigma_1, \ldots, \sigma_m$ the decreasingly ordered singular values of B.

The spectrum of the preconditioned matrix can be more easily analyzed by means of a particular spectral mapping, which we introduce next. We shall then derive estimates for the location of the eigenvalues of (1.2).

We first observe that $(\mathcal{H} + \alpha I)(\mathcal{S} + \alpha I) = \mathcal{H}\mathcal{S} + \alpha(\mathcal{H} + \mathcal{S}) + \alpha^2 I$. By collecting the terms with $(\mathcal{H} + \mathcal{S})$ we can write the eigenvalue problem (1.2) as

$$\left(1 - \frac{1}{2}\eta\right)(\mathcal{H} + \mathcal{S})x = \frac{\eta\alpha}{2}\left(I + \frac{1}{\alpha^2}\mathcal{H}\mathcal{S}\right)x. \tag{2.2}$$

If $1 - \frac{1}{2}\eta = 0$ then $\eta = 2$. For $1 - \frac{1}{2}\eta \neq 0$ we set

$$\theta := \frac{\eta\alpha}{2 - \eta}, \qquad \text{from which} \qquad \eta = 2 - \frac{2\alpha}{\theta + \alpha}. \tag{2.3}$$

Therefore, (2.2) can be written as $(\mathcal{H} + \mathcal{S})x = \theta\left(I + \frac{1}{\alpha^2}\mathcal{H}\mathcal{S}\right)x$.

By explicitly writing the term $\mathcal{H}\mathcal{S}$, the eigenproblem above becomes

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} x = \theta \begin{pmatrix} I & \frac{1}{\alpha^2} A B^T \\ 0 & I \end{pmatrix} x \qquad \Leftrightarrow \qquad \mathcal{A}x = \theta \mathcal{G} x.$$

The equivalent eigenproblem $\mathcal{G}^{-1}\mathcal{A}x = \theta x$ can be explicitly written as

$$\begin{pmatrix} A + \frac{1}{\alpha^2} A B^T B & B^T \\ -B & 0 \end{pmatrix} x = \theta x. \tag{2.4}$$

Therefore, the two eigenproblems (1.2) and (2.4) have the same eigenvectors, while the eigenvalues are related by (2.3). Our spectral analysis aims at describing the behavior of the spectrum of $\mathcal{G}^{-1}\mathcal{A}$, from which considerations on the spectrum of (1.2) can be derived.

LEMMA 2.1. *Assume A is symmetric and positive semidefinite. Let $K = I + \frac{1}{\alpha^2} B^T B$. For each eigenpair $(\eta, [u; v])$ of (1.2), η is either $\eta = 2$ or it can be written as $\eta = 2 - \frac{2\alpha}{\alpha + \theta}$ where $\theta \neq 0$ satisfies the following:*

1. *If $\Im(\theta) \neq 0$ then*

$$\Re(\theta) = \frac{1}{2}\frac{u^* K A K u}{u^* K u}, \qquad |\theta|^2 = \frac{u^* K B^T B u}{u^* K u}. \tag{2.5}$$

2. *If $\Im(\theta) = 0$ then*

$$\min\left\{\lambda_n, \frac{\sigma_m^2}{\lambda_1\left(1 + \frac{\sigma_1^2}{\alpha^2}\right)}\right\} \le \theta \le \rho.$$

Proof. The first statement of the lemma was already shown by means of the mapping in (2.3). We are thus left with proving the estimates for θ. First of all, note that $\theta \neq 0$ or else $\eta = 0$, which is not possible since $\mathcal{P}^{-1}\mathcal{A}$ is nonsingular.

Let $x = [u; v] \neq 0$ be the complex eigenvector associated with θ. We explicitly observe that $K = I + \frac{1}{\alpha^2}B^TB$ is symmetric positive definite and that KB^TB is symmetric. We shall make use of the following properties of K,

$$\lambda_{\max}(K) = 1 + \frac{\sigma_1^2}{\alpha^2}, \qquad \lambda_{\min}(K) \geq 1, \tag{2.6}$$

where the inequality becomes an equality whenever B is not square. In addition,

$$\lambda_n \leq \frac{u^*KAKu}{u^*K^2u} \leq \lambda_1, \tag{2.7}$$

and using $KB^TB = \alpha^2(K^2 - K)$,

$$0 \leq \frac{u^*KB^TBu}{u^*K^2u} = \alpha^2\frac{u^*K^2u - u^*Ku}{u^*K^2u} = \alpha^2\left(1 - \frac{u^*Ku}{u^*K^2u}\right) \leq \alpha^2, \quad \forall u \neq 0. \tag{2.8}$$

The two matrix equations in (2.4) are given by

$$\left(A + \frac{1}{\alpha^2}AB^TB\right)u + B^Tv = \theta u \tag{2.9}$$

$$-Bu = \theta v. \tag{2.10}$$

It must be $u \neq 0$ otherwise (2.10) would imply $\theta = 0$ or $v = 0$, neither of which can be satisfied. For $u \neq 0$ and $v = 0$, from (2.9), θ must satisfy $AKu = \theta u$ and $Bu = 0$. Since K is symmetric and positive definite, we can write $K^{\frac{1}{2}}AK^{\frac{1}{2}}\hat{u} = \theta\hat{u}$, $\hat{u} = K^{\frac{1}{2}}u$, from which it follows that θ is real and satisfies $0 < \theta \leq \lambda_1\|K^{\frac{1}{2}}\|^2 = \lambda_1\lambda_{\max}(I + \frac{1}{\alpha^2}B^TB)$.

We now assume $u \neq 0 \neq v$. Using (2.10), we write $v = -\theta^{-1}Bu$, which, substituted into (2.9), yields $\theta A(I + \frac{1}{\alpha^2}B^TB)u - B^TBu = \theta^2 u$. By multiplying this equation from the left by u^*K we obtain

$$\theta u^*KAKu - u^*KB^TBu = \theta^2 u^*Ku. \tag{2.11}$$

Let $\theta = \theta_1 + \imath\theta_2$. For A symmetric, the quadratic equation (2.11) has real coefficients so that its roots are given by

$$\theta_\pm = \frac{1}{2}\frac{u^*KAKu}{u^*Ku} \pm \sqrt{\frac{1}{4}\left(\frac{u^*KAKu}{u^*Ku}\right)^2 - \frac{u^*KB^TBu}{u^*Ku}}. \tag{2.12}$$

Eigenvalues with nonzero imaginary part arise if the discriminant is negative.

Case $\theta_2 \neq 0$. It must be

$$(u^*KAKu)^2 - 4(u^*Ku)(u^*KB^TBu) < 0, \tag{2.13}$$

and from (2.12) we get $\theta_1 = \frac{1}{2}\frac{u^*KAKu}{u^*Ku}$. By substituting θ_1 in (2.12), we obtain $\theta_2^2 + \theta_1^2 = \frac{u^*KB^TBu}{u^*Ku}$.

Case $\theta_2 = 0$. In this case, from (2.12) it follows that $\theta = \theta_1 > 0$. For $Bu = 0$, from (2.10) it follows $v = 0$ ($\theta \neq 0$) and the reasoning for $v = 0$ applies.

We now assume that $Bu \neq 0$. We have

$$-\theta_1^2 u^* K u + \theta_1 u^* KAKu = u^* KB^T Bu > 0,$$

where the last inequality follows from (2.8). Since $\theta_1 > 0$, the inequality $\theta_1 u^* KAKu - \theta_1^2 u^* Ku > 0$ is satisfied for $u^* KAKu - \theta_1 u^* Ku > 0$, hence $\theta_1 < \lambda_1 \lambda_{\max}(K) \equiv \rho$.

To prove the lower bound on θ, write the equation (2.9) as $(AK - \theta I)u = -B^T v$. If θ is an eigenvalue of AK, then $\theta \geq \lambda_n \lambda_{\min}(K) \geq \lambda_n$. Otherwise, $(AK - \theta I)$ is invertible, so that $u = -(AK - \theta I)^{-1} B^T v$, which, substituted into (2.11) yields

$$B(AK - \theta I)^{-1} B^T v = \theta v \qquad \Leftrightarrow \qquad BK^{-1}(A - \theta K^{-1})^{-1} B^T v = \theta v. \tag{2.14}$$

Let $B^T = [W_1, W_2] \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T$ be the singular value decomposition of B^T, and note that

$$K = [W_1, W_2] \begin{pmatrix} I + \frac{1}{\alpha^2}\Sigma^2 & 0 \\ 0 & I \end{pmatrix} [W_1^T, W_2^T]^T,$$
$$BK^{-1} = Q \begin{pmatrix} \Sigma(I + \frac{1}{\alpha^2}\Sigma^2)^{-1} & 0 \end{pmatrix} [W_1^T, W_2^T]^T \equiv QD^{-1}\Sigma W_1^T,$$

where $D = I + \frac{1}{\alpha^2}\Sigma^2$. Problem (2.14) can be thus written as $QD^{-1}\Sigma W_1^T(A - \theta K^{-1})^{-1} W_1 \Sigma Q^T v = \theta v$, or, equivalently,

$$\Sigma W_1^T (A - \theta K^{-1})^{-1} W_1 \Sigma w = \theta D w, \qquad w = Q^T v. \tag{2.15}$$

We multiply both sides from the left by w^* and we notice that the left-hand side is positive for any $w \neq 0$. If $\theta \geq \lambda_{\min}(AK) \geq \lambda_n$, then λ_n is the sought after lower bound. Assume now that $\theta < \lambda_{\min}(AK)$. Then, the matrix $A - \theta K^{-1}$ is symmetric and positive definite. Therefore,

$$\begin{aligned} w^*\Sigma W_1^T (A - \theta K^{-1})^{-1} W_1 \Sigma w &\geq \lambda_{\min}((A - \theta K^{-1})^{-1}) \|W_1 \Sigma w\|^2 \\ &\geq \lambda_{\min}((A - \theta K^{-1})^{-1}) \sigma_m^2 \|w\|^2, \end{aligned} \tag{2.16}$$

and we have

$$\begin{aligned} \lambda_{\min}((A - \theta K^{-1})^{-1}) &= \frac{1}{\lambda_{\max}(A - \theta K^{-1})} \geq \frac{1}{\lambda_1 - \theta \lambda_{\min}(K^{-1})} \\ &= \frac{1}{\lambda_1 - \frac{\theta}{\lambda_{\max}(K)}} = \frac{1}{\lambda_1 - \frac{\theta}{\tau}}, \end{aligned}$$

where $\tau = \lambda_{\max}(K) = \left(1 + \frac{\sigma_1^2}{\alpha^2}\right)$. This, together with (2.16), provides a lower bound for the left-hand side of (2.15). Using $\theta w^* D w \leq \theta \tau \|w\|^2$ and recalling that $\lambda_1 \tau - \theta > 0$, from (2.15) we obtain

$$\frac{\sigma_m^2}{\lambda_1 - \frac{\theta}{\tau}} \leq \theta \tau \qquad \Leftrightarrow \qquad \theta^2 + \sigma_m^2 \leq \tau \lambda_1 \theta.$$

Since $\theta^2 > 0$, we get $\sigma_m^2 \leq \tau \lambda_1 \theta$, and the final bound follows. □

The quantities in part 1 of the lemma can be also bounded with techniques similar to those for the real case. However, in the next theorem, we derive sharper bounds for complex η than those one would obtain by using estimates for complex θ.

THEOREM 2.2. *Under the hypotheses and notation of Lemma 2.1, the eigenvalues of problem (1.2) are such that*

1. If $\Im(\eta) \neq 0$ *then*

$$\frac{(\alpha + \frac{1}{2}\lambda_n)\lambda_n}{3\alpha^2} < \Re(\eta) < \min\left\{2, \frac{4\alpha}{\alpha+\lambda_n}\right\} \tag{2.17}$$

$$\frac{\lambda_n^2}{3\alpha^2 + \frac{1}{4}\lambda_n^2} < |\eta|^2 \leq \frac{4\alpha}{\alpha + \alpha(1+\frac{\sigma_1^2}{\alpha^2})^{-1} + \lambda_n}. \tag{2.18}$$

2. If $\Im(\eta) = 0$ *then* $\eta > 0$ *and*

$$\min\left\{\frac{2\lambda_n}{\alpha+\lambda_n}, \frac{2\frac{\sigma_m^2}{\rho}}{\alpha + \frac{\sigma_m^2}{\rho}}\right\} \leq \eta \leq \frac{2\rho}{\alpha+\rho} < 2 \tag{2.19}$$

where $\rho = \lambda_1(1+\frac{\sigma_1^2}{\alpha^2})$.

Proof. We have that η is real if and only if θ is real. Assume $\Im(\eta) \neq 0$ and write $\theta = \theta_1 + \imath\theta_2$. Let also $\tau = (1+\frac{\sigma_1^2}{\alpha^2})$.

Using the definition of θ in (2.3) we obtain

$$\Re(\eta) = 2\frac{\alpha\theta_1 + |\theta|^2}{\alpha^2 + 2\alpha\theta_1 + |\theta|^2},$$

that is, $(\alpha^2 + 2\alpha\theta_1 + |\theta|^2)\Re(\eta) = 2\alpha\theta_1 + 2|\theta|^2$. We substitute the quantities in (2.5) to get $(\alpha^2 u^*Ku + \alpha u^*KAKu + u^*KB^TBu)\Re(\eta) = \alpha u^*KAKu + 2u^*KB^TBu$. Note that $\alpha^2 u^*Ku + u^*KB^TBu = \alpha^2 u^*K^2u$. We divide by $x^*K^2u > 0$ to obtain

$$\left(\alpha^2 + \alpha\frac{u^*KAKu}{u^*K^2u}\right)\Re(\eta) = \alpha\frac{u^*KAKu}{u^*K^2u} + 2\frac{u^*KB^TBu}{u^*K^2u}.$$

We recall that for $\Im(\eta) \neq 0$ relation (2.13) holds, which implies

$$\frac{(u^*KAKu)^2}{(u^*K^2u)^2} < 4\frac{(u^*Ku)}{u^*K^2u}\frac{(u^*KB^TBu)}{u^*K^2u} \leq 4\alpha^2, \tag{2.20}$$

and

$$\frac{(u^*KB^TBu)}{u^*K^2u} > \frac{1}{4}\frac{(u^*KAKu)^2}{(u^*K^2u)^2}\frac{(u^*K^2u)}{u^*Ku} \geq \frac{1}{4}\lambda_n^2. \tag{2.21}$$

Therefore, by applying (2.7), (2.20) and (2.8) we obtain

$$(\alpha^2 + \alpha\lambda_n)\Re(\eta) < \alpha(2\alpha) + 2\alpha^2 \qquad \Leftrightarrow \qquad \Re(\eta) < \frac{4\alpha}{\alpha+\lambda_n}.$$

By applying once more (2.20), (2.7) and (2.21), we also get

$$(\alpha^2 + \alpha(2\alpha))\Re(\eta) > \alpha\lambda_n + \frac{1}{2}\lambda_n^2 \qquad \Leftrightarrow \qquad \Re(\eta) > \frac{(\alpha+\frac{1}{2}\lambda_n)\lambda_n}{3\alpha^2},$$

which provide the upper and lower bounds for $\Re(\eta)$.

To complete the proof of the first statement, we write $|\eta|^2$ using (2.3), to obtain

$$(\alpha^2 + 2\alpha\theta_1)|\eta|^2 = (4 - |\eta|^2)|\theta|^2.$$

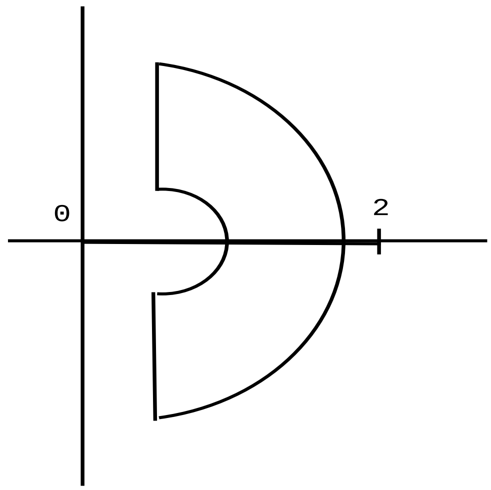

FIG. 2.1. *Inclusion region for the typical spectrum of the preconditioned matrix.*

Substituting (2.5) as before and dividing by u^*K^2u, it yields

$$\left(\alpha^2 \frac{u^*Ku}{u^*K^2u} + \alpha \frac{u^*KAKu}{u^*K^2u}\right) |\eta|^2 = (4 - |\eta|^2) \frac{u^*KB^TBu}{u^*K^2u}.$$

Note that $4 - |\eta|^2 > 0$. As before, we bound $|\eta|^2$ from both sides, keeping in mind (2.6), (2.7), (2.8), (2.21) and (2.20), to get

$$(\frac{1}{\tau}\alpha^2 + \alpha\lambda_n)|\eta|^2 \leq 4\alpha^2 - |\eta|^2\alpha^2 \qquad \Leftrightarrow \qquad |\eta|^2 \leq \frac{4\alpha}{\alpha + \alpha(1 + \frac{\sigma_1^2}{\alpha^2})^{-1} + \lambda_n},$$

and

$$(\alpha^2 + \alpha(2\alpha))|\eta|^2 > \frac{1}{4}\lambda_n^2(4 - |\eta|^2) \qquad \Leftrightarrow \qquad |\eta|^2 > \frac{\lambda_n^2}{3\alpha^2 + \frac{1}{4}\lambda_n^2}.$$

This completes the proof of the first part.

Assume now that η is real. Then, from the corresponding bound for real θ in Lemma 2.1 and the fact that $\eta = \phi(\theta) = \frac{2\theta}{\alpha+\theta}$ is a strictly increasing function of its argument, we obtain the desired bounds on η. □

A few comments are in order. We start by noticing that in general, real eigenvalues η may well cover the whole open interval $(0, 2)$, depending on the parameter α. Our numerical experiments show that these bounds are indeed sharp for several values of α (cf. section 4).

Although much less sharp in general, we also found the bounds for eigenvalues with nonzero imaginary part of interest. The lower estimate for $|\eta|$ indicates that nonreal eigenvalues are not close to the origin, especially for small α. In addition, they are located in a section of an annulus as in Figure 2.1. It is also apparent from the proof, cf. the bound in (2.20), that complex eigenvalues cannot arise for values of α smaller than one half the smallest eigenvalue of A. We shall formalize this statement in Theorem 3.1.

3. Conditions for a real spectrum and clustering properties. We next show that under suitable conditions, the spectrum of the nonsymmetric preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ is real. We stress the fact that a real spectrum is a welcome property, because it enables the efficient use of short-recurrence Krylov subspace methods such as Bi-CGSTAB; see, e.g., [9, page 139].

THEOREM 3.1. *Assume the hypotheses and notation of Lemma 2.1 hold and assume in addition that A is symmetric positive definite. If $\alpha \le \frac{1}{2}\lambda_n$, then all eigenvalues η are real.*

Proof. We prove our assertion for the eigenvalues θ, from which the statement for η will follow. Let $x = [u; v]$ be an eigenvector associated with θ. For $u \neq 0, v = 0$ we already showed that the spectrum is real, while $u = 0$ implies $v = 0$, a contradiction. We now assume $u \neq 0 \neq v$.

The eigenvalues θ are the roots of the equation (2.11),whose roots can be expressed as in (2.12). These are all real if the discriminant is nonnegative. Equivalently,

$$\theta \in \mathbb{R} \quad \text{if} \quad (u^*KAKu)^2 \ge 4(u^*Ku)\,(u^*KB^TBu) \quad \forall u \neq 0.$$

Since $u^*K^2u > 0$ for $u \neq 0$, we write the problem above as

$$\theta \in \mathbb{R} \quad \text{if} \quad \frac{(u^*KAKu)^2}{(u^*K^2u)^2} \ge 4\frac{u^*Ku}{u^*K^2u}\,\frac{u^*KB^TBu}{u^*K^2u} \quad \forall u \neq 0.$$

We have $\frac{(u^*KAKu)^2}{(u^*K^2u)^2} \ge \lambda_n^2$, and $\frac{u^*Ku}{u^*K^2u} \le \lambda_{\min}(K)^{-1} \le 1$. Therefore, using (2.8), if $\alpha \le \frac{1}{2}\lambda_n$, we have

$$\frac{(u^*KAKu)^2}{(u^*K^2u)^2} \ge \lambda_n^2 \ge 4\cdot 1\cdot \alpha^2 \ge 4\frac{u^*Ku}{u^*K^2u}\,\frac{u^*KB^TBu}{u^*K^2u} \quad \forall u \neq 0. \tag{3.1}$$

The discriminant is nonnegative, therefore all roots of (2.12) are real, and so are the eigenvalues θ. □

Under additional assumptions on the spectrum of the block matrices, it is possible to provide a less strict condition on α. This is stated in the following corollary.

COROLLARY 3.2. *Under the hypotheses and notation of Theorem 3.1, assume that $4\sigma_1^2 - \lambda_n^2 > 0$. If $\alpha \le \frac{\lambda_n\sigma_1}{\sqrt{4\sigma_1^2-\lambda_n^2}}$ then all eigenvalues η are real.*

Proof. Using (2.8), we can write

$$\frac{u^*KB^TBu}{u^*K^2u} = \alpha^2\left(1 - \frac{u^*Ku}{u^*K^2u}\right) \le \alpha^2\left(1 - \frac{1}{1+\frac{\sigma_1^2}{\alpha^2}}\right) = \alpha^2\frac{\sigma_1^2}{\alpha^2+\sigma_1^2}.$$

Therefore, if $\lambda_n^2 \ge 4\alpha^2\frac{\sigma_1^2}{\alpha^2+\sigma_1^2}$, the bound equivalent to (3.1) follows. Moreover, we note that under the assumption that $4\sigma_1^2 - \lambda_n^2 > 0$,

$$\lambda_n^2 \ge 4\alpha^2\frac{\sigma_1^2}{\alpha^2+\sigma_1^2} \qquad \Leftrightarrow \qquad \alpha^2 \le \frac{\lambda_n^2\sigma_1^2}{4\sigma_1^2-\lambda_n^2}. \qquad \square$$

It is interesting to observe that if $\sigma_1^2 = \lambda_1$, the condition $4\sigma_1^2 - \lambda_n^2 > 0$ corresponds to the following inequality,

$$\frac{\lambda_1}{\lambda_n} > \frac{1}{4}\lambda_n,$$

which is easily satisfied since usually λ_n is small and λ_1 is much bigger than λ_n. Note that such setting is very common in the Stokes problem, where A is a discretization of a (vector) Laplacian and BB^T can also be regarded as a discrete Laplacian.

The following result shows that the eigenvalues form two tight clusters as $\alpha \to 0$. This is an important property from the point of view of convergence of preconditioned Krylov subspace methods. This result extends and sharpens the clustering result obtained in [3] (using different tools) for the special case of Poisson's equation in saddle point form.

PROPOSITION 3.3. *Assume A is symmetric and positive definite. For sufficiently small $\alpha > 0$, the eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ cluster near zero and two. More precisely, for small $\alpha > 0$, $\eta \in (0, \varepsilon_1) \cup (2-\varepsilon_2, 2)$, with $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_1, \varepsilon_2 \to 0$ for $\alpha \to 0$.*

Proof. We assume α is small, and in particular $\alpha \le \frac{1}{2}\lambda_n$, therefore all eigenvalues are real. Let $[u; v]$ be an eigenvector of (2.4) and let $\theta_\pm$ be the roots of equation (2.11). These are given by (2.12). Collecting u^*Ku and dividing and multiplying (2.12) by $u^*K^2u > 0$, we obtain

$$\theta_\pm = \frac{u^*K^2u}{u^*Ku}\left(\frac{1}{2}\frac{u^*KAKu}{u^*K^2u} \pm \sqrt{\frac{1}{4}\left(\frac{u^*KAKu}{u^*K^2u}\right)^2 - \frac{u^TKu}{u^*K^2u}\frac{u^TKB^TBu}{u^*K^2u}}\right) \equiv \frac{u^*K^2u}{u^*Ku}\nu_\pm.$$

We recall the bounds in (2.7) and (2.8), while $1 \le \frac{u^TK^2u}{u^*Ku} \le (1+\frac{\sigma_1^2}{\alpha^2})$ for any $u \neq 0$. Note that $(1+\frac{\sigma_1^2}{\alpha^2}) = O(\alpha^{-2})$ as $\alpha \to 0$. We thus have

$$\nu_+ \to \frac{u^*KAKu}{u^*K^2u} \qquad \text{and} \qquad \nu_- = O(\alpha^4) \qquad \text{for} \quad \alpha \to 0,$$

from which it follows that

$$\eta_+ = 2 - \frac{2}{1+\frac{\theta_+}{\alpha}} \to 2 \qquad \text{and} \qquad \eta_- = 2 - \frac{2}{1+\frac{\theta_-}{\alpha}} \to 0 \qquad \text{for} \quad \alpha \to 0. \qquad \square$$

It is important to remark that the occurrence of a gap in the spectrum for small α can be deduced from known results for overdamped systems. Indeed, equation (2.11) stems from the quadratic eigenvalue problem

$$\theta^2 Ku - \theta KAKu + KB^TBu = 0.$$

The eigenproblem above has $2n$ eigenvalues, $n-m$ of which are zero, corresponding to the dimension of the null space of KB^TB. The remaining $n+m$ eigenvalues coincide with the eigenvalues of our problem (2.4). By introducing $\widetilde{\theta} = -\theta$, we obtain the quadratic symmetric eigenproblem (see [5])

$$\widetilde{\theta}^2 Ku + \widetilde{\theta} KAKu + KB^TBu = 0, \qquad K > 0, \quad KAK > 0, \quad KB^TB \ge 0.$$

It can be shown, see e.g. [5, Theorem 13.1], that if the discriminant is positive, that is if $(u^*KAKu)^2 - 4(u^*Ku)(u^*KB^TBu) > 0$ for any $u \neq 0$, then all eigenvalues $\widetilde{\theta}$ are real and nonpositive. Moreover, the spectrum is split in two parts, each of which contains n eigenvalues.[1]

In our context, and in light of Proposition 3.3, the result above implies that m eigenvalues η will cluster towards zero, while n eigenvalues η will cluster around 2, for sufficiently small α.

[1]Note that in the statement of Theorem 13.1 in [5], matrix KB^TB is required to be positive definite rather than just semidefinite. However, the result is still true under the weaker assumption $KB^TB \ge 0$; see also the treatment in [8] and references therein.

TABLE 4.1
Real bounds in (2.19) vs. actual eigenvalues, Stokes problem.

α	Lower bound	$\eta_{\min}$	$\eta_{\max}$	Upper bound
0.001	2.328E-07	5.063E-04	1.9999	1.9999
0.01	2.334E-06	1.697E-03	1.9999	1.9999
0.1	2.010E-05	2.235E-04	1.9929	1.9929
0.2	2.829E-05	1.120E-04	1.9608	1.9608
0.3	2.841E-05	7.473E-05	1.9134	1.9135
0.4	2.590E-05	5.606E-05	1.8633	1.8635
0.5	2.302E-05	4.485E-05	1.8150	1.8154
0.6	2.041E-05	3.738E-05	1.7696	1.7702
0.7	1.820E-05	3.204E-05	1.7271	1.7278
0.8	1.635E-05	2.803E-05	1.6871	1.6880
0.9	1.480E-05	2.492E-05	1.6494	1.6504
1.0	1.350E-05	2.243E-05	1.6137	1.6147
2.0	7.057E-06	1.121E-05	1.3327	1.3344
5.0	2.859E-06	4.486E-06	0.8826	0.8838

4. Numerical experiments. In this section we present the results of a few numerical tests aimed at assessing the tightness of our bounds. The first problem we consider is a saddle point system arising from a finite element discretization of the Stokes problem. This problem was generated using the IFISS software written by Howard Elman, Alison Ramage and David Silvester [7]. Here $n = 578$, $m = 254$, $\lambda_n = 0.0763666$, $\lambda_1 = 3.949253$, $\sigma_1 = 0.247606661$, and $\sigma_m = 0.005319517$. Note that the B matrices (discrete divergence operators) generated by this software are rank deficient; we obtained a full rank matrix by dropping the two last rows of B.

In Table 4.1 we compare the lower and upper bounds given in Theorem 2.2 with the actual values of the smallest and largest eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$, which in this case are all real. While the upper bound is always very tight, the lower bound is relatively loose for tiny values of α, and becomes more accurate as α increases.

For $\alpha \approx 0.01$ or smaller, the eigenvalues form two tight clusters near 0 and 2, containing m and n eigenvalues, respectively, as predicted by Proposition 3.3. Numerical experiments not reported here suggest that the number of iterations for a Krylov subspace method like GMRES or Bi-CGSTAB is minimized for $\alpha \approx 10^{-3}$; for $\alpha < 10^{-4}$ the convergence is markedly slower, perhaps due to the increasing ill-conditioning of $\mathcal{P}^{-1}\mathcal{A}$ [4]. In our experiments we did not use any scalings prior to form the preconditioner. Diagonal scaling can be used to reduce the condition number of A before forming the preconditioner, resulting in faster convergence; see the results in [4].

Next, we consider a saddle point system arising from the discretization of a groundwater flow problem using mixed-hybrid finite elements [6]. In the example at hand $n = 270$, $m = 207$, $n + m = 477$ and $\mathcal{A}$ contains $1,746$ nonzeros. Here we have $\lambda_n = 0.0017$, $\lambda_1 = 0.010$, $\sigma_1 = 2.611$ and $\sigma_m = 0.19743$.

In this case there are nonreal eigenvalues (except for very small α). In Table 4.2 we compare the lower and upper bounds given in Theorem 2.2 with the actual values of the smallest and largest *real* eigenvalues of $\mathcal{P}^{-1}\mathcal{A}$ while in Tables 4.3–4.4 we provide the analogous results for the real part and modulus of the nonreal eigenvalues.

Table 4.2
Bounds in (2.19) vs. actual real eigenvalues, groundwater flow problem.

α	Lower bound	$\eta_{\min}$	$\eta_{\max}$	Upper bound
0.001	0.001143	0.181818	2.000000	2.000000
0.01	0.011371	0.310869	1.999893	1.999971
0.05	0.055576	0.070481	1.985944	1.996341
0.1	0.032787	0.035865	0.137154	1.971127
0.3	0.011050	0.012099	0.047856	1.437903
0.5	0.006644	0.007277	0.028988	0.722331
1.0	0.003328	0.003645	0.014599	0.145003
3.0	0.001110	0.001217	0.004890	0.011648
5.0	0.000666	0.000730	0.002937	0.005078

Table 4.3
Bounds in (2.17) vs. actual real part of nonreal eigenvalues, groundwater flow problem.

α	Lower bound	$\min \Re(\eta)$	$\max \Re(\eta)$	Upper bound
0.001	-	-	-	-
0.01	-	-	-	-
0.05	0.011296	1.823080	1.962387	2.000000
0.1	0.005602	1.571808	1.975776	2.000000
0.3	0.001857	0.608980	1.966375	2.000000
0.5	0.001113	0.274840	1.924906	2.000000
1.0	0.000556	0.078255	1.742401	2.000000
3.0	0.000185	0.009779	0.862083	2.000000
5.0	0.000111	0.003810	0.428775	2.000000

Table 4.4
Bounds in (2.18) vs. actual modulus of nonreal eigenvalues, groundwater flow problem.

α	Lower bound	$\min \lvert\eta\rvert$	$\max \lvert\eta\rvert$	Upper bound
0.001	-	-	-	-
0.01	-	-	-	-
0.05	0.019244	1.860113	1.963349	1.967129
0.1	0.009622	1.753875	1.977199	1.982111
0.3	0.003207	1.093125	1.979200	1.981669
0.5	0.001924	0.731979	1.959713	1.962379
1.0	0.000962	0.386709	1.865509	1.881779
3.0	0.000321	0.131260	1.312480	1.596393
5.0	0.000192	0.078883	0.925533	1.496510

One can see that the location of the real eigenvalues is well detected with our bounds. In particular, the lower bound is quite sharp, whereas the upper bound gets looser when the whole spectrum becomes complex ($\alpha \geq 0.05$), providing again sharp estimates for large values of α.

The real part of the eigenvalues changes considerably as α varies, clustering on different regions of the interval $(0, 2)$. Unfortunately, on this example, our bounds for $\Re(\eta)$ do not seem to be able to capture the cluster migration from the right to the left end of the interval, yielding bounds that include most of the reference interval.

In Table 4.4, sharper estimates can be observed for the magnitude of the nonreal eigenvalues.

Numerical experiments indicate that the optimal value of α is between 0.001 and 0.01, corresponding to 11-12 iterations of preconditioned GMRES. For α in this range, the eigenvalues are all real and fall in two clusters. The lower bounds, although not very sharp, suggest that the leftmost cluster will not be too close to zero, and can be used to predict a good choice of α.

5. Conclusions. In this paper we have provided bounds and clustering results for the spectra of preconditioned matrices arising from the application of the Hermitian/skew-Hermitian splitting preconditioner to symmetric saddle point problems. Numerical experiments have been used to illustrate the capability of our estimates to locate the actual spectral region. We have also shown that for small α, all the eigenvalues are real and fall in two clusters, one near 0 and the other near 2. In addition, we found a connection with the quadratic eigenvalue problems arising in the theory of overdamped systems; it is possible that exploitation of this connection may lead to further insight into the spectral properties of preconditioned saddle point problems.

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