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THE ANTI-REFLECTIVE TRANSFORM AND REGULARIZATION BY FILTERING

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Abstract. Filtering methods are used in signal and image restoration to reconstruct an approximation of a signal or image from degraded measurements. Filtering methods rely on computing a singular value decomposition or a spectral factorization of a large structured matrix. The structure of the matrix depends in part on imposed boundary conditions. Anti-reflective boundary conditions preserve continuity of the image and its derivative at the boundary, and have been shown to produce superior reconstructions compared to other commonly used boundary conditions, such as periodic, zero and reflective. The purpose of this paper is to analyze the eigenvector structure of matrices that enforce anti-reflective boundary conditions. In particular, a new anti-reflective transform is introduced, and an efficient approach to computing filtered solutions is proposed. Numerical tests illustrate the performance of the discussed methods.

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1. Introduction. In this paper we consider structured matrices that arise from the discretization of large scale ill-posed inverse problems,

$$\mathbf{g} = A\mathbf{f} + \eta. \tag{1.1}$$

Given the vector $\mathbf{g}$ and matrix A, the aim is to compute an approximation of the unknown vector $\mathbf{f}$. The vector η represents unknown errors (e.g., measurement or discretization errors and noise) in the observed data. These problems arise in many applications, including image reconstruction, image deblurring, geophysics, parameter identification and inverse scattering; cf. [3, 10, 12, 13, 19]. We are mainly interested in problems that arise in spatially invariant signal and image restoration, where the observed data is

$$g_i = \sum_{j \in \mathbf{Z}^d} f_j h_{i-j} + \eta_i \,,$$

and the dimension $d = 1$ for signals (such as voice), and $d = 2$ or 3 for images. The d-dimensional tensor $\mathbf{h} = [h_i]$ represents the blurring operator, and is called the *point spread function* (PSF). Notice that we have an infinite summation because a true signal or image scene does not have a finite boundary. However, the data g_i is collected only at a finite number of values, and thus represents only a finite region of an infinite scene. Boundary conditions (BCs) are used to artificially describe the scene outside the viewable region. The PSF and the imposed BCs together define the matrix A.

Typically the matrix A is very ill-conditioned and the degenerating subspace largely intersects the high frequency space: consequently regularization techniques are

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used to compute stable approximations of $\mathbf{f}$ with controlled noise levels [10, 11, 12, 19]. Many choices of regularization can be employed, such as TSVD, Tikhonov, and total variation [10, 12, 19]. Analysis and implementation of regularization methods can often be simplified by computing a spectral (or singular value) decomposition of A. Unfortunately this may be very difficult for large scale problems, unless the matrix has exploitable structure. For example, if A is circulant then the spectral decomposition can be computed efficiently with the fast Fourier transform (FFT) [6]. In image deblurring, circulant structures arise when enforcing periodic boundary conditions. Periodic boundary conditions are convenient for computational reasons, but it is difficult to justify their use in a physical sense for most problems.

Other boundary conditions, which better describe the scene outside the viewable region have been proposed. For example, reflective boundary conditions assume the scene outside the viewable region is a reflection of the scene inside the viewable region. In this case the matrix A has a Toeplitz-plus-Hankel structure. If the blur satisfies a strong symmetry condition, $h_i = h_{|i|}$ for all $i \in \mathbf{Z}^d$, then the spectral decomposition of A can be computed very efficiently using the fast discrete cosine transform (DCT)[15].

More recently, new *anti-reflective* boundary conditions (AR-BCs) have been proposed [18, 1] and studied [2, 7, 8, 16], which have the advantage that continuity of the image, and of the normal derivative, are preserved at the boundary. This regularity, which is not shared with zero or periodic BCs, and only partially shared with reflective BCs, significantly reduces ringing artifacts that may occur with other boundary conditions. The matrix structure arising from the imposition of AR-BCs is Toeplitz-plus-Hankel (as in the case of reflective BCs), plus an additional structured low rank matrix. By linearity of the boundary conditions with respect to the PSF, it is evident that the set of AR-BC matrices is a vector space. Unfortunately it is not closed under multiplication or inversion. However, if we restrict our attention to strongly symmetric PSFs and assume that the PSF satisfies a mild finite extent condition (more precisely $h_i = 0$ if $|i_j| \geq n-2$, $i = (i_1, \ldots, i_d)$ for some $j \in \{1, \ldots, d\}$), then any of the resulting AR-BC matrices belong to a d-level commutative matrix algebra denoted by $\mathcal{AR}^{(d)}$, see [2]. Certain computations involving matrices in $\mathcal{AR}^{(d)}$ can be done efficiently. For example, matrix-vector products can be implemented using FFTs, and solution of linear systems and eigenvalue computation involving these matrices can be done efficiently using mainly fast sine transforms (FSTs), see [1, 2].

The new contribution of this paper is the analysis of the eigenvector structure of $\mathcal{AR}^{(d)}$ matrices. The main result concerns the definition of the AR-transform, which carries interesting functional information, is fast ($O(n^d \log(n))$ real operations) and structured, but it is not orthogonal. Then we use the resulting eigenvector structure to define filtering-based regularization methods to reconstruct approximate solutions of (1.1).

The paper is organized as follows. In Section 2, we review the main features of the $\mathcal{AR}$ matrix algebra that are essential for describing and analyzing AR-BC matrices. In Section 3 we introduce AR-BCs and in Section 4 we discuss the spectral features of the involved matrices and define an AR-transform. In Section 5 we describe an efficient approach to compute a filtered (regularized) solution of (1.1). Some numerical results are reported in Section 6, and final remarks are given in Section 7, together with a concise treatment of the multidimensional case.

In the rest of the paper we consider essentially only the one-dimensional problem (that is, $d = 1$), to simplify the notation and mathematical analysis. However, the results generalize to higher dimensions, $d > 1$; comments and results for extending

the analysis are provided in Section 7.

2. The algebra of matrices induced by AR-BCs. This section is devoted to describing the algebra $\mathcal{AR}_n \equiv \mathcal{AR}$, $n \geq 3$. The notation introduced in this section will be used throughout the paper, and is essential for the description, given in Section 3, of the $n \times n$ matrices arising from the imposition of AR-BCs.

2.1. The τ algebra. Let Q be the type I sine transform matrix of order n (see [4]) with entries

$$[Q]_{i,j} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{ji\pi}{n+1}\right), \quad i,j = 1,\ldots,n. \tag{2.1}$$

It is known that the real matrix Q is orthogonal and symmetric ($Q^{-1} = Q^T = Q$). For any n-dimensional real vector $\mathbf{v}$, the matrix-vector multiplication $Q\mathbf{v}$ (DST-I transform) can be computed in $O(n\log(n))$ real operations by using the algorithm FST-I.

Let τ be the space of all the matrices that can be diagonalized by Q:

$$\tau = \{QDQ : D \text{ is a real diagonal matrix of size } n\}. \tag{2.2}$$

Let $X = QDQ \in \tau$, then $QX = DQ$. Consequently, if we let $\mathbf{e}_1$ denote the first column of the identity matrix, then the relationship $QX\mathbf{e}_1 = DQ\mathbf{e}_1$ implies that the eigenvalues $[D]_{i,i}$ of X are given by $[D]_{i,i} = [Q(X\mathbf{e}_1)]_i/[Q\mathbf{e}_1]_i$, $i = 1,\ldots,n$. Therefore the eigenvalues of X can be obtained by applying a DST-I transform to the first column of X and, in addition, any matrix in τ is uniquely determined by its first column.

Now we report a characterization of the τ class, which is important for analyzing the structure of AR-BC matrices. Let us define the shift of any vector $\mathbf{v} = [v_0,\ldots,v_{n-1}]^T$ as $\sigma(\mathbf{v}) = [v_1, v_2, \ldots, v_{n-1}, 0]^T$. According to a Matlab like notation, we define $T(\mathbf{x})$ to be the n-by-n symmetric Toeplitz matrix whose first column is $\mathbf{x}$ and $H(\mathbf{x},\mathbf{y})$ to be the n-by-n Hankel matrix whose first and last column are $\mathbf{x}$ and $\mathbf{y}$, respectively. Every matrix of the class (2.2) can be written as (see [4])

$$T(\mathbf{v}) - H(\sigma^2(\mathbf{v}), J\sigma^2(\mathbf{v})), \tag{2.3}$$

where $\mathbf{v} = [v_0,\ldots,v_{n-1}]^T \in \mathbf{R}^n$ and J is a matrix with entries $[J]_{s,t} = 1$ if $s+t = n+1$ and zero otherwise. We refer to J as a "flip" matrix because the multiplication $J\mathbf{x}$ has the effect of flipping the entries of the vector $\mathbf{x}$ in an up/down direction. The structure defined by (2.3) means that matrices in the τ class are special instances of Toeplitz-plus-Hankel matrices.

Moreover, the eigenvalues of the τ matrix in (2.3) are given by the cosine function

$$v(y) = \sum_{|j|\leq n-1} v_j \exp(\mathbf{i}jy), \qquad y \in G_n \equiv \left\{\frac{k\pi}{n+1},\ k = 1,\ldots,n\right\}, \tag{2.4}$$

where $\mathbf{i}^2 = -1$ and $v_j = v_{|j|}$ for $|j| \leq n-1$. The τ matrix in (2.3) is denoted by $\tau(v)$ and is called the τ matrix generated by the function or *symbol* $v = v(\cdot)$.

2.2. The AR-algebras $\mathcal{AR}$. Let $h = h(\cdot)$ be a real-valued cosine polynomial of degree l and let $\tau_k(h) \equiv Q\mathrm{diag}_{y\in G_k}(h(y))Q$ (note that $\tau_k(h)$ coincides with the matrix in (2.3)–(2.4) with $v(\cdot) = h(\cdot)$, when $l \leq k-1$). Then the Fourier coefficients

of h are such that $h_i = h_{-i} \in \mathbf{R}$ with $h_i = 0$ if $|i| > l$, and for $k = n-2$ we can define the one-level $AR_n(\cdot)$ operator

$$AR_n(h) = \begin{bmatrix} h(0) & & \\ v_{n-2}(h) & \tau_{n-2}(h) & Jv_{n-2}(h) \\ & & h(0) \end{bmatrix}, \tag{2.5}$$

where J is the flip matrix, $v_{n-2}(h) = \tau_{n-2}((\phi(h))(\cdot))\mathbf{e}_1$ and

$$(\phi(h))(y) = \frac{h(y) - h(0)}{2(\cos(y) - 1)}. \tag{2.6}$$

It is interesting to observe that $h(y) - h(0)$ has a zero of order at least 2 at zero (since $h = h(\cdot)$ is a cosine polynomial) and therefore $\phi(h) = (\phi(h))(\cdot)$ is still a cosine polynomial of degree $l-1$, whose value at zero is $-h''(0)/2$ (in other words the function is well defined at zero).

As proved in [2], with the above definition of the operator $AR_n(\cdot)$, we have

1. $\alpha AR_n(h_1) + \beta AR_n(h_2) = AR_n(\alpha h_1 + \beta h_2)$,
2. $AR_n(h_1)AR_n(h_2) = AR_n(h_1 h_2)$,

for real α and β and for cosine functions $h_1 = h_1(\cdot)$ and $h_2 = h_2(\cdot)$.

These properties allow us to define $\mathcal{AR}$ as the algebra (closed under linear combinations, product and inversion) of matrices $AR_n(h)$, with h being a cosine polynomial. By standard interpolation arguments it is easy to see that $\mathcal{AR}$ can be defined as the set of matrices $AR_n(h)$, where h is a cosine polynomial of degree at most $n-3$. Therefore, it is clear that $\dim(\mathcal{AR}) = n-2$. Moreover the algebra $\mathcal{AR}$ is commutative thanks to point 2, since $h_1(y)h_2(y) = h_2(y)h_1(y)$ at every y in the domain. Consequently, if matrices of the form $AR_n(h)$ are diagonalizable, then they must have the same set of eigenvectors [14]. This means there must exist an "anti-reflective transform" that diagonalizes the matrices in $\mathcal{AR}$. Unfortunately this transform fails to be unitary, since the matrices in $\mathcal{AR}$ are generically not normal. However the AR transform is close in rank to an orthogonal linear mapping. Development of this transform is the main contribution of this paper, and is discussed in detail in Section 4.

3. AR-BCs and the AR-BC matrices. In this section we describe the anti-reflective BCs. We have already mentioned that, in the generic case, periodic and zero Dirichlet BCs introduce a discontinuity in the signal, while the reflective BCs preserve the continuity of the signal, but introduce a discontinuity in the derivative. Our approach is to use an anti-reflection: in this way, at the boundaries, instead of having a mirror-like symmetry (reflective BCs), we impose a global symmetry around the boundary points. This latter choice corresponds to a central symmetry around the considered boundary point. If f_1 is the left boundary point and f_n is the right one, then the external points f_{1-j} and f_{n+j}, $j \geq 1$, are computed as a function of the internal points according to the rules $f_{1-j} - f_1 = -(f_{j+1} - f_1)$ and $f_{n+j} - f_n = -(f_{n-j} - f_n)$. If the support of the centered blurring function is $q = 2m+1 \leq n$, then for $j = 1, \ldots, m$, we have

$$f_{1-j} = 2f_1 - f_{j+1}, \qquad f_{n+j} = 2f_n - f_{n-j}.$$

Following the analysis given in [18], if the blurring function (the PSF) $\mathbf{h}$ is symmetric (i.e., $h_i = h_{-i}$, $\forall\, i \in \mathbf{Z}$), if $h_i = 0$ for $|i| \geq n-2$ (degree condition), and $\mathbf{h}$ is normalized so that $\sum_{i=-m}^{m} h_i = 1$, then the structure of the $n \times n$ anti-reflective blurring matrix

A is

$$A = \begin{bmatrix} z_0 & \mathbf{0}^T & \mathbf{0} \\ z_1 & & z_m \\ \vdots & \hat{A} & \vdots \\ z_m & & z_1 \\ \mathbf{0} & \mathbf{0}^T & z_0 \end{bmatrix}, \tag{3.1}$$

where $A_{1,1} = A_{n,n} = 1$, $z_i = h_i + 2\sum_{k=i+1}^{m} h_k$, $\hat{A}$ has size $n-2$ and

$$\hat{A} = T(\mathbf{u}) - H(\sigma^2(\mathbf{u}), J\sigma^2(\mathbf{u})), \tag{3.2}$$

with $\mathbf{u} = [h_0, h_1, \ldots, h_m, 0, \ldots, 0]^T$. According to the brief discussion of Section 2.1, relation (3.2) implies that $\hat{A} = \tau_{n-2}(h)$ with $h(y) = h_0 + 2\sum_{k=1}^{m} h_k \cos(ky)$ (see (2.3) and (2.4)). Moreover in [2] it is proved that $A = AR_n(h)$.

4. Eigenvalues and eigenvectors of AR-BC matrices. We first describe the spectrum of AR-BC matrices, under the usual mild degree condition (that is, the PSF $\mathbf{h}$ has finite support), with symmetric, normalized PSFs. Then we describe the eigenvector structure and we introduce the AR-transform. Finally, we discuss how to efficiently compute filtered solutions of (1.1) when the blurring matrix A is in the $\mathcal{AR}^{(d)}$ algebra.

4.1. Eigenvalues of $AR_n(\cdot)$ operators. The spectral structure of any AR-BC matrix, with symmetric PSF $\mathbf{h}$, is concisely described in the following result.

THEOREM 4.1. [2] *Let the blurring function (PSF) $\mathbf{h}$ be symmetric (i.e., $h_s = h_{-s}$), normalized, and satisfying the usual degree condition. Then the eigenvalues of the $n \times n$ AR-BC blurring matrix A in (3.1), $n \geq 3$, are given by $h(0) = 1$ with multiplicity two and $\{h(y)|_{y \in G_{n-2}}\}$.*

The proof can be easily derived by (2.5) which shows that the eigenvalues of $AR_n(h)$ are $h(0)$ with multiplicity 2 and those of $\tau_{n-2}(h)$; i.e. $h(\cdot)$ evaluated at every $y \in G_{n-2}$ with multiplicity 1 each.

4.2. The AR-transform and its functional interpretation. Here we will determine the eigenvectors of every matrix $AR_n(h)$. In particular, we show that every AR-BC matrix is diagonalizable, and we demonstrate independence of the eigenvectors from the symbol h. With reference to the notation in (2.1)–(2.4), calling $\mathbf{q}_j^{(n-2)}$ the j-th column of Q_{n-2}, and $x_j^{(n-2)}$ the j-th point of G_{n-2}, $j = 1, \ldots, n-2$, we have

$$\begin{aligned} AR_n(h) \begin{bmatrix} 0 \\ \mathbf{q}_j^{(n-2)} \\ 0 \end{bmatrix} &= \begin{bmatrix} h(0) & & \\ v_{n-2}(h) & \tau_{n-2}(h) & Jv_{n-2}(h) \\ & & h(0) \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{q}_j^{(n-2)} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \tau_{n-2}(h)\mathbf{q}_j^{(n-2)} \\ 0 \end{bmatrix} = h\left(x_j^{(n-2)}\right) \begin{bmatrix} 0 \\ \mathbf{q}_j^{(n-2)} \\ 0 \end{bmatrix}, \end{aligned} \tag{4.1}$$

since $\mathbf{q}_j^{(n-2)}$ is an eigenvector of $\tau_{n-2}(h)$ and $h\big(x_j^{(n-2)}\big)$ is the related eigenvalue. Due to the centro-symmetry of the involved matrix, if $[\,1\,,\,\mathbf{p}\,,\,0\,]^T$ is an eigenvector of $AR_n(h)$ related to the eigenvalue $h(0)$, then the other is its flip, i.e., $[\,0\,,\,J\mathbf{p}\,,\,1\,]^T$. Let

us look for this eigenvector, by imposing the equality

$$AR_n(h)\begin{bmatrix} 1 \\ \mathbf{p} \\ 0 \end{bmatrix} = h(0)\begin{bmatrix} 1 \\ \mathbf{p} \\ 0 \end{bmatrix}$$

which is equivalent seeking a vector $\mathbf{p}$ that satisfies

$$v_{n-2}(h) + \tau_{n-2}(h)\mathbf{p} = h(0)\mathbf{p}.$$

Since $v_{n-2}(h) = \tau_{n-2}(\phi(h))\mathbf{e}_1$ by definition of the operator $v_{n-2}(\cdot)$ (see (2.5) and the lines below), and, because of the algebra structure of τ_{n-2} and thanks to (2.6), we deduce that the vector $\mathbf{p}$ satisfies the relation

$$\tau_{n-2}(h - h(0))\left[-L_{n-2}^{-1}\mathbf{e}_1 + \mathbf{p}\right] = \mathbf{0} \tag{4.2}$$

where L_{n-2} is the discrete one-level Laplacian, i.e., $L_{n-2} = \tau_{n-2}(2 - 2\cos(\cdot))$. Therefore by (4.2) the solution is given by $\mathbf{p} = L_{n-2}^{-1}\mathbf{e}_1 + \mathbf{r}$ where $\mathbf{r}$ is any vector belonging to the kernel of $\tau_{n-2}(h - h(0))$. If $\tau_{n-2}(h - h(0))$ is invertible (as it happens for every nontrivial PSF, since the unique maximum of the function is reached at $y = 0$, which is not a grid point of G_{n-2}), then the solution is unique. Otherwise $\mathbf{r}$ will belong to the vector space generated by those vectors $\mathbf{q}_j^{(n-2)}$ for which the index j is such that $h(x_j^{(n-2)}) = h(0)$. However, the contribution contained in $\mathbf{r}$ was already considered in (4.1), and therefore $\mathbf{p} = L_{n-2}^{-1}\mathbf{e}_1$ is the only solution that carries new information. Hence, independently of h, we have

$$AR_n(h)\begin{bmatrix} 1 & & \\ \mathbf{p} & Q_{n-2} & J\mathbf{p} \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \mathbf{p} & Q_{n-2} & J\mathbf{p} \\ & & 1 \end{bmatrix}\begin{bmatrix} h(0) & & \\ & \underset{y\in G_{n-2}}{\text{diag}}\,(h(y)) & \\ & & h(0) \end{bmatrix}.$$

Now we observe that the j-th eigenvector is unitary, $j = 2, \ldots, n-1$, because Q_{n-2} is unitary: we wish to impose the same condition on the first and the last eigenvector. The interesting fact is that $\mathbf{p}$ has an explicit expression. By using standard finite difference techniques, it follows that $p_j = 1 - j/(n-1)$ so that the first vector is exactly the sampling of the function $1 - x$ on the grid $j/(n-1)$ for $j = 0, \ldots, n-1$. Its Euclidean norm is $\alpha_n = \sqrt{\sum_{j=0}^{n-1} j^2/(n-1)} \sim \sqrt{n/3}$, where, for nonnegative sequences β_n, γ_n, the relation $\gamma_n \sim \beta_n$ means $\gamma_n = \beta_n(1 + o(1))$. In this way, the (normalized) AR-transform can be defined as

$$T_n = \begin{bmatrix} \alpha_n^{-1} & & \\ \alpha_n^{-1}\mathbf{p} & Q_{n-2} & \alpha_n^{-1}J\mathbf{p} \\ & & \alpha_n^{-1} \end{bmatrix}. \tag{4.3}$$

REMARK 4.2. With the normalization condition in (4.3), all the columns of T_n are unitary. However orthogonality is only partially fulfilled since it holds for the central columns, while the first and last columns are not orthogonal to each other, and neither one is orthogonal to the central columns. We can solve the first problem: the sum of the first and of the last column (suitably normalized) and the difference of the first and the last column (suitably normalized) become orthonormal, and are still

eigenvectors related to the eigenvalue $h(0)$. However, since $\mathbf{q}_1^{(n-2)}$ has only positive components and the vector space generated by the first and the last column of T_n contains positive vectors, it follows that T_n cannot be made orthonormal just by operating on the first and the last column. Indeed, we do not want to change the central block of T_n since it is related to a fast $O(n\log(n))$ real transform and hence, necessarily, we cannot get rid of this quite mild lack of orthogonality. Finally, it is worth mentioning that the inverse transform is also described in terms of the same block structure since

$$\widetilde{T}_n \equiv T_n^{-1} = \begin{bmatrix} \alpha_n & & \\ -Q_{n-2}\mathbf{p} & Q_{n-2} & -Q_{n-2}J\mathbf{p} \\ & & \alpha_n \end{bmatrix}. \tag{4.4}$$

REMARK 4.3. There is a suggestive functional interpretation of the transform T_n. When considering periodic BCs, the transform of the related matrices is the Fourier transform: its j-th column vector, up to a normalizing scalar factor, can be viewed as a sampling, over a suitable uniform gridding of $[0, 2\pi]$, of the frequency function $\exp(-\mathbf{i}jy)$. Analogously, when imposing reflective BCs with a symmetric PSF, the transform of the related matrices is the cosine transform: its j-th column vector, up to a normalizing scalar factor, can be viewed as a sampling, over a suitable uniform gridding of $[0, \pi]$, of the frequency function $\cos(jy)$. Here the imposition of the anti-reflective BCs can be functionally interpreted as a linear combination of sine functions and of linear polynomials (whose use is exactly required for imposing C^1 continuity at the borders). This intuition becomes evident in the expression of T_n in (4.3). Indeed, by defining the one-dimensional grid $\widetilde{G}_n = G_{n-2} \cup \{0, 1\} = \{j\pi/(n-1) : \ j = 0, \ldots, n-1\} \subset [0, \pi]$, we infer that the first column of T_n is given by $\alpha_n^{-1}(1 - y/\pi)|_{\widetilde{G}_n}$, the j-th column of T_n, $j = 2, \ldots, n-1$, is given by $\sqrt{2/(n-1)}(\sin(jy))|_{\widetilde{G}_n}$, and finally the last column of T_n is given by $\alpha_n^{-1}(y/\pi)|_{\widetilde{G}_n}$ i.e.

$$T_n = \left[\, 1 - \frac{y}{\pi}\,,\, \sin(y)\,,\, \cdots\,,\, \sin((n-2)y)\,,\, \frac{y}{\pi}\,\right]\Big|_{\widetilde{G}_n} \cdot \operatorname{diag}\left(\alpha_n^{-1},\, \sqrt{\frac{2}{n-1}}I_{n-2},\, \alpha_n^{-1}\right). \tag{4.5}$$

REMARK 4.4. As discussed in Remark 4.3, there is a natural interpretation in terms of frequencies when considering one-dimensional periodic and reflective BCs. The eigenvalue obtained as a sampling of the symbol h at a grid-point close to zero, i.e. close to the maximum point of h, has an associated eigenvector that corresponds to low frequency (signal space) information, while the eigenvalue obtained as a sampling of the symbol h at a grid-point far away from zero (and, in particular, close to π), has an associated eigenvector that corresponds to high frequency (noise space) information. Concerning anti-reflective BCs, the same situation occurs when dealing with the frequency eigenvectors $\sqrt{2/(n-1)}(\sin(jy))|_{\widetilde{G}_n}$, $j = 2, \ldots, n-1$. The other two exceptional eigenvectors generate the space of linear polynomials and therefore they correspond to low frequency information: this intuition is well supported by the fact that the related eigenvalue is $h(0)$, i.e. the maximum and the infinity norm of h, and by the fact that AR-BCs are more precise than other classical BCs.

THEOREM 4.5. [$AR_n(\cdot)$ Jordan Canonical Form.] *With the notation and assumptions of Theorem 4.1, the $n \times n$ AR-BC blurring matrix A in* (3.1)*, $n \geq 3$, coincides*

with

$$AR_n(h) = T_n \mathrm{diag}_{y \in \widehat{G}_n}\ (h(y))\ \widetilde{T}_n, \tag{4.6}$$

where T_n and $\widetilde{T}_n$ are defined in (4.3) and (4.4), while $\widehat{G}_n = \{0\} \cup G_{n-2} \cup \{0\}$.

5. Filtering methods for AR-BC matrices. As mentioned in Section 1, regardless of the imposed boundary conditions, matrices A that arise in signal and image restoration are typically severely ill-conditioned, and regularization is needed in order to compute a stable approximation of the solution of (1.1). A class of regularization methods is obtained through spectral filtering [12, 13]. Specifically, if

$$A = T_n D_n T_n^{-1}, \quad T_n = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 & \cdots & \mathbf{t}_n \end{bmatrix}, \quad \widetilde{T}_n = T_n^{-1} = \begin{bmatrix} \tilde{\mathbf{t}}_1^T \\ \tilde{\mathbf{t}}_2^T \\ \vdots \\ \tilde{\mathbf{t}}_n^T \end{bmatrix}$$

is the spectral decomposition of A, then a spectral filter solution is given by

$$\mathbf{f}_{\mathrm{reg}} = \sum_{i=1}^{n} \phi_i \frac{\tilde{\mathbf{t}}_i^T \mathbf{g}}{d_i} \mathbf{t}_i\,, \tag{5.1}$$

where $D_n = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i = h\left(\frac{(i-1)\pi}{n-1}\right)$, $i = 2, \ldots, n-1$, $d_1 = d_n = h(0)$, and ϕ_i are filter factors that satisfy

$$\phi_i \approx \begin{cases} 1 & \text{if } d_i \text{ is large,} \\ 0 & \text{if } d_i \text{ is small.} \end{cases}$$

The small eigenvalues correspond to eigenvectors with high frequency components, and are typically associated with the noise space, while the large eigenvalues correspond to eigenvectors with low frequency components, and are associated with the signal space. Thus filtering methods attempt to reconstruct signal space components of the solution, while avoiding reconstruction of noise space components.

For example, the filter factors for two well known filtering methods, truncated spectral value decomposition (TSVD) and Tikhonov regularization, are

$$\phi_i^{\mathrm{tsvd}} = \begin{cases} 1 & \text{if } d_i \geq \delta, \\ 0 & \text{if } d_i < \delta \end{cases} \qquad \text{and} \qquad \phi_i^{\mathrm{tik}} = \frac{d_i^2}{d_i^2 + \lambda}\,,\ \lambda > 0, \tag{5.2}$$

where the problem dependent *regularization parameters* δ and λ must be chosen if d_i are the singular values [13]. Several techniques can be used to estimate appropriate choices for the regularization parameters, including generalized cross validation (GCV), L-curve, and the discrepancy principle [10, 12, 19].

In our case, the notation in (5.2) defines a slightly abuse of notation, because the eigenvalues d_i are not the singular values: in fact the Jordan canonical form (CF) in (4.6) is different from the singular value decomposition (SVD), since the transform T_n is not orthogonal (indeed it is a rank-2 correction of a symmetric orthogonal matrix). Therefore note that the use of ϕ_i^{tsvd} in (5.1) defines the filtering of the eigenvalues in the Jordan canonical form instead of the more classical filtering of the singular values in the SVD. However, we note that in general determining a truncated SVD

can be computationally very expensive, especially in the multidimensional case and also in the strongly symmetric case. Moreover, quite surprisingly, a recent and quite exhaustive set of numerical tests, both in the case of signals and images by Rossi (see [17]), has shown that the truncated Jordan canonical form is more or less equivalent to the truncated SVD in terms of quality of the restored object: indeed this is a delicate issue that deserves more attention in the future.

Furthermore, also the so-called Tikhonov regularization needs a further discussion in this direction. Indeed its definition is related to the solution of the linear system $(A^TA+\lambda^2 I)\mathbf{f}_{\text{tik}} = A^T\mathbf{g}$, but the $\mathcal{AR}$ algebra is not closed under transposition. Hence we cannot use the Jordan canonical form for computing the solution of this linear system in a fast and stable way. In [9] it was proposed to replace A^T by A: in such a way the associated Tikhonov-like linear system becomes $(A^2+\lambda I)\mathbf{f}_{\text{reg}} = A\mathbf{g}$, and now $\mathbf{f}_{\text{reg}}$ is the one in (5.1) with the filter factors ϕ_i^{tik}. In [7] it has been shown that the considered approach, called reblurring, arises when looking at the regularized solution of the continuous problem and then followed by the anti-reflective approximation of each operator separately.

Our final aim is to compute (5.1) in a fast and stable way. We can follow two strategies.

Strategy 1. This is the classic approach implemented for instance with periodic BCs by using three FFTs. In our case we employ the AR-transform, its inverse, and a fast algorithm for computing the eigenvalues.

Algorithm 1:

1. $\tilde{\mathbf{g}} = \widetilde{T}_n\mathbf{g}$,
2. $\mathbf{d} = [h(0),\, \hat{\mathbf{d}}^T,\, h(0)]^T$, where $\hat{\mathbf{d}} = [d_2,\, \ldots,\, d_{n-1}]^T$ are the eigenvalues of $\tau_{n-2}(h)$ that can be computed by a fast sine transform (FST),
3. $\tilde{\mathbf{f}} = (\boldsymbol{\phi}\,./\,\mathbf{d})\,.*\,\tilde{\mathbf{g}}$, where the dot operations are component-wise,
4. $\mathbf{f}_{\text{reg}} = T_n\tilde{\mathbf{f}}$.

The product $T_n\tilde{\mathbf{f}}$ can be clearly computed in a fast and stable way by one FST. Indeed for all $\mathbf{x} \in \mathbf{R}^n$

$$T_n\mathbf{x} = \alpha_n^{-1}x_1\begin{bmatrix}1\\ \mathbf{p}\\ 0\end{bmatrix} + \begin{bmatrix}0\\ Q_{n-2}\mathbf{x}(2:n-1)\\ 0\end{bmatrix} + \alpha_n^{-1}x_n\begin{bmatrix}0\\ J\mathbf{p}\\ 1\end{bmatrix},$$

where $\mathbf{x}(2:n-1)$ in Matlab notation is the vector $\mathbf{x}$ with components indexed from 2 to $n-1$. A similar strategy can be followed for computing the matrix-vector product $\widetilde{T}_n\mathbf{g}$. Instead of $\alpha_n^{-1}\mathbf{p}$ there is $\mathbf{v} = -Q_{n-2}\mathbf{p}$ and instead of $\alpha_n^{-1}J\mathbf{p}$ there is $\mathbf{w} = -Q_{n-2}J\mathbf{p}$. Recalling that $\mathbf{p} = L_{n-2}^{-1}\mathbf{e}_1$ the two vectors $\mathbf{v}$, and $\mathbf{w}$ can be explicitly computed obtaining $v_i = (2n-2)^{-1/2}\cot(\frac{i\pi}{2n-2})$, for $i = 1, \ldots, n-2$ and $\mathbf{w} = \text{diag}_{i=1,\ldots,n-2}(-1)^{i+1}\mathbf{v}$.

Strategy 2. The discussion in Remark 4.4 allows us to describe a slightly different approach for computing filtered solutions. In particular, from Remark 4.4, we see that the eigenvalues $d_1 = d_n = h(0)$ have eigenvectors essentially belonging to the signal space. Hence we set a priori $\phi_1 = \phi_n = 1$, and rewrite the filtered solution as

$$\mathbf{f}_{\text{reg}} = \frac{\tilde{\mathbf{t}}_1^T\mathbf{g}}{d_1}\mathbf{t}_1 + \frac{\tilde{\mathbf{t}}_n^T\mathbf{g}}{d_n}\mathbf{t}_n + \sum_{i=2}^{n-1}\phi_i\frac{\tilde{\mathbf{t}}_i^T\mathbf{g}}{d_i}\mathbf{t}_i\,.$$

Now observe that $\tilde{\mathbf{t}}_1 = \mathbf{e}_1$, $\tilde{\mathbf{t}}_n = \mathbf{e}_n$, and for $i = 2, 3, \ldots, n-1$, $\mathbf{t}_i = [0,\, \mathbf{q}_{i-1}^T,\, 0]^T$, where $\mathbf{q}_j$ are columns of the DST-I matrix Q_{n-2}. Thus, the

filtered solution can be written as

$$\mathbf{f}_{\text{reg}} = \frac{1}{h(0)}\left(g_1\mathbf{t}_1 + g_n\mathbf{t}_n\right) + \begin{bmatrix} 0 \\ \hat{\mathbf{f}}_{\text{reg}} \\ 0 \end{bmatrix}.$$

Let $\mathbf{g} = [g_1,\, \hat{\mathbf{g}}^T,\, g_n]^T$, then

$$\begin{aligned}\hat{\mathbf{f}}_{\text{reg}} &= \sum_{i=2}^{n-1} \phi_i \frac{\tilde{\mathbf{t}}_i^T \mathbf{g}}{d_i}\mathbf{q}_{i-1} \\ &= \sum_{i=2}^{n-1} \frac{\phi_i}{d_i}([-Q_{n-2}\mathbf{p}]_{i-1}g_1 + q_{i-1}^T\hat{\mathbf{g}} - [Q_{n-2}J\mathbf{p}]_{i-1}g_n)\mathbf{q}_{i-1} \\ &= Q_{n-2}\mathbf{y}\end{aligned}$$

where

$$\mathbf{y} = \text{diag}_{i=2,\dots,n-1}\left(\frac{\phi_i}{d_i}\right)(Q_{n-2}\hat{\mathbf{g}} - g_1 Q_{n-2}\mathbf{p} - g_n Q_{n-2}J\mathbf{p}).$$

Therefore

$$\hat{\mathbf{f}}_{\text{reg}} = Q_{n-2}\text{diag}_{i=2,\dots,n-1}\left(\frac{\phi_i}{d_i}\right)Q_{n-2}\tilde{\mathbf{g}}, \quad \text{where} \quad \tilde{\mathbf{g}} = \hat{\mathbf{g}} - g_1\mathbf{p} - g_n J\mathbf{p}.$$

The algorithm can be summarized as following

Algorithm 2:

1. $\tilde{\mathbf{g}} = \hat{\mathbf{g}} - g_1\mathbf{p} - g_n J\mathbf{p}$,
2. $\hat{\mathbf{f}}_{\text{reg}} = Q_{n-2}\text{diag}_{i=2,\dots,n-1}\left(\frac{\phi_i}{d_i}\right)Q_{n-2}\tilde{\mathbf{g}}$ by three FSTs,
3. $\mathbf{f}_{\text{reg}} = \frac{1}{h(0)}\left(\begin{bmatrix} 0 \\ \hat{\mathbf{f}}_{\text{reg}} \\ 0 \end{bmatrix} + g_1\begin{bmatrix} 1 \\ \mathbf{p} \\ 0 \end{bmatrix} + g_n\begin{bmatrix} 0 \\ J\mathbf{p} \\ 1 \end{bmatrix}\right)$.

REMARK 5.1. The matrix $Q_{n-2}\text{diag}_{i=2,\dots,n-1}(\phi_i/d_i)Q_{n-2}$ is the τ matrix with eigenvalues ϕ_i/d_i, for $i = 2,\dots,n-1$. Therefore step 2 in the above algorithm is equivalent to regularizing a linear system with coefficient matrix $\tau_{n-2}(h)$ corresponding to the inner part of $A = AR_n(h)$. It is straightforward that this strategy is exactly the approach used in [5] with homogeneous AR-BCs. Obviously, as already discussed in Remark 4.2, the two eigenvectors that complete the sine basis can be chosen in several ways: for instance in [5], instead of $[1,\, \mathbf{p}^T,\, 0]^T$ and $[0,\, J\mathbf{p}^T,\, 1]^T$, the authors prefer to consider the first vector and the vector $\mathbf{e}$ with all components equal to one, since

$$\mathbf{e} \equiv \begin{bmatrix} 1 \\ \mathbf{p} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ J\mathbf{p} \\ 1 \end{bmatrix}.$$

We can compare the two strategies when ϕ_i are the two classic choices in (5.2). Concerning the spectral truncation performed with ϕ_i^{tsvd}, for every choice of δ we have $\delta \le d_1 = d_n = \max_{i=1,\dots,n} d_i$ and then $\phi_1^{\text{tsvd}} = \phi_n^{\text{tsvd}} = 1$. Therefore the two strategies are exactly the same. On the other hand, for Tikhonov regularization

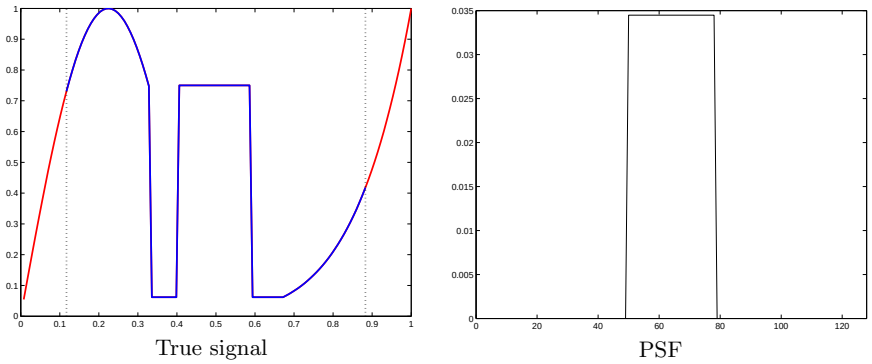


FIG. 6.1. *True signal and out of focus PSF.*

$\phi_1^{\text{tik}} = \phi_n^{\text{tik}} = \frac{h(0)^2}{h(0)^2+\lambda} \neq 1$ and the two strategies are slightly different as already observed in [5]. Indeed the first one arises from the reblurring approach proposed in [9], while the second one is the same as described in [5] (see Remark 5.1).

REMARK 5.2. The difference between the two strategies increases with λ, hence it is more evident when the problem requires a substantial amount of regularization, while it is negligible for small values of λ.

REMARK 5.3. The only difference between the two approaches is that the second imposes $\phi_1 = \phi_n = 1$ a priori. Therefore, for the implementation, we can also use the Algorithm 1 where at step 3 we add $\phi_1 = \phi_n = 1$. In this way it computes exactly the same $\mathbf{f}_{\text{reg}}$ of the Algorithm 2.

6. Numerical experiments. We do not compare the AR-BCs with the other classic BCs, like periodic or reflective, since the topic and related issues have been already widely discussed in several other works (see e.g. [9, 7, 5]), where the advantage, in terms of the restored image quality, of the application of AR-BCs has been emphasized.

For Tikhonov regularization, a comparison between the two strategies described in the previous section is already provided in [5] Section 6. However, we report an example where, according to Remark 5.2, we explicitly compare both strategies varying λ. Since from a computational point of view they are equivalent, we will compare only the quality of the restored signals.

In our example we use the true signal and the out of focus PSF shown in Figure 6.1. The out of focus blurring is well-known to be severely ill-conditioned. The two dotted vertical lines shown in the figure of the true signal denote the field of view of our signal; that is, the signal inside the dotted lines represents that part of the signal that can be directly observed, while the signal extending outside the dotted lines represents information that cannot be directly observed, and which must be approximated through boundary conditions. To the blurred signal we add white Gaussian noise (i.e., normally distributed random values with mean 0 and variance 1) with a percentage $\|\eta\|_2/\|\mathbf{f}_{\text{blur}}\|_2$, where $\mathbf{f}_{\text{blur}}$ is the (noise free) blurred signal and η is the noise. We consider two different levels of noise, 1% and 10%. The observed signals are shown in Figure 6.2.

Clearly the problem with 10% noise requires a stronger regularization than the problem with 1% noise. For both strategies, in Figure 6.3 we show the restored signals, while in Figure 6.4 we report the logarithmic plot of the relative restoration errors

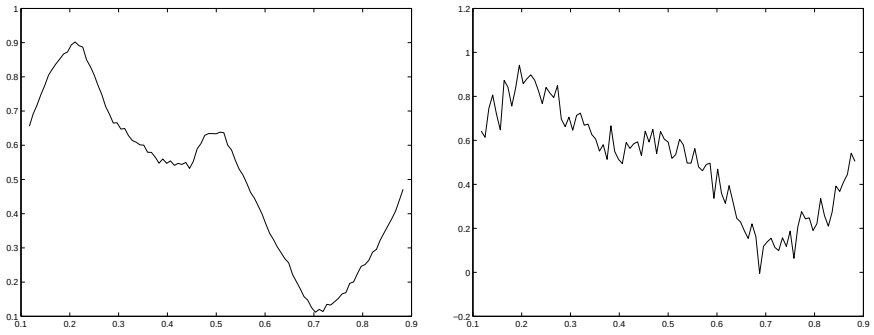


FIG. 6.2. *Observed signals.*

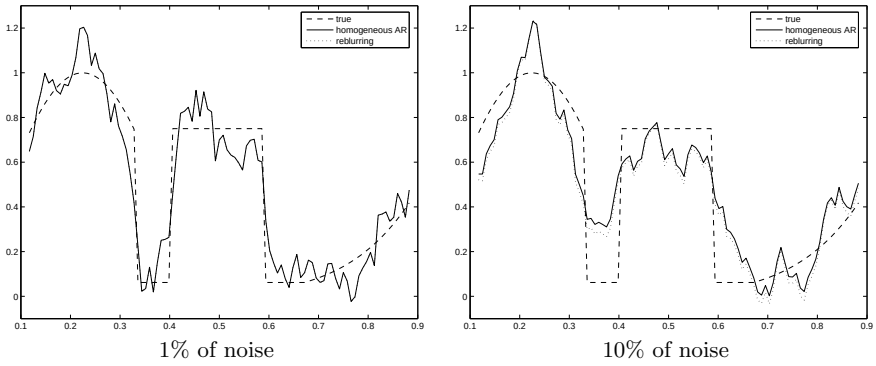
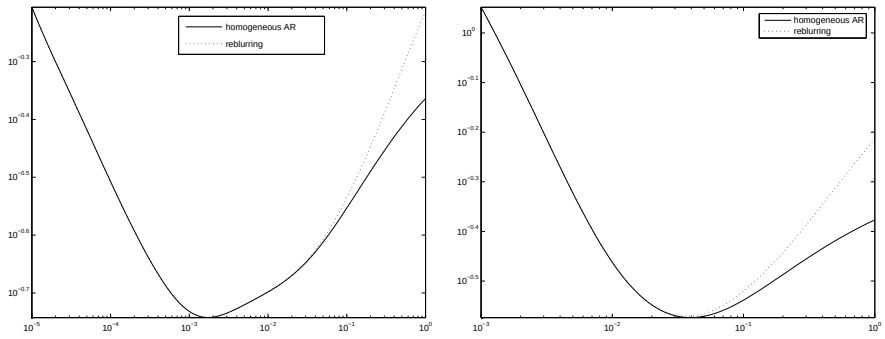


FIG. 6.3. *Restored signals for both strategies.*

varying λ. In the legends the first strategy is called "re-blurring" according to the terminology of [9] where this idea was proposed, while the second strategy is called "homogeneous AR" according to the terminology in [5], where this variation of the theme was discussed. For this example, we note that for a low level of noise, i.e. in our case 1%, the two strategies are equivalent. Indeed the two restored signals are not distinguishable in Figure 6.3. Moreover in Figure 6.4 we observe that the second strategy becomes superior with respect to the first only for $\lambda > 10^{-2}$, while the optimum is reached for $\lambda < 10^{-2}$. On the other hand, for higher levels of noise, i.e. for instance 10%, we need a more substantial regularization and hence a larger value of λ. From Figure 6.4, by looking in a neighborhood of the optimal value of λ, we notice that the second procedure becomes more precise and in fact the restored signal is computed with a lower error norm. However, in Figure 6.3 we can see that the quality of the restored signal is not sensibly improved.

Finally, we compare the two strategies by varying the noise level. Figure 6.5 shows in logarithmic scale the optimal relative restoration error changing the noise level for both techniques. When the noise is lower than 10%, the two strategies achieve about the same minimal restoration error. For a noise level greater than 10%, we observe a different minimum and the second proposal seems to be slightly better.

From the previous example, the second strategy seems to be slightly superior with respect to the first one, when the problem requires stronger regularization. On



1% of noise 10% of noise

FIG. 6.4. *Relative restoration errors vs* λ.

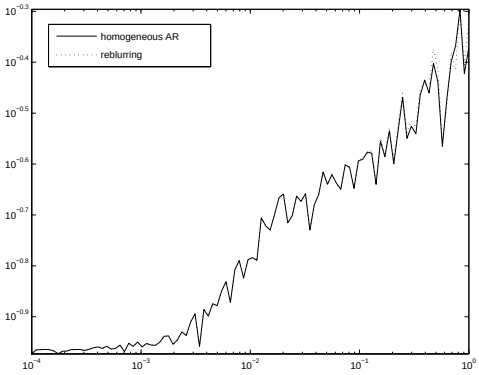


FIG. 6.5. *Optimal relative restoration errors vs noise.*

the other hand, when the optimal value of λ is small, the considered procedures are essentially similar, according to the theoretical discussion in Section 5. However a further comparison in terms of algorithmic implementation and restoration results should be done in the future in order to compare the two proposals with special attention to the more challenging multidimensional case.

7. Multilevel extension and conclusions. Here we provide some concluding remarks and especially hints and comments on the extension of our findings to d-dimensional objects with $d > 1$. Note when $d = 1$, $\mathbf{h}$ is a vector, when $d = 2$, $\mathbf{h}$ is a 2-D array, when $d = 3$, $\mathbf{h}$ is a 3-D tensor, etc. For $d = 1$ and with reference to the previous sections, we have proved that, thanks to the definition of a (fast) AR-transform, it is possible to define a truncated spectral decomposition. However we are well-aware that the real challenge is represented by a general extension to the multidimensional setting. This is the topic that we briefly discuss in the rest of the section.

With reference to Section 2.2 we propose a (canonical) multidimensional extension of the algebras $\mathcal{AR}$ and of the operators $AR_{\boldsymbol{n}}(\cdot)$, $\boldsymbol{n} = (n_1, \ldots, n_d)$: the idea is to use tensor products. If $h = h(\cdot)$ is d-variate real-valued cosine polynomial, then its Fourier coefficients form a real d-dimensional tensor which is strongly symmetric. In addition, $h(\mathbf{y})$, $\mathbf{y} = (y_1, \ldots, y_d)$, can be written as a linear combination of terms of the form

$m(\mathbf{y}) = \prod_{j=1}^{d} \cos(\alpha_j y_j)$ where any α_j is a nonnegative integer. Therefore, we define

$$AR_{\boldsymbol{n}}(m(\mathbf{y})) = AR_{n_1}(\cos(\alpha_1 y_1)) \otimes \cdots \otimes AR_{n_d}(\cos(\alpha_d y_d)), \tag{7.1}$$

where $\otimes$ denotes Kronecker product, and we force

$$AR_{\boldsymbol{n}}(\alpha h_1 + \beta h_2) = \alpha AR_{\boldsymbol{n}}(h_1) + \beta AR_{\boldsymbol{n}}(h_2) \tag{7.2}$$

for every real α and β and for ever d-variate real-valued cosine polynomials $h_1 = h_1(\cdot)$ and $h_2 = h_2(\cdot)$. It is clear that the request that $AR_{\boldsymbol{n}}(\cdot)$ is a linear operator (for $d > 1$, we impose this property in (7.2) by definition) is sufficient for defining completely the operator in the d-dimensional setting.

With the above definition of the operator $AR_{\boldsymbol{n}}(\cdot)$, we have

1. $\alpha AR_{\boldsymbol{n}}(h_1) + \beta AR_{\boldsymbol{n}}(h_2) = AR_{\boldsymbol{n}}(\alpha h_1 + \beta h_2)$,
2. $AR_{\boldsymbol{n}}(h_1) AR_{\boldsymbol{n}}(h_2) = AR_{\boldsymbol{n}}(h_1 h_2)$,

for real α and β and for cosine functions $h_1 = h_1(\cdot)$ and $h_2 = h_2(\cdot)$.

The latter properties of algebra homomorphism allows to define a commutative algebra $\mathcal{AR}$ of the matrices $AR_{\boldsymbol{n}}(h)$, with $h(\cdot)$ being a d-variate cosine polynomial. By standard interpolation arguments it is easy to see that $\mathcal{AR}$ can be defined as the set of matrices $AR_{\boldsymbol{n}}(h)$, where h is a d-variate cosine polynomial of degree at most $n_j - 3$ in the j-th variable for every j ranging in $\{1, \ldots, d\}$: we denote the latter polynomial set by $\mathcal{P}^{(d,\text{even})}_{\boldsymbol{n}-2\boldsymbol{e}}$, with $\boldsymbol{e}$ being the vector of all ones. Here we have to be a bit careful in specifying the meaning of algebra when talking of polynomials. More precisely, for $h_1, h_2 \in \mathcal{P}^{(d,\text{even})}_{\boldsymbol{n}-2\boldsymbol{e}}$ the product $h_1 \cdot h_2$ is the unique polynomial $h \in \mathcal{P}^{(d,\text{even})}_{\boldsymbol{n}-2\boldsymbol{e}}$ satisfying the following interpolation condition

$$h(y) = z_y, \quad z_y \equiv h_1(y) h_2(y), \quad \forall y \in G^{(d)}_{n-2}. \tag{7.3}$$

If the degree of h_1 plus the degree of h_2 in the j-th variable does not exceed $n_j - 2$, $j = 1, \ldots, d$, then the uniqueness of the interpolant implies that h coincides with the product between polynomials in the usual sense. The uniqueness holds also for $d \geq 2$ thanks to the tensor form of the grid $G^{(d)}_{n-2}$ (see [2] for more details). The very same idea applies when considering inversion. In conclusion, with this careful definition of the product/inversion and with the standard definition of addition, $\mathcal{P}^{(d,\text{even})}_{\boldsymbol{n}-2\boldsymbol{e}}$ has become an algebra, showing the vector-space dimension equal to $(n_1 - 2) \cdot (n_2 - 2) \cdot \ldots \cdot (n_d - 2)$ which coincides with that of $\mathcal{AR}_{\boldsymbol{n}}$.

Without loss of generality and for the sake of notational clarity, in the following we assume $n_j = n$ for $j = 1, \ldots, d$. Thanks to the tensor structure emphasized in (7.1)–(7.2), and by using Theorem 4.5 for every term $AR_n(\cos(\alpha_j y_j))$, $j = 1, \ldots, d$, of $AR_{\boldsymbol{n}}(m)$ the d-level extension of such a theorem easily follows. More precisely, if h is a d-variate real-valued cosine symbol related to a d-dimensional strongly symmetric and normalized mask $\mathbf{h}$, then

$$AR_{\boldsymbol{n}}(h) = T^{(d)}_{\boldsymbol{n}} D \tilde{T}^{(d)}_{\boldsymbol{n}}, \qquad T^{(d)}_{\boldsymbol{n}} = T_n \otimes \cdots \otimes T_n, \qquad \tilde{T}^{(d)}_{\boldsymbol{n}} = \tilde{T}_n \otimes \cdots \otimes \tilde{T}_n, \tag{7.4}$$

(d times) where D is the diagonal matrix containing the eigenvalues of $AR_{\boldsymbol{n}}(h)$. The description of D in d dimensions is quite involved when compared with the case $d = 1$ reported in Theorem 4.1.

For a complete analysis of the spectrum of $AR_{\boldsymbol{n}}(h)$ we refer the reader to [2]. Here we give details on a specific aspect. More precisely we attribute a correspondence in a precise and simple way among eigenvalues and eigenvectors, by making recourse only

to the main d-variate symbol $h(\cdot)$. Let $\mathbf{x}_{\boldsymbol{n}} = \mathbf{x}_n^{(1)} \otimes \mathbf{x}_n^{(2)} \otimes \cdots \otimes \mathbf{x}_n^{(d)}$ be a column of $T_{\boldsymbol{n}}^{(d)}$, with $\mathbf{x}_n^{(j)} \in \{\alpha_n^{-1}[1, p^T, 0]^T, \alpha_n^{-1}[0, (p')^T, 1]^T\}$ or $\mathbf{x}_n^{(j)} = [0, \mathbf{q}_{v_j}^T, 0]^T$, $1 \le v_j \le n-2$ and $\mathbf{q}_{v_j}$ is the (v_j)-th column of Q_{n-2}, for $j = 1, \ldots, d$. Let

$$\mathcal{F}_{\mathbf{x}_{\boldsymbol{n}}} = \{j \mid \mathbf{x}_n^{(j)} = \alpha_n^{-1}[1, p^T, 0]^T \ \text{ or } \ \mathbf{x}_n^{(j)} = \alpha_n^{-1}[0, (p')^T, 1]^T\} \subset \{1, \ldots, d\},$$

with $\mathbf{x}_{\boldsymbol{n}}$ being the generic eigenvector, i.e., the generic column of $T_{\boldsymbol{n}}^{(d)}$. The eigenvalue related to $\mathbf{x}_{\boldsymbol{n}}$ is

$$\lambda = h\left(y_1^{(n)}, \ldots, y_d^{(n)}\right) \tag{7.5}$$

where $y_j^{(n)} = 0$ for $j \in \mathcal{F}_{\mathbf{x}_{\boldsymbol{n}}}$ and $y_j^{(n)} = \frac{\pi v_j}{n-1}$ for $j \notin \mathcal{F}_{\mathbf{x}_{\boldsymbol{n}}}$. Therefore for every fixed integer $d \ge 1$, by defining the multi-set

$$\hat{G}_{\boldsymbol{n}}^{(d)} = \hat{G}_n^{(1)} \times \cdots \times \hat{G}_n^{(1)} \qquad d \text{ times}, \tag{7.6}$$

where $\hat{G}_n^{(1)} = \hat{G}_n = \{0\} \cup G_{n-2} \cup \{0\}$, the following compact and elegant result can be stated (its proof is omitted since it is simply the combination of the eigenvalue analysis in [2], of Theorem 4.5, and of the previous tensor arguments).

THEOREM 7.1. [$AR_{\boldsymbol{n}}(\cdot)$ Jordan Canonical Form.] *The $n^d \times n^d$ AR-BC blurring matrix A, obtained when using a strongly symmetric d-dimensional mask $\mathbf{h}$ such that $h_i = 0$ if $|i_j| \ge n-2$ for some $j \in \{1, \ldots, d\}$ (the d-dimensional degree condition), $n \ge 3$, coincides with*

$$AR_{\boldsymbol{n}}(h) = T_{\boldsymbol{n}}^{(d)} \mathrm{diag}_{\mathbf{y} \in \hat{G}_{\boldsymbol{n}}^{(d)}} \left(h(\mathbf{y})\right) \tilde{T}_{\boldsymbol{n}}^{(d)}, \tag{7.7}$$

where the relevant objects $T_{\boldsymbol{n}}^{(d)}$, $\tilde{T}_{\boldsymbol{n}}^{(d)}$, $\hat{G}_{\boldsymbol{n}}^{(d)}$ are defined in (7.4) *and* (7.6)*, and where the ordering of the entries $\mathbf{y}$ in the multi-set $\hat{G}_{\boldsymbol{n}}^{(d)}$ is the one indicated in* (7.5) *and connecting an arbitrary column $\mathbf{x}_n^{(1)} \otimes \mathbf{x}_n^{(2)} \otimes \cdots \otimes \mathbf{x}_n^{(d)}$ of $T_{\boldsymbol{n}}^{(d)}$ with $\mathbf{y} \in \hat{G}_{\boldsymbol{n}}^{(d)}$.*

This shows that the study of the anti-reflective transform is not only of interest in itself both from theoretical and computational viewpoints, but it is also useful in simplifying the analysis and the interpretation of the eigenvalues studied in [2]: in this sense compare the elegant result in Theorem 7.1 and the quite tricky combinatorial representation in [2].

It is finally worth observing that, except for more involved multi-index notations, both Algorithm 1 and Algorithm 2 in Section 5 are plainly generalized in the multilevel setting, maintaining a cost proportional to three d-level FSTs of size $(n-2)^d$, and the key tool is the simplified eigenvalue-eigenvector correspondence concisely indicated in Theorem 7.1. Indeed, for the Algorithm 1 the only difficult task is computation in step 2, where we have to compute the eigenvalues in the right order. For this task we refer to [2], where an algorithm is proposed and studied: more specifically the related procedure in [2] is based on a single d-level FST of size $(n-2)^d$ plus lower order computations. For the second strategy in Section 5, we can operate as suggested in Remark 5.3. We employ Algorithm 1 where we fix a priori $\phi_{\boldsymbol{k}} = 1$, when the corresponding grid point $\hat{G}_{\boldsymbol{n}}^{(d)}$ is equal to the d-dimensional null zero (notice that, according to (7.6) and to the definition of $\hat{G}_n^{(1)}$, there exist 2^d of such points in the d-dimensional setting, i.e., when considering the multi-set $\hat{G}_{\boldsymbol{n}}^{(d)}$).

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