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by

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TURÁN'S THEOREM FOR PSEUDO-RANDOM GRAPHS

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Dedicated to Professor R.L. Graham on the occasion of his 70th birthday

Abstract. The generalized Turán number $\mathrm{ex}(G,H)$ of two graphs G and H is the maximum number of edges in a subgraph of G not containing H. When G is the complete graph K_m on m vertices, the value of $\mathrm{ex}(K_m,H)$ is $(1-1/(\chi(H)-1)+o(1))\binom{m}{2}$, where $o(1)\to 0$ as $m\to\infty$, by the Erdős–Stone–Simonovits Theorem.

In this paper we give an analogous result for triangle-free graphs H and pseudo-random graphs G. Our concept of pseudo-randomness is inspired by the *jumbled* graphs introduced by Thomason [31]. A graph G is (q,α)-*bi-jumbled* if

$$\big|e_G(X,Y)-q|X||Y|\big|\le\alpha\sqrt{|X||Y|}$$

for every two sets of vertices X, $Y\subset V(G)$. Here $e_G(X,Y)$ is the number of pairs (x,y) such that $x\in X$, $y\in Y$, and $xy\in E(G)$. This condition guarantees that G and the binomial random graph with edge probability q share a number of properties.

Our results imply that, for example, for any triangle-free graph H with maximum degree Δ and for any $\delta>0$ there exists $\gamma>0$ so that the following holds: any large enough m-vertex, $(q,\gamma q^{\Delta+1/2}m)$-bi-jumbled graph G satisfies

$$\mathrm{ex}(G,H)\le\left(1-\frac{1}{\chi(H)-1}+\delta\right)\big|E(G)\big|.$$

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1. Introduction

We say that a graph is *H-free* if it does not contain a copy of a given graph H as a subgraph (not necessarily induced). A classical area of extremal graph theory investigates numerical and structural results concerning H-free graphs. A basic problem in this area is to determine, or estimate, the maximum number of edges $\mathrm{ex}(m,H)$ that an H-free graph on m vertices may have. When H is a complete graph, we know the value of $\mathrm{ex}(m,H)$ precisely, by Turán's theorem [32] (hence we refer to $\mathrm{ex}(m,H)$ as the *Turán number of H*). When H is arbitrary, an asymptotic solution to this problem is given by the celebrated Erdős–Stone–Simonovits theorem, at least when $\chi(H) \geq 3$.

Theorem 1 (Erdős, Stone, Simonovits [12, 10])**.** *For every graph H with chromatic number $\chi(H)$,*

$$\mathrm{ex}(m,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{m}{2}, \tag{1}$$

where $o(1) \to 0$ as $m \to \infty$.

Here we are interested in a variant of the function $\mathrm{ex}(m,H)$. Denote by $\mathrm{ex}(G,H)$ the maximum number of edges that an H-free subgraph of a given graph G may have, i.e.,

$$\mathrm{ex}(G,H) = \max\left\{|E(G')|\colon H \not\subset G' \subset G\right\}.$$

For instance, if $G = K_m$, the complete graph on m vertices, then $\mathrm{ex}(K_m,H)$ is the usual Turán number $\mathrm{ex}(m,H)$. Furthermore, by considering a random partition of vertices of G into $\chi(H)-1$ parts, one easily observes that

$$\mathrm{ex}(G,H) \geq \left(1 - \frac{1}{\chi(H)-1}\right)|E(G)| \tag{2}$$

holds for any G and H.

Let us mention a few problems and results concerning the "generalized Turán function" $\mathrm{ex}(G,H)$. The case in which G is the n-dimensional hypercube Q^n and H is a short even cycle, say, the four-cycle C_4, was raised by Erdős in the mid-70s, who conjectured that $\mathrm{ex}(Q^n, C_4) = (1/2+o(1))|E(Q^n)|$ as $n \to \infty$ (see [8]; for the best result in this direction, see Chung [4]).

Three results for the case in which G is a random graph are due to Frankl and Rödl [13], Babai, Simonovits, and Spencer [3], and Füredi [14]. In [13], the authors investigate $\mathrm{ex}(\mathcal{G}(m,q), K_3)$ for q around $m^{-1/2}$, where $\mathcal{G}(m,q)$ denotes the binomial random graph on m vertices with edge probability q, and we write K_t for the complete graph on t vertices. In [3], the authors investigate in detail the function $\mathrm{ex}(\mathcal{G}(m,1/2), K_3)$. In [14], the author investigates $\mathrm{ex}(\mathcal{G}(m,q), C_4)$ for q around $m^{-2/3}$.

More recently, there has been a more systematic study of the function $\mathrm{ex}(\mathcal{G}(m,q), H)$. For a discussion on this topic, the reader is referred to [17, Chapter 8], [20], and [15], and the references therein. Roughly speaking, the main problem is to identify the threshold for $q = q(m)$ for the property that the lower bound in (2) should be asymptotically tight, giving the almost sure value of $\mathrm{ex}(\mathcal{G}(m,q), H)$. The thresholds for $H = K_3$ and $H = C_4$ are $m^{-1/2}$ and $m^{-2/3}$, and these are determined in [13] and in [14] cited above. The original conjecture about the threshold for H arbitrary, which is still open, may be found in [19].

For deterministic graphs G the known results in this direction are for the so called (m,d,λ)-graphs. Let G be a graph and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ be the eigenvalues of its adjacency matrix. We say that G is an (m,d,λ)-graph if it has m vertices, it is d-regular and $\max\{\lambda_2, -\lambda_m\} \leq \lambda$. Sudakov, Szabó, and Vu [29] proved that, for $t \geq 3$,

$$\mathrm{ex}(G, K_t) = \left(1 - \frac{1}{t-1} + o(1)\right)|E(G)| \tag{3}$$

for any (m,d,λ)-graph G, as long as

$$d^{t-1}/m^{t-2} \gg \lambda, \tag{4}$$

that is, $\lim_{m\to 0} m^{t-2}\lambda/d^{t-1} = 0$. A result of Krivelevich, Sudakov, and Szabó [27, Theorem 1.2], inspired by a construction of Alon [1], implies that condition (4) is essentially best possible for $t = 3$ and basically any $d = d(n) = \Omega(n^{2/3})$. For $t > 3$ it is not known whether condition (4) is optimal. For a very recent result in this direction, see Chung [5].

In this paper we prove a similar result for triangle-free graphs H. First, we need to define some simple graph parameters. For any graph H, we define the *degeneracy* d_H of H by

$$d_H = \max\{\delta(H')\colon H' \subset H\},$$

where $\delta(H')$ denotes the minimum degree of the graph H'. We remark that d_H is the smallest integer d for which there exists an ordering $u_1, \ldots, u_h$ of the vertices of H ($h = |V(H)|$) in which u_i has at most d neighbors among $u_1, \ldots, u_{i-1}$, for every $1 \le i \le h$. Any such ordering is called a *d-degenerate ordering.*

Moreover, let

$$D_H = \min\{2d_H, \Delta(H)\},$$

where $\Delta(H)$ stands for the maximum degree of H. We shall later make use of the following simple fact: for any d_H-element set F of vertices of H, there is a D_H-degenerate ordering $u_1, \ldots, u_h$ of $V(H)$ with $F = \{u_1, \ldots, u_{d_H}\}$. Finally, let

$$\nu_H = \frac{1}{2}(d_H + D_H + 1). \tag{5}$$

Remark 2. *The parameters D_H and ν_H are somewhat artificial. For simplicity, the reader may prefer to replace D_H by $\Delta(H)$ in* (5) *at the first reading, although some statements below are considerably weaker with this change. For example, if $H = K_{2,t}$ and t is large, then $d_H = 2$, $D_H = 4$, $\nu_H = 7/2$, and condition* (6) *becomes much more restrictive with ν_H replaced by $(d_H + \Delta(H) + 1)/2 = (t + 3)/2 \gg 7/2$.*

Our main theorem, to be given in a short while, implies the following result for (m, d, λ)-graphs.

Theorem 3. *Let a triangle-free graph H and $\delta > 0$ be given. Then there exists $\gamma = \gamma(\delta, H) > 0$ such that for any function $d = d(m) < m$ there is $M_0 = M_0(\delta, H, d)$ such that any (m, d, λ)-graph G with $m \ge M_0$ vertices and satisfying*

$$\gamma d^{\nu_H}/m^{\nu_H - 1} \ge \lambda \tag{6}$$

has

$$\mathrm{ex}(G, H) \le \left(1 - \frac{1}{\chi(H) - 1} + \delta\right)|E(G)|.$$

Notice that to satisfy (6) for graphs with $\lambda = O(\sqrt{d})$ (e.g., for Ramanujan graphs), it suffices to have $d^{2\nu_H - 1} = d^{D_H + d_H} \gg m^{D_H + d_H - 1} = m^{2\nu_H - 2}$.

Theorem 3 is a consequence of a more general theorem for pseudo-random graphs; see Theorem 5 in the next section.

2. The result for pseudo-random graphs

A graph is *pseudo-random* if it resembles (in some well-defined sense) a random graph of the same density. The systematic study of such graphs was initiated by Thomason [31], who introduced the notion of *jumbled* graphs. A graph G is *(q, α)-jumbled* if for every $X \subset V(G)$ we have

$$\left| e_G(X) - q\binom{|X|}{2} \right| \le \alpha |X|,$$

where $e_G(X)$ denotes the number of edges in G with both endpoints in X.

Here we shall use a closely related concept of pseudo-randomness. For any sets X, $Y \subset V$, we write

$$E_G(X,Y) = \{(x,y)\colon x \in X,\ y \in Y,\ \{x,y\} \in E(G)\}.$$

We also set $e_G(X,Y) = |E_G(X,Y)|$ and $d_G(X,Y) = e_G(X,Y)/|X||Y|$. Note that each edge in $X \cap Y$ is counted twice in $e_G(X,Y)$. Thus, if $X = Y$ then $e_G(X) = e_G(X,X)/2$ is the number of edges in G with both endpoints in X. We drop the subscript G whenever there is no danger of confusion.

Definition 4. *We say that a graph G is (q,α)-bi-jumbled if for every X, $Y \subset V(G)$ we have*

$$\big|e_G(X,Y) - q|X||Y|\big| \le \alpha\sqrt{|X||Y|}. \tag{7}$$

It is easy to see that every (q,α)-bi-jumbled graph is also $(q,(\alpha+q)/2)$-jumbled. Since we consider only $0 \le q \le 1 < \alpha$, we immediately have that every (q,α)-bi-jumbled graph is (q,α)-jumbled.

To put (7) into some context, on the one hand, we observe that the random graph $\mathcal{G}(m,q)$ is almost surely (q,α)-bi-jumbled for $\alpha = O(\sqrt{qm})$ if, say, $qm \gg \log m$.[1] On the other hand, Erdős and Spencer [11] (see also Theorem 5 in [9]) observed that there exists $c > 0$ such that every m-vertex graph with density q contains two disjoint sets X and Y for which $\big|e(X,Y) - q|X||Y|\big| \ge c\sqrt{qm}\sqrt{|X||Y|}$, as long as $q(1-q) \ge 1/m$.

We may finally state our main theorem, Theorem 5. Our main result shows that if α is sufficiently small, then (2) is asymptotically optimal for any sufficiently large (q,α)-bi-jumbled graph G and any triangle-free graph H.

Theorem 5. *Let a triangle-free graph H and $\delta > 0$ be given. Then there exists $\gamma = \gamma(\delta, H) > 0$ with the following property: For any function $q = q(m)$ there is $M_0 = M_0(\delta, H, q)$ such that any $(q, \gamma q^{\nu_H} m)$-bi-jumbled graph G with $m \ge M_0$ vertices has*

$$\mathrm{ex}(G,H) \le \left(1 - \frac{1}{\chi(H)-1} + \delta\right)|E(G)|. \tag{8}$$

Remark 6. *From $c\sqrt{qm} \le \alpha = \gamma q^{\nu_H} m$ we deduce that the inequality $q \ge q_{\gamma,H}(m) := (c^{-2}\gamma^2 m)^{-1/(2\nu_H - 1)}$ must hold for any m-vertex, $(q,\gamma q^{\nu_H} m)$-bi-jumbled graph. Hence, we may assume that the function q in Theorem 5 satisfies $q \ge q_{\gamma,H}(m)$ for otherwise there is no $(q,\gamma q^{\nu_H} m)$-bi-jumbled graph G and the statement of Theorem 5 holds trivially.*

Theorem 5 gives a sufficient condition on the bi-jumbledness of G for (8) to hold. Assuming the best possible bi-jumbledness $\alpha = O(\sqrt{qm})$, the bi-jumbledness hypothesis $\alpha = \gamma q^{\nu_H} m$ in Theorem 5 gives a condition on the density q for (8) to hold. We illustrate this on the concrete example $H =$

[1] This fact may be checked by combining Lemma 3.8 in [16] and the fact that, almost surely, all vertices of $\mathcal{G}(m,q)$ have degree $(1+o(1))qm$ for $qm \gg \log m$.

C_5. We have $\nu_{C_5} = 5/2$. Theorem 5 implies that, for any $(q, O(\sqrt{qm}))$-bi-jumbled graph G, we have $\mathrm{ex}(G, C_5) = (1/2 + o(1))|E(G)|$ if $q^{5/2}m \gg \sqrt{qm}$, that is, $q \gg m^{-1/4}$. However, a fairly simple argument based on the Sparse Regularity Lemma (Proposition 8) yields that the much weaker condition $q \gg m^{-1/2}$ actually suffices.

Let us mention that there exists a C_5-free, $(q, O(\sqrt{qm}))$-bi-jumbled graph A_5 with $q = \Theta(m^{-3/5})$. Thus, the "threshold" for this problem lies between $m^{-3/5}$ and $m^{-1/2}$. Hence, even in this case, we do not have an optimal result. For a general odd cycle $C_{2\ell+1}$, the "threshold" is between $m^{-1+2/(2\ell+1)}$ and $m^{-1+1/\ell}$. The lower end of the gap is proved considering a $C_{2\ell+1}$-free, $(q, O(\sqrt{qm}))$-bi-jumbled graph $A_{2\ell+1}$ with $q = \Theta(m^{-1+2/(2\ell+1)})$. The existence of the graphs $A_{2\ell+1}$ may be proved suitably adapting a beautiful construction of Alon [1] (see also [26, Section 3, Example 10]).

We believe that it would be interesting to weaken the bi-jumbledness hypothesis in Theorem 5. Moreover, it would be interesting to drop the triangle-freeness condition on the graph H. In this direction, we only mention that in [23] a result similar to Theorem 5 for arbitrary graphs H is proved, but a stronger bi-jumbledness hypothesis on G is required. We finish our introduction deducing Theorem 3 from Theorem 5 and giving a brief description of the structure of this paper.

Proof of Theorem 3. It is a well-known fact (see Corollary 9.2.5 in [2]) that any (m, d, λ)-graph G is $(d/m, \lambda)$-bi-jumbled. Therefore, the graph G will be $(d/m, \gamma(d/m)^{\nu_H} m)$-bi-jumbled if

$$\lambda \leq \gamma(d/m)^{\nu_H} m, \tag{9}$$

which is equivalent to (6). □

This paper is organized as follows. In the next section we present additional definitions and notation and we attempt to give a fairly detailed outline of the proof of Theorem 5. In Section 3 we also state all the auxiliary lemmas (including what we call the Regularity-to-Pair Lemma, the Pair-to-Tuple Lemma, and the Embedding Lemma) needed for the proof of Theorem 5. In Section 4 we give the proof of Theorem 5 and in Section 5 we prove some technical facts needed in the proofs of the Embedding Lemma and the Pair-to-Tuple Lemma. Sections 6 and 7 contain the proofs of the Embedding Lemma and the Pair-to-Tuple Lemma.

3. An outline of the proof of Theorem 5 and auxiliary results

In this section we introduce all the necessary tools for the proof of Theorem 5. We also try to motivate these tools by discussing the underlying ideas in the proof.

3.1. **Additional definitions and notation.** We start with some basic notation. Let ℓ be a positive integer. We denote by $[\ell]$ the set $\{1, 2, \dots, \ell\}$. For a multiset $I = \{i_1, \dots, i_r\}$ we write $I \subset [\ell]$ to mean that $i_1, \dots, i_r \in [\ell]$.

We also adopt the convention that we always write the elements of I in non-decreasing order, i.e., $i_1 \leq \ldots \leq i_r$. For three real numbers a, b, and c, the expression $a = b \pm c$ means $b - c \leq a \leq b + c$. We also write a/bc instead of $a/(bc)$ whenever there is no danger of confusion. For functions $f = f(n)$ and $g = g(n)$ we write $f \gg g$ and $g \ll f$ if $\lim_{n\to\infty} f(n)/g(n) = +\infty$. For clarity, we omit inessential floor and ceiling brackets.

Let $G = (V, E)$ be a graph. For a vertex $x \in V$ let $N(x)$ be the set of all neighbors of x in G. If $U \subset V$ then $N_U(x)$ denotes the set of neighbors of $x \in V$ belonging to U, that is, $N_U(x) = N(x) \cap U$. For an r-set $X = \{x_1, \ldots, x_r\} \subset V$ and a set $U \subset V$, we let $N(X) = N(x_1, \ldots, x_r) = N(x_1) \cap \ldots \cap N(x_r)$ and $N_U(X) = N_U(x_1, \ldots, x_r) = N(x_1) \cap \ldots \cap N(x_r) \cap U$.

We say that J is an *(ℓ, n, p)-partite graph* if J is ℓ-partite with $V(J) = \bigcup_{j=1}^{\ell} V_j$, $|V_j| = n$ for all $j \in [\ell]$, and $e(J[V_i, V_j]) = pn^2$ for all $i \neq j \in [\ell]$.

For an (ℓ, n, p)-partite graph J, an integer $r \geq 1$, and a multiset $I = \{i_1, \ldots, i_r\} \subset [\ell]$, denote by $\mathcal{T}(I)$ the set of all r-tuples $(x_1, \ldots, x_r) \in V_{i_1} \times \cdots \times V_{i_r}$ such that $x_i \neq x_j$ for all $1 \leq i < j \leq r$. Note that

$$(n - r)^r < (n - r + 1)^r \leq |\mathcal{T}(I)| \leq n^r, \tag{10}$$

since each x_i can be chosen in at least $n - r + 1$ ways to avoid $x_1, \ldots, x_{i-1}$.

Now we define two important properties of (ℓ, n, p)-partite graphs.

Definition 7. *An (ℓ, n, p)-partite graph J has property* $\mathrm{TUPLE}_\ell(\varepsilon, d)$ *if for every integer $1 \leq r \leq d$, every multiset $I = \{i_1, \ldots, i_r\} \subset [\ell]$, and for all $j \in [\ell] \setminus I$, we have*

$$\left|N_{V_j}(x_1, \ldots, x_r)\right| = (1 \pm \varepsilon)p^r n$$

for all but at most εn^r r-tuples $(x_1, \ldots, x_r) \in \mathcal{T}(I)$.

When $d = 2$ we say that J satisfies the pair condition $\mathrm{PAIR}_\ell(\varepsilon)$.

3.2. **An outline of the proof of Theorem 5.** We now outline the proof of Theorem 5. We hope that this will motivate the somewhat technical looking auxiliary lemmas that will be required. Our proof strategy is natural and proceeds as follows: consider an arbitrary spanning subgraph G' of G with

$$|E(G')| \geq \left(1 - \frac{1}{\chi(H) - 1} + \delta\right) |E(G)|. \tag{11}$$

Suppose first that the density q of G is constant, independent of n. We apply Szemerédi's regularity lemma [30] with ε significantly smaller than δ and q, and obtain a partition $\bigcup_{j=0}^{t} V_j$ of the vertex set of G' into a bounded number of parts t so that all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are *ε-regular*, i.e., for all $V_i' \subset V_i$ and $V_j' \subset V_j$ with $|V_i'| \geq \varepsilon|V_i|$ and $|V_j'| \geq \varepsilon|V_j|$, we have

$$\left|d_G(V_i, V_j) - d_G(V_i', V_j')\right| \leq \varepsilon.$$

Standard arguments (see, e.g., [7, Section 7.5, pp. 186–187]) show that there exist $\ell = \chi(H)$ sets (without loss of generality, $V_1, \ldots, V_\ell$) so that there is an (ℓ, n, p)-partite graph $J \subset G'$ with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$ such that $p = \alpha q$, α is considerably larger than ε, and each pair (V_i, V_j) is ε-regular.[2] It is a well-known fact (e.g., Theorem 3.1 in [25]) that J satisfying the conditions above also contains a copy of H. We remark that this statement (sometimes called the embedding lemma) is ensured by the regularity of J itself.

For the case $q = o(1)$ we utilize the same approach. First we apply a version of the regularity lemma for sparse graphs (Proposition 8) to G' and obtain an (ε, G', q)-regular partition $V_0 \cup V_1 \cup \ldots \cup V_t$ of $V(G')$ (see Section 3.3 for the relevant definitions). In the same way as above (see Section 4 for the details) we obtain an (ℓ, n, p)-partite graph J with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$ with $p = \alpha q$ for some constant $\alpha > 0$ and such that each pair (V_i, V_j) is (ε, q)-regular. Unfortunately, the embedding lemma required in this context does not hold (see, e.g., [20, Theorem B′]).

One way to deal with this problem is to restrict the choice of G to certain classes of graphs (such as random graphs) and to prove an appropriate embedding lemma that works for their subgraphs G' (for instance, see Theorem B″ in [20] and Lemmas 2.2 and 2.2′ in [22]). In this paper, roughly speaking, we follow an approach in [24]. An embedding lemma in [24] is as follows:

(*) *Let H be a triangle-free graph and C a positive constant. If J is an n-vertex graph with density $p = p(n) = |E(J)|\binom{n}{2}^{-1} \gg n^{-1/D_H}$ satisfying properties* $\mathrm{BDD}(C, D_H)$ *and* PAIR *given below, then J contains H as a subgraph, as long as n is sufficiently large.*

$\mathrm{BDD}(C, D_H)$: $|N_J(x_1, \ldots, x_r)| \leq C^r p^r n$ holds for all mutually distinct vertices $x_1, \ldots, x_r \in V(J)$ and $1 \leq r \leq D_H$.

PAIR: $|N_J(x_1, x_2)| = (1 + o(1))p^2 n$ holds for all but at most $o(n^2)$ pairs $\{x_1, x_2\} \subset V(J)$.

Going back to the proof of Theorem 5, we recall that we had arrived at an (ℓ, n, p)-partite graph $J \subset G'$ with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$ such that $p = \alpha q$ with α a positive constant and each pair (V_i, V_j) is (ε, q)-regular. The first discrepancy that one notices between our current set-up and the hypotheses in (*) is that our J is ℓ-partite, whereas in (*) we do not have an ℓ-partite

[2] From $\bigcup_{j=0}^{t} V_j$ we construct a "cluster" graph F_c on $[t]$ whose edges are those pairs $\{i, j\}$ for which the pair (V_i, V_j) is ε-regular and has large density. The graph F_c has enough edges for us to apply Turán's Theorem and obtain a copy of $K_{\chi(H)}$. To deduce this we need $d_{G'}(V_i, V_j) \leq (1 + o(1))q$ for all $i \neq j$, which may be guaranteed by the bi-jumbledness of G. This copy of $K_{\chi(H)}$ corresponds to a subgraph of G' that may be further reduced or "sliced" (see Lemma 9) to the $(\chi(H), n, p)$-partite graph J we are looking for.

graph. As the reader may guess, albeit cumbersome, this difference is not essential, and one may in fact prove an appropriate "ℓ-partite version" of (*). In the discussion that follows, for simplicity, when not important, we shall blur this discrepancy and we shall ignore the fact that we have an ℓ-partite graph J at hand.

A more substantial discrepancy occurs in the hypotheses $\mathrm{BDD}(C, D_H)$ and PAIR in (*): in our proof of Theorem 5, we have arrived at an ℓ-partite (ε, q)-regular graph J.

Achieving PAIR. The reader may be familiar with the fact that in the case of dense graphs, the $o(1)$-regularity of a pair (V_i, V_j) and property $\mathrm{PAIR}(o(1))$ are equivalent (in a certain precise sense, see [31] and [6] for details). Unfortunately, this equivalence breaks down in the sparse setting, as observed in Theorem A′ in [20]. Therefore, achieving hypothesis PAIR in (*) requires some work. This will be accomplished by making use of Proposition 10 below, which, roughly speaking, states that one recovers the fact that $o(1)$-regularity implies PAIR if one has a graph J that is a subgraph of a suitably bi-jumbled graph Γ, as long as one has a positive fraction α of the edges of Γ in J.

Loosening BDD. Let us now discuss hypothesis $\mathrm{BDD}(C, D_H)$ in (*). Basically, if G has property $\mathrm{BDD}(C, D_H)$, then any subgraph $J \subset G$ with a positive fraction of the edges of G has $\mathrm{BDD}(C', D_H)$ for some $C' \geq C$. As it turns out, the constant C' ends up depending on some other parameters in the proof in such a way that we are not able to use this simple hereditary property of BDD. Therefore, we take a different route. For every $1 \leq r \leq D_H$, define an r-uniform hypergraph $\mathcal{B}_r$ on the vertex set of J, putting an r-set $B \subset V(J)$ in $\mathcal{B}_r$ if the joint neighborhood $N_J(B)$ of B in J violates the upper bound in the definition of $\mathrm{BDD}(C, D_H)$, that is, $|N_J(B)| > C^r p^r n$. A simple consequence of the bi-jumbledness of G is that the hypergraphs $\mathcal{B}_r$ are in a certain sense locally sparse: for every $1 \leq r \leq D_H$, if an $(r-1)$-set is not a member of $\mathcal{B}_{r-1}$, then it cannot be contained in many members of $\mathcal{B}_r$ (see Lemma 14). This sparseness of the $\mathcal{B}_r$ turns out to be enough for our purposes.

The embedding lemma. As the discussion above suggests, the embedding lemma that we shall make use of is an ℓ-partite variant of (*), with the BDD hypothesis replaced by the hypothesis that J should be a subgraph of a suitably bi-jumbled graph G, with a positive fraction α of the edges of G in J. From this hypothesis, one obtains PAIR and the local sparseness of the $\mathcal{B}_r$. For the precise statement of this embedding lemma, see Proposition 12.

Before we finish this outline, we just mention a step in the proof of this embedding lemma. We remark that we shall use the bi-jumbledness of G to show that, in fact, PAIR implies the following property:

$\mathrm{TUPLE}(d_H)$: $|N_J(x_1, \dots, x_r)| = (1 + o(1))p^r n$ holds for all but at most $o(n^r)$ r-sets $\{x_1, \dots, x_r\} \subset V(J)$ for any $1 \leq r \leq d_H$

(see Proposition 11). Going from PAIR to TUPLE(d_H) is also an important step in the proof of (*). For this step, hypothesis BDD is used in [24]; here, in the proof of Proposition 11, we again replace BDD with the local sparseness of the $\mathcal{B}_r$.

3.3. **The Sparse Regularity Lemma.** Let $G = (V, E)$ be a graph. Suppose $0 < q \leq 1$, $\xi > 0$ and $C > 1$. For two disjoint subsets X, Y of V, we let

$$d_{G,q}(X,Y) = \frac{e_G(X,Y)}{q|X||Y|},$$

which we refer to as the *q-density* of the pair (X, Y).

We say that G is a *(ξ, C)-bounded* graph with respect to density q if for all pairwise disjoint X, $Y \subset V$, with $|X|$, $|Y| \geq \xi|V|$, we have $e_G(X,Y) \leq Cq|X||Y|$.

For $\varepsilon > 0$ fixed and X, $Y \subset V$, $X \cap Y = \emptyset$, we say that the pair (X, Y) is *(ε, q)-regular* if for all $X' \subset X$ and $Y' \subset Y$ with

$$|X'| \geq \varepsilon|X| \quad \text{and} \quad |Y'| \geq \varepsilon|X|,$$

we have

$$|d_{G,q}(X,Y) - d_{G,q}(X',Y')| \leq \varepsilon.$$

Note that for $q = 1$ we get the well-known definition of ε-regularity [30].

When G is (ℓ, n, p)-partite with ℓ-partition $\bigcup_{i=1}^{\ell} V_i$ we say that G is (ε, q)-regular if all pairs (V_i, V_j), $1 \leq i < j \leq \ell$, are (ε, q)-regular.

Let $\bigcup_{j=0}^{t} V_j$ be a partition of V. We call V_0 the *exceptional class*. This partition is called *(ε, t)-equitable* if $|V_0| \leq \varepsilon|V|$ and $|V_1| = \ldots = |V_t|$.

We say that an (ε, t)-equitable partition $\bigcup_{j=0}^{t} V_j$ of V is *(ε, G, q)-regular* if all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq k$, are (ε, q)-regular. Now we can state a variant of Szemerédi's regularity lemma [30] for sparse graphs (see, e.g., [18, 21]).

Proposition 8 (Sparse Regularity Lemma). *For any $\varepsilon > 0$, $C > 1$, and $t_1 \geq 1$, there exist constants $T_1 = T_1(\varepsilon, C, t_1)$, $\xi = \xi(\varepsilon, C, t_1) \leq \min\{1/2T_1, \varepsilon\}$, and $M_1 = M_1(\varepsilon, C, t_1)$ such that any graph G with at least M_1 vertices that is (ξ, C)-bounded with respect to density $0 < q \leq 1$ admits an (ε, t)-equitable (ε, G, q)-regular partition of its vertex set with $t_1 \leq t \leq T_1$.*

After applying the lemma above we obtain (ε, q)-regular bipartite graphs with different densities. The next lemma will allow us to change these densities to a particular value without losing regularity.

Lemma 9 (Slicing Lemma, [22]). *For every $0 < \alpha, \varepsilon \leq 1$, $C > 1$, and a function $q = q(n)$ satisfying $qn \gg 1$ there exists $n_0 = n_0(\alpha, \varepsilon, C, q)$ such that if $B = (U \cup W, E)$ is a bipartite graph satisfying*

(i) $|U| = |W| = n > n_0$,

(ii) *$\alpha q|U||W| \le e_B(U,W) \le Cq|U||W|$, and*
(iii) *B is (ε, q)-regular,*

then there exists an $(3\varepsilon, q)$-regular subgraph $B' = (U \cup W, E') \subset B$ such that $e_{B'}(U,W) = \alpha q|U||W|$.

3.4. **Regularity and the pair condition.** The next proposition shows that, under certain restrictions, a regular (ℓ, n, p)-partite graph also has property PAIR_ℓ.

Proposition 10 (Regularity-to-Pair Lemma, [23]). *For any $0 < \alpha, \varrho \le 1$ and any integer $\ell > 1$, there exist $\delta = \delta(\alpha, \varrho, \ell) > 0$ and $\gamma = \gamma(\alpha, \varrho, \ell) > 0$ such that for every function $q = q(n)$ there exists an $n_0 = n_0(\alpha, \varrho, \ell, q) > 1$ for which the following holds.*

Let Γ be a $(q, \gamma q^2 \ell n)$-bi-jumbled graph on ℓn vertices with $n > n_0$. Suppose J is a (δ, q)-regular (ℓ, n, p)-partite subgraph of Γ satisfying $p \ge \alpha q$. Then J also has property $\mathrm{PAIR}_\ell(\varrho)$.

We remark that if p is a constant then the above statement holds for any $(\delta, 1)$-regular (ℓ, n, p)-partite graph J (i.e., J need not to be a subgraph of some bi-jumbled graph Γ).

Clearly, any graph having property $\mathrm{TUPLE}_\ell(\varepsilon, d)$, $d \ge 2$, also satisfies $\mathrm{PAIR}_\ell(\varepsilon)$. The next proposition shows that under certain conditions the converse is also true.

Proposition 11 (Pair-to-Tuple Lemma). *Let $d \ge 1$ and $\ell \ge 2$ be integers. Then for every $0 < \alpha, \varepsilon \le 1$ there exist $\delta = \delta(d, \ell, \alpha, \varepsilon) > 0$ and $\gamma = \gamma(d, \ell, \alpha, \varepsilon) > 0$ such that for every function $p = p(n)$ satisfying $p^d n \gg 1$ there is $N_0 = N_0(d, \ell, \alpha, \varepsilon, p)$ with the following property: any (ℓ, n, p)-partite graph J with $n > N_0$ satisfying*

(i) *for all $U \subset V_i$ and $W \subset V_j$, $i \ne j \in [\ell]$, we have*

$$e_J(U,W) \le \frac{p}{\alpha}|U||W| + \gamma \left(\frac{p}{\alpha}\right)^{(d+3)/2} n\sqrt{|U||W|}, \tag{12}$$

(ii) *$\mathrm{PAIR}_\ell(\delta)$,*

also satisfies $\mathrm{TUPLE}_\ell(\varepsilon, d)$.

We will apply Proposition 11 with $d = d_H$ to an (ℓ, n, p)-partite graph J that is obtained from a subgraph G' of a $(q, \gamma q^{(d_H+3)/2} m)$-bi-jumbled graph G, as explained in Section 3.2. Moreover, we shall have $p = \alpha q$ for some constant $\alpha > 0$. Condition (12) will follow from the upper bound in the $(q, \gamma q^{(d_H+3)/2} m)$-bi-jumbledness hypothesis on G (see (7)), by substituting p/α for q. The proof of Proposition 11 appears in Section 7.

3.5. **An embedding lemma for ℓ-partite bi-jumbled graphs.** In this section we state the adjusted version of one of the main results from [24] (see (*)) discussed in Section 3.2. Given an ℓ-partite graph H with ℓ-partition $V(H) = \bigcup_{j=1}^{\ell} U_j$, an *embedding* of H in an (ℓ, n, p)-partite graph J is an injective, edge preserving map $f\colon V(H) \to V(J)$ such that $f(U_j) \subset V_j$

for all $1 \le j \le \ell$. The next proposition shows that (12), TUPLE_ℓ, and large enough density guarantee an embedding of any triangle-free graph H in J.

Proposition 12 (Embedding Lemma)**.** *Let H be a fixed triangle-free, ℓ-partite graph with h vertices and e edges. Then for all $0 < \alpha, \eta \le 1$ there exist $\varepsilon = \varepsilon(H, \alpha, \eta) > 0$ and $\gamma = \gamma(H, \alpha, \eta) > 0$ such that for any function $p = p(n)$ satisfying $p^{d_H} n \gg 1$ there is $N_1 = N_1(H, \alpha, \eta, p) > 0$ for which the following holds. Suppose that*

(a) *J is an (ℓ, n, p)-partite graph and $n > N_1$,*
(b) *for all $U \subset V_i$ and $W \subset V_j$, $i \ne j \in [\ell]$, we have*

$$e_J(U, W) \le \frac{p}{\alpha}|U||W| + \gamma \left(\frac{p}{\alpha}\right)^{\nu_H} n\sqrt{|U||W|}, \tag{13}$$

(c) *J satisfies $\mathrm{TUPLE}_\ell(\varepsilon, d_H)$.*

Then the number of embeddings of H in J is at least $(1-\eta)p^e n^h$.

The proof of Proposition 12 is given in Section 6.

4. Proof of the main result

Now we are ready to prove the main result, Theorem 5. When $D_H = 1$, the graph H is a matching and we just need $qm > (4/\delta)|E(H)|$ to prove Theorem 5. Indeed, for any graph G with $|E(G)| = q\binom{m}{2} > (2/\delta)|E(H)|(m-1)$ edges, let G' be any spanning subgraph with at least $\delta|E(G)| > 2|E(H)|(m-1)$ edges. In this subgraph we find a copy of the matching H greedily.

When q is constant, Theorem 5 follows from an easy application of Szemerédi's regularity lemma (see Section 3.2 for some details). Hence, we shall henceforth suppose that $q = o(1)$, $D_H \ge 2$, and (cf. (5)) $\nu_H \ge 2$. Now we proceed with the details of the proof.

Proof of Theorem 5. Let H be a fixed, triangle-free graph with h vertices and $e \ge 1$ edges. Without loss of generality, let δ be a constant such that $1/(\chi(H) - 1) > \delta > 0$. We start our proof by choosing the constants. Since Theorem 5 and Propositions 10, 11, and 12 involve a double alternation ("$\forall\exists\forall\exists$"), this choice will consist of two rounds. In the first round (A)-(G) we address the choice of γ. After this we get the function q and we deal with the choice of M_0 in the second round (H)-(L).

(A) Set $\alpha = \delta/16$, $\eta = 1/2$, and $\ell = \chi(H) > 1$.
(B) Proposition 12 (Embedding Lemma) applied with $\alpha_{\mathrm{EL}} = \alpha$ and $\eta_{\mathrm{EL}} = \eta$ yields $\varepsilon_{\mathrm{EL}}$ and γ_{EL}.
(C) Then we apply Proposition 11 (Pair-to-Tuple Lemma) with $d = d_H$, $\alpha_{\mathrm{P2T}} = \alpha$, $\varepsilon_{\mathrm{P2T}} = \varepsilon_{\mathrm{EL}}$, and obtain δ_{P2T} and γ_{P2T}.
(D) Proposition 10 (Regularity-to-Pair Lemma) applied with $\alpha_{\mathrm{R2P}} = \alpha$ and $\varrho_{\mathrm{R2P}} = \delta_{\mathrm{P2T}}$, yields δ_{R2P} and γ_{R2P}.
(E) We define

$$\varepsilon = \min\left\{\frac{\delta_{\mathrm{R2P}}}{3}, \frac{\delta}{160}\right\}. \tag{14}$$

(F) We then apply Proposition 8 (Sparse Regularity Lemma) with $\varepsilon_{\mathrm{RL}} = \varepsilon$, $C_{\mathrm{RL}} = 1+\delta/4$, and $t_{1,\mathrm{RL}} = \max\{2\ell/\delta^{1/2}, 80/\delta\}$ and obtain $T_{1,\mathrm{RL}} \geq t_{1,\mathrm{RL}}$, $\xi_{\mathrm{RL}} \leq \min\{1/2T_{1,\mathrm{RL}}, \varepsilon\}$, and $M_{1,\mathrm{RL}}$.

(G) Finally, we fix the constant γ promised by Theorem 5 and set

$$\gamma = \frac{1-\varepsilon}{T_{1,\mathrm{RL}}} \min\left\{\gamma_{\mathrm{EL}}, \gamma_{\mathrm{P2T}}, \gamma_{\mathrm{R2P}}, \delta\xi_{\mathrm{RL}}/4\right\} \tag{15}$$

Now let $q = q(m) = o(1)$ be a function satisfying $q \geq q_{\gamma,H}(m)$ (see Remark 6). From this we have $q^{d_H} m \gg 1$.

(H) Proposition 12 (Embedding Lemma) applied with $\alpha_{\mathrm{EL}} = \alpha$, $\eta_{\mathrm{EL}} = \eta$, and $p_{\mathrm{EL}} = \alpha q$ yields N_{EL}.

(I) Then we apply Proposition 11 (Pair-to-Tuple Lemma) with $d = d_H$, $\alpha_{\mathrm{P2T}} = \alpha$, $\varepsilon_{\mathrm{P2T}} = \varepsilon_{\mathrm{EL}}$, $p_{\mathrm{P2T}} = p_{\mathrm{EL}} = \alpha q$ and obtain N_{P2T}.

(J) Proposition 10 (Regularity-to-Pair Lemma) applied with $\alpha_{\mathrm{R2P}} = \alpha$, $\varrho_{\mathrm{R2P}} = \delta_{\mathrm{P2T}}$, and $q_{\mathrm{R2P}} = q$ yields n_{R2P}.

(K) We use Lemma 9 (Slicing Lemma) with $\alpha_{\mathrm{SL}} = \alpha$, $C_{\mathrm{SL}} = 1+\delta/4$, $\varepsilon_{\mathrm{SL}} = \varepsilon$, and $q_{\mathrm{SL}} = q$ to obtain n_{SL}.

(L) Finally, we define

$$M_0 = \max\left\{1 + \frac{16}{\delta^2}, M_{1,\mathrm{RL}}, \max\{N_{\mathrm{EL}}, N_{\mathrm{P2T}}, n_{\mathrm{R2P}}, n_{\mathrm{SL}}\} \cdot T_{1,\mathrm{RL}}\right\}.$$

Let G be any $(q, \gamma q^{\nu_H} m)$-bi-jumbled graph with $m \geq M_0$ vertices, and let G' be an arbitrary spanning subgraph of G with

$$|E(G')| \geq \left(1 - \frac{1}{\ell-1} + \delta\right)|E(G)|. \tag{16}$$

By $(q, \gamma q^{\nu_H} m)$-bi-jumbledness of G and $\nu_H \geq 2$, we have

$$|E(G)| \geq \frac{qm^2 - \gamma q^{\nu_H} m^2}{2} \overset{(15)}{\geq} \left(1 - \frac{\delta}{4}\right) q \binom{m}{2}.$$

Hence (16) implies

$$|E(G')| \geq \left(1 - \frac{1}{\ell-1} + \frac{3\delta}{4}\right) q \binom{m}{2}. \tag{17}$$

We claim that G' is $(\xi_{\mathrm{RL}}, C_{\mathrm{RL}})$-bounded with respect to q. Indeed, for any two sets X, $Y \subset V(G')$, $|X|$, $|Y| \geq \xi_{\mathrm{RL}} m$, we have

$$\begin{aligned} e_{G'}(X,Y) &\leq e_G(X,Y) \leq q|X||Y| + \gamma q^{\nu_H} m \sqrt{|X||Y|} \\ &\leq \left(1 + \gamma\sqrt{\frac{m}{|X|}\frac{m}{|Y|}}\right) q|X||Y| \leq \left(1 + \frac{\gamma}{\xi_{\mathrm{RL}}}\right) q|X||Y| \overset{(15)}{\leq} C_{\mathrm{RL}} q|X||Y|. \end{aligned}$$

Since G' has at least $M_0 \geq M_{1,\mathrm{RL}}$ vertices, we can apply Proposition 8 (Sparse Regularity Lemma) to G' with parameters $\varepsilon_{\mathrm{RL}} = \varepsilon$, C_{RL}, and $t_{1,\mathrm{RL}}$ defined in (F). This yields an (ε, q)-regular (ε, G', t)-equitable partition $\bigcup_{i=0}^{t} V_i$

of $V(G')$ such that $t_{1,\mathrm{RL}} \le t \le T_{1,\mathrm{RL}}$ and $|V_1| = \ldots = |V_t| = n$, where $(1-\varepsilon)m/t \le n \le m/t$.

Let G_c be the subgraph of G' obtained by removing all edges in the following four sets:

$B_1 = \{e \in E(G') : \; e \cap V_0 \neq \emptyset\}$,

$B_2 = \bigcup_{1\le i\le t} \{e \in E(G') : \; e \subset V_i\}$,

$B_3 = \bigcup_{1\le i<j\le t} \{e \in E_{G'}(V_i,V_j) \colon (V_i,V_j) \text{ is not } (\varepsilon,q)\text{-regular in } G'\}$,

$B_4 = \bigcup_{1\le i<j\le t} \big\{e \in E_{G'}(V_i,V_j) \colon e_{G'}(V_i,V_j) < \alpha q n^2\big\}$.

Since G' is $(\xi_{\mathrm{RL}}, C_{\mathrm{RL}})$-bounded, $\xi_{\mathrm{RL}} \le 1/2T_{1,\mathrm{RL}}$, and

$$\xi_{\mathrm{RL}} m \le m/2T_{1,\mathrm{RL}} \le (1-\varepsilon)m/t \le n \le m/t,$$

for $i = 1,2,3$, $|B_i|$ can be bounded from above as follows:

$$|B_1| \le C_{\mathrm{RL}} q \varepsilon m^2,\; |B_2| \le C_{\mathrm{RL}} q \left(\frac{m}{t}\right)^2 \cdot t, \text{ and } |B_3| \le C_{\mathrm{RL}} q \left(\frac{m}{t}\right)^2 \cdot \varepsilon \binom{t}{2}.$$

By the definition of B_4, we have

$$|B_4| \le \alpha q \left(\frac{m}{t}\right)^2 \cdot \binom{t}{2}.$$

Combining the above inequalities with $m \ge 2$, $\varepsilon \le \delta/160$, $C_{\mathrm{RL}} = 1 + \delta/4$, $t \ge t_{\mathrm{RL},1} \ge 80/\delta$, $\alpha = \delta/16$, and $\delta < 1$ yields

$$\begin{aligned} e(G') - e(G_c) &\le \left(C_{\mathrm{RL}}\left(\varepsilon + \frac{1}{t} + \frac{\varepsilon}{2}\right) + \frac{\alpha}{2}\right) q m^2 \\ &\le \left(C_{\mathrm{RL}}\left(2\varepsilon + \frac{1}{t}\right) + \frac{\alpha}{2}\right) \cdot 4q \binom{m}{2} \\ &\le \left((4+\delta)\left(\frac{\delta}{80} + \frac{\delta}{80}\right) + \frac{\delta}{8}\right) \cdot q\binom{m}{2} \\ &\le \frac{\delta}{4} q \binom{m}{2}. \end{aligned} \tag{18}$$

From (17) and (18), we obtain

$$e(G_c) \ge \left(1 - \frac{1}{\ell - 1} + \frac{\delta}{2}\right) q \binom{m}{2}. \tag{19}$$

We define the *cluster* graph F_c of G' as the graph with vertex set $V(F_c) = \{1,\ldots,t\}$ and edge set

$$E(F_c) = \big\{\{i,j\} \colon \; (V_i,V_j) \text{ is } (\varepsilon,q)\text{-regular in } G' \text{ and } e_{G_c}(V_i,V_j) \ge \alpha q n^2\big\}.$$

We claim that F_c contains a copy of K_ℓ. To prove this observe first that, since G' is $(\xi_{\mathrm{RL}}, C_{\mathrm{RL}})$-bounded with respect to q, we have $e_{G_c}(V_i,V_j) = e_{G'}(V_i,V_j) \le C_{\mathrm{RL}} q n^2 \le C_{\mathrm{RL}} q (m/t)^2$ for every $1 \le i < j \le t$ such that $e_{G_c}(V_i,V_j) \neq 0$.

Now using (19), the fact that $C_{\mathrm{RL}} = 1 + \delta/4$, and the definition of F_c, we get

$$e(F_c) \geq \frac{e(G_c)}{C_{\mathrm{RL}} q(m/t)^2} \geq \left(1 - \frac{1}{\ell - 1} + \frac{\delta}{2}\right)\left(1 - \frac{1}{m}\right)\left(1 + \frac{\delta}{4}\right)^{-1} \frac{t^2}{2}.$$

Since $m \geq M_0 > 16/\delta^2$, we have

$$e(F_c) > \left(1 - \frac{1}{\ell - 1} + \frac{\delta}{2}\right)\left(1 - \frac{\delta}{4}\right)\frac{t^2}{2} > \left(1 - \frac{1}{\ell - 1}\right)\frac{t^2}{2}.$$

The above implies that F_c contains K_ℓ as a subgraph by Turán's theorem [32] (see also Exercise 7 on p. 189 in [7]). From this we now deduce that G' contains H as a subgraph.

Without loss of generality assume $\{1, \ldots, \ell\} \subset V(F_c)$ is the vertex set of a copy of K_ℓ in F_c. Since G' is $(\xi_{\mathrm{RL}}, C_{\mathrm{RL}})$-bounded with respect to q, it follows from the definition of F_c that the subgraphs $G_c[V_i, V_j] = G'[V_i, V_j]$, $1 \leq i < j \leq \ell$, satisfy

(i) $|V_i| = |V_j| = n$,
(ii) $C_{\mathrm{RL}} q n^2 \geq e(G_c[V_i, V_j]) \geq \alpha q n^2$,
(iii) $G_c[V_i, V_j]$ is (ε, q)-regular.

Therefore, we can apply Lemma 9 (Slicing Lemma) with $\varepsilon_{\mathrm{SL}} = \varepsilon \leq \delta_{\mathrm{R2P}}/3$ (see (14)), $\alpha_{\mathrm{SL}} = \alpha$, $C_{\mathrm{SL}} = C_{\mathrm{RL}} = 1 + \delta/4$, and $q_{\mathrm{SL}} = q$. This yields subgraphs $J_{ij} \subset G_c[V_i, V_j]$, $1 \leq i < j \leq \ell$, such that

(iv) $e_{J_{ij}}(V_i, V_j) = \alpha q n^2$,
(v) J_{ij} is $(3\varepsilon, q)$-regular and $3\varepsilon \leq \delta_{\mathrm{R2P}}$.

Now let Γ be the subgraph of G induced on $V_1 \cup \ldots \cup V_\ell$ and let J be the subgraph of Γ defined by

$$J = \bigcup_{1 \leq i < j \leq \ell} J_{ij}.$$

Note that since G is $(q, \gamma q^{\nu_H} m)$-bi-jumbled, $n \geq (1 - \varepsilon)m/t$, and $\nu_H \geq 2$ the graph Γ is $(q, (\gamma t/(1 - \varepsilon)\ell) q^2 \ell n)$-bi-jumbled. It follows from (15) and $t \leq T_{1,\mathrm{RL}}$ that $\gamma t/(1 - \varepsilon)\ell \leq \gamma_{\mathrm{R2P}}$. Consequently, Γ is $(q, \gamma_{\mathrm{R2P}} q^2 \ell n)$-bi-jumbled.

Furthermore, by (iv), J is (ℓ, n, p)-partite with $p = p(n) = \alpha q(n)$. By (v), J is also $(\delta_{\mathrm{R2P}}, q)$-regular.

Clearly we can apply Proposition 10 (Regularity-to-Pair lemma) with parameters $\alpha_{\mathrm{R2P}} = \alpha$, $\varrho_{\mathrm{R2P}} = \delta_{\mathrm{P2T}}$, and $q_{\mathrm{R2P}} = q$ and deduce that J satisfies $\mathrm{PAIR}_\ell(\delta_{\mathrm{P2T}})$.

To conclude that J satisfies the conditions of Proposition 11 (Pair-to-Tuple Lemma), we just need to show that for all $U \subset V_i$ and $W \subset V_j$, $i \neq j \in [\ell]$, we have

$$e_J(U, W) \leq \frac{p}{\alpha}|U||W| + \gamma_{\mathrm{P2T}} \left(\frac{p}{\alpha}\right)^{(d_H+3)/2} n\sqrt{|U||W|}.$$

From the $(q, \gamma q^{\nu_H} m)$-bi-jumbledness of G, $p = \alpha q$, $\nu_H = (d_H + D_H + 1)/2$, and $D_H \geq 2$, we obtain

$$e_J(U,W) \leq e_G(U,W) \leq \frac{p}{\alpha}|U||W| + \gamma \left(\frac{p}{\alpha}\right)^{(D_H+d_H+1)/2} m\sqrt{|U||W|} \quad (20)$$

$$\leq \frac{p}{\alpha}|U||W| + \gamma \left(\frac{p}{\alpha}\right)^{(d_H+3)/2} m\sqrt{|U||W|}. \quad (21)$$

Hence, we just need to show that $\gamma m \leq \gamma_{\mathrm{P2T}} n$. This, however, follows from (15), $m \leq nt/(1-\varepsilon)$, and $t \leq T_{1,\mathrm{RL}}$:

$$\gamma m \leq \frac{(1-\varepsilon)\gamma_{\mathrm{P2T}}}{T_{1,\mathrm{RL}}} \cdot \frac{nt}{1-\varepsilon} \leq \gamma_{\mathrm{P2T}} n. \quad (22)$$

Thus, we can apply Proposition 11 to J with $d = d_H$, $\varepsilon_{\mathrm{P2T}} = \varepsilon_{\mathrm{EL}}$, and $p_{\mathrm{P2T}} = \alpha q$ to infer that J satisfies property $\mathrm{TUPLE}_\ell(\varepsilon_{\mathrm{P2T}}, d_H)$.

Finally, we verify the hypothesis of Proposition 12 (Embedding Lemma). The graph J is (ℓ, n, p)-partite and has property $\mathrm{TUPLE}_\ell(\varepsilon_{\mathrm{EL}}, d_H)$. Similarly as in (22) we obtain $\gamma m \leq \gamma_{\mathrm{EL}} n$. This together with (20) show that J also satisfies (13). Hence, the conditions of Proposition 12 are met and we can conclude that $J \subset G'$ contains at least $(1 - \eta_{\mathrm{EL}})p^e n^h = p^e n^h/2 \geq 1$ copies of H. □

5. Sets with large neighborhoods

The motivation for the results in this section already appeared in the outline of the proof of Theorem 5. Indeed, recall that, in our discussion in Section 3.2, we defined the hypergraphs $\mathcal{B}_r$ $(1 \leq r \leq D_H)$, whose members are the r-sets B of vertices of J with the joint neighborhood $N_J(B)$ overshooting a certain bound. In what follows, we shall make this more precise and we shall prove the "local sparseness" condition of the $\mathcal{B}_r$ mentioned in Section 3.2.

Definition 13. *Let J be an (ℓ, n, p)-partite graph with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$. For a given $C > 1$, we say that an s-set $S \subset V(J)$ is C-exceptionally neighborly if its common neighborhood in V_j satisfies*

$$|N_J(S) \cap V_j| > C^s p^s n$$

for some $j \in [\ell]$ so that $S \cap V_j = \emptyset$. The set S is C-reasonable if it is not C-exceptionally neighborly.

The following lemma states that, under the technical hypothesis (23), in an (ℓ, n, p)-partite graph J there are only $O(\gamma^2 p^{2d-1-s} n)$ ways how to extend a C-reasonable $(s-1)$-set into a C-exceptionally neighborly s-set, where the implicit constant in the big O notation depends only on C, s, d, and α.

Lemma 14. *For a given $0 < \alpha \leq 1$ and $d > 0$ suppose that an (ℓ, n, p)-partite graph J with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$ also satisfies the property that for* *all $U \subset V_i$ and $W \subset V_j$, $i \neq j \in [\ell]$, we have*

$$e_J(U,W) \leq \frac{p}{\alpha}|U||W| + \gamma\left(\frac{p}{\alpha}\right)^d n\sqrt{|U||W|}\,. \tag{23}$$

Let s be a positive integer, let $S \subset V(J)$ be any $(s-1)$-set, let $i \neq j \in [\ell]$ be such that $S \cap V_j = \emptyset$, and suppose C is such that $1/p > C > 1/\alpha \geq 1$. Set

$$W_{ij}(S) = \Big\{w \in V_i \setminus S \colon \big|N_J(S \cup \{w\}) \cap V_j\big| > C^s p^s n\Big\}.$$

If $\big|N_J(S) \cap V_j\big| \leq C^{s-1}p^{s-1}n$, then

$$|W_{ij}(S)| \leq \gamma^2(1/\alpha)^{2d}(C - 1/\alpha)^{-2}C^{-(s-1)}p^{2d-1-s}n.$$

Proof. Let $U \subset V_j$ be a set of vertices of J containing the common neighborhood of S and of size $|U| = C^{s-1}p^{s-1}n$. Note that the definition of $W_{ij}(S)$ implies that

$$e_J(U, W_{ij}(S)) > C^s p^s n|W_{ij}(S)| = Cp|U||W_{ij}(S)|.$$

From inequality (23), we deduce that

$$(C - 1/\alpha)p|U||W_{ij}(S)| < \gamma(p/\alpha)^d n\sqrt{|U||W_{ij}(S)|},$$

whence, recalling that $|U| = C^{s-1}p^{s-1}n$, the claimed bound on $|W_{ij}(S)|$ follows. □

We now state three corollaries that we later use in our proofs. Although these corollaries hold in more general settings, we prefer to present them in the exact way they are used later.

Corollary 15. *For a given $0 < \alpha \leq 1$ and a graph H, suppose that an (ℓ, n, p)-partite graph J with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$ satisfies the property that for all $U \subset V_i$ and $W \subset V_j$, $i \neq j \in [\ell]$, we have*

$$e_J(U,W) \leq \frac{p}{\alpha}|U||W| + \gamma\left(\frac{p}{\alpha}\right)^{\nu_H} n\sqrt{|U||W|}\,.$$

Set $C = 2/\alpha > 1$ and assume $p < \alpha/2$ holds. Let S be an arbitrary subset of $V(J)$ not containing any C-exceptionally neighborly s'-set S' for every $1 \leq s' \leq D_H$. Then, for every $i \in [\ell]$, the number of vertices $y \in V_i \setminus S$ such that $S \cup \{y\}$ contains a C-exceptionally neighborly s'-set S' for some $1 \leq s' \leq D_H$ is at most

$$2^{|S|}\ell c p^{2\nu_H - 1 - D_H} n, \tag{24}$$

where

$$c = \gamma^2(1/\alpha)^{2\nu_H - 2}.$$

Proof. Fix $i \in [\ell]$. Since S contains no C-exceptionally neighborly s'-set, $1 \leq s' \leq D_H$, the set $S \cup \{y\}$ $(y \in V_i \setminus S)$ will contain a C-exceptionally neighborly s'-set with $1 \leq s' \leq D_H$ if and only if there exists a set of the form $S' \cup \{y\}$ with $S' \subset S$ that is C-exceptionally neighborly.

Fix $S' \subset S$ and suppose $|S'| = s' - 1$, where $1 \le s' \le D_H$. We apply Lemma 14, and conclude that the number of $y \in V_i \setminus S$ such that $S' \cup \{y\}$ is C-exceptionally neighborly is at most $\ell \cdot cp^{2\nu_H - 1 - s'}n$ (we multiply by ℓ to account for all possible $j \in [\ell]$). We now take the union over all $S' \subset S$, $|S'| \le D_H - 1$, and get (24), as required. □

Corollary 16. *For a given $0 < \alpha \le 1$ and $d > 0$ suppose that an (ℓ, n, p)-partite graph J with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$ satisfies the property that for all $U \subset V_i$ and $W \subset V_j$, $i \ne j \in [\ell]$, we have*

$$e_J(U, W) \le \frac{p}{\alpha}|U||W| + \gamma\left(\frac{p}{\alpha}\right)^{(d+3)/2} n\sqrt{|U||W|}\,.$$

Set $C = 2/\alpha > 1$ and assume $p < \alpha/2$ holds. Then

(a) *all but at most $\ell\gamma^2(1/\alpha)^{d+1}p^{d+1}n$ vertices $x \in V_i$ satisfy $|N_J(x) \cap V_j| \le Cpn$ for every $j \ne i \in [\ell]$;*
(b) *let $x \in V_i$ be a vertex satisfying $|N_J(x) \cap V_j| \le Cpn$ for every $j \ne i$. Then all but at most $\ell\gamma^2(1/2\alpha^d)p^d n$ vertices $x' \in V_i \setminus \{x\}$ satisfy $|N_J(x, x') \cap V_j| \le C^2p^2n$ for every $j \ne i$.*

Outline of the proof. The first part of this corollary follows from Lemma 14 applied with $s = 1$, $S = \emptyset$, and d replaced with $(d+3)/2$, by summing $|W_{ij}(S)|$ for all $j \in [\ell]$, $j \ne i$. In the second part we use $s = 2$ and $S = \{x\}$ instead of $S = \emptyset$. □

6. Proof of the Embedding Lemma (Proposition 12)

The proof of Proposition 12, our embedding lemma, will generally follow the same lines as the proof of (*), discussed in Section 3.2. We start with some preliminary definitions and facts.

6.1. **The Extension Lemma and clean embeddings.**

We first fix a setup under which we shall work in this section.

Setup 17. *Let H and J be graphs such that*

(a) *J is (ℓ, n, p)-partite with ℓ-partition $\bigcup_{j=1}^{\ell} V_j$;*
(b) *H has h vertices, e edges, and an ℓ-partition $V(H) = \bigcup_{j=1}^{\ell} U_j$.*

Recall that an *embedding* of H in J is an injective, edge-preserving map $f : V(H) \to V(J)$ such that $f(U_j) \subset V_j$ for all $1 \le j \le \ell$.

For a given $C > 1$ we say that the embedding f of H in J is (D_H, C)-*reasonable* if $f(H)$ contains no C-exceptionally neighborly set of size at most D_H. Denote by $\mathcal{R}(H, J; D_H, C)$ the set of all (D_H, C)-reasonable embeddings of H in J.

Moreover, for $t \in [h]$ and t-tuples $F = (u_1, \ldots, u_t) \in V(H)^t$ and $X = (x_1, \ldots, x_t) \in V(J)^t$, let $\mathcal{R}(H, J, F, X; D_H, C)$ denote the set of all (D_H, C)-reasonable embeddings $f \in \mathcal{R}(H, J; D_H, C)$ such that $f(u_i) = x_i$ for all $i \in [t]$. Clearly, we may always assume that all u_i, $1 \le i \le t$, and all x_i, $1 \le i \le t$, are distinct. Set $F^{\rm set} = \{u_1, \ldots, u_t\}$ and $X^{\rm set} = \{x_1, \ldots, x_t\}$.

Below, for any graph H' and any t-tuple F of vertices of H', we write $w(H', F)$ for the number of edges in H' that do not have both endpoints in $F^{\rm set}$. That is,

$$w(H', F) = |E(H')| - |E(H'[F^{\rm set}])|.$$

Let $u_1, \ldots, u_h$ be the vertices of H. We denote by H_i, $1 \le i \le h$, the subgraph induced by $u_1, \ldots, u_i$, i.e., $H_i = H[\{u_1, \ldots, u_i\}]$. Recall that the ordering $u_1, \ldots, u_h$ is d-degenerate if $\deg_{H_i}(u_i) \le d$ for all $1 \le i \le h$. We now state the following lemma.

Lemma 18 (Extension Lemma)**.** *Let $C > 1$ be a given constant. Suppose $0 \le t \le \max\{2, d_H\}$, and let $F \in V(H)^t$ and $X \in V(J)^t$ be fixed. Then*

$$|\mathcal{R}(H, J, F, X; D_H, C)| \le C^{(h-t)D_H} p^{w(H,F)} n^{h-t}.$$

In particular, if $F^{\rm set} \subset V(H)$ is a stable set, then

$$|\mathcal{R}(H, J, F, X; D_H, C)| \le C^{(h-t)D_H} p^e n^{h-t}.$$

Proof. It is observed in [24] that there is a D_H-degenerate ordering $u_1, \ldots, u_h$ of the vertices of H with $F^{\rm set} = \{u_1, \ldots, u_t\}$. Fix such an ordering. We shall prove

(*) for all $t \le i \le h$, we have

$$|\mathcal{R}(H_i, J, F, X; D_H, C)| \le C^{(i-t)D_H} p^{w(H_i,F)} n^{i-t}, \tag{25}$$

where $H_i = H[\{u_1, \ldots, u_i\}]$.

We prove (*) by induction on i. The case in which $i = t$ is clear. Now suppose that $t < i \le h$ and that (25) holds for all values smaller than i. Since our ordering $u_1, \ldots, u_h$ of the vertices of H is D_H-degenerate, we have $\deg_{H_i}(u_i) \le D_H$. Therefore, if we let $r = \deg_{H_i}(u_i) \le D_H$, then any (D_H, C)-reasonable embedding of H_{i-1} can be extended in at most $C^r p^r n$ ways to a (D_H, C)-reasonable embedding of H_i. Using the induction hypothesis and the fact that $w(H_i, F) = w(H_{i-1}, F) + r$, we get

$$\begin{aligned} |\mathcal{R}(H_i, J, F, X; D_H, C)| &\le C^r p^r n \cdot |\mathcal{R}(H_{i-1}, J, F, X; D_H, C)| \\ &\le C^{D_H} p^r n \cdot C^{(i-1-t)D_H} p^{w(H_{i-1},F)} n^{i-1-t} \\ &= C^{(i-t)D_H} p^{w(H_h,F)} n^{i-t}, \end{aligned}$$

verifying (25). This completes the induction step. Our lemma follows from setting $i = h$ in (25). □

Now we derive two corollaries of Lemma 18. Denote by $\mathcal{R}_{\rm ni}(H, J; D_H, C)$ the set of all mappings $f \in \mathcal{R}(H, J; D_H, C)$ for which $f(H)$ is a non-induced copy of H in J. The next corollary shows that the set $\mathcal{R}_{\mathrm{ni}}(H,J;D_H,C)$ is small.

Corollary 19. *Let $C>1$ and $\eta>0$ be fixed and let $p=p(n)=o(1)$ be a function of n. Then there exists an integer $n_2=n_2(p)$ such that if graphs J and H satisfy Setup 17 for $n>n_2$, then*

$$|\mathcal{R}_{\mathrm{ni}}(H,J;D_H,C)|\le \eta p^e n^h. \tag{26}$$

Proof. Let $\eta>0$, $C>1$, integers $h,\ell\ge 1$, and a function $p=p(n)=o(1)$ be given. Let $n_1>0$ be such that

$$p(n)\le \frac{\eta}{h^2 C^{(h-2)(h-1)}} \tag{27}$$

for every $n>n_1$.

Suppose that graphs H and J satisfy Setup 17 with $n\ge n_1$. The case in which $h=1$ or H is a complete graph is clear, hence we assume $h\ge 2$ and $H\neq K_h$. To count non-induced (D_H,C)-reasonable embeddings of H in J, we select an edge $\{x,x'\}\in E(J)$ and a pair u, u' of distinct, non-adjacent vertices of H. By Lemma 18 applied to $F=(u,u')$ and $X=(x,x')$, the number of (D_H,C)-reasonable embeddings $f\colon V(H)\to V(J)$ such that $f(u)=x$ and $f(u')=x'$ is at most $C^{(h-2)D_H}p^e n^{h-2}$.

Since $\{x,x'\}\in E(J)$ can be selected in at most pn^2 ways, the ordered pair X can be selected in at most $2pn^2$ ways. Similarly, F can be selected in at most $2\binom{h}{2}$ ways. Therefore,

$$|\mathcal{R}_{\mathrm{ni}}(H,J;D_H,C)|\le 4pn^2\binom{h}{2}\cdot C^{(h-2)D_H}p^e n^{h-2}< h^2 C^{(h-2)D_H}p^{e+1}n^h.$$

The inequality $|\mathcal{R}_{\mathrm{ni}}(H,J;D_H,C)|\le \eta p^e n^h$ follows from $D_H\le\Delta(H)\le h-1$ and (27). □

The next two definitions introduce several important terms for our proof of Proposition 12.

Definition 20. *For $\varepsilon>0$, we call an s-set S ε-untypical if $S\cap V_j=\emptyset$ for some $j\in[\ell]$ and*

$$|N_J(S)\cap V_j|\neq(1\pm\varepsilon)p^s n\,.$$

To give some intuition behind Definition 21(i) below, we first recall that we are dealing with a triangle-free graph H, and hence the neighborhood of a vertex of H is stable. In view of Corollary 19, we may and shall basically disregard non-induced embeddings of H in J. Putting these two observations together, we see that we may disregard embeddings f of H in J in which we have a vertex u in $V(H)$ with $f(N_H(u))$ non-stable. Finally, we remark that, in the inductive proof that will follow, we shall be interested in avoiding ε-untypical sets for $f(N_H(u))$.

Definition 21. *Let graphs J and H be as in Setup 17 and let $u_1,\dots,u_h$ be any d_H-degenerate ordering of the vertices of H. For (i)–(iii) below, we suppose that $1<i\le h$.*

(i) *An embedding* $f\colon V(H_{i-1})\to V(J)$ *is* ε*-polluted if the set* $f(N_{H_i}(u_i))$ *is stable but it is* ε*-untypical. Otherwise* f *is called* ε*-clean.*

(ii) *Set*

$$\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)=\{f\in\mathcal{R}(H_{i-1},J;D_H,C)\colon f\text{ is }\varepsilon\text{-polluted}\}.$$

(iii) *Finally, we say that* $f\colon V(H_{i-1})\to V(J)$ *is* ε*-perfect if* f *is* ε*-clean and* $f(H_{i-1})$ *is an induced copy of* H_{i-1} *in* J*. We also set*

$$\mathcal{R}_{\mathrm{perf}}(H_{i-1},J;D_H,C)=\{f\in\mathcal{R}(H_{i-1},J;D_H,C)\colon f\text{ is }\varepsilon\text{-perfect}\}.$$

In Corollary 22 below, we estimate the size of $\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)$ for $1<i\le h$.

Corollary 22. *Let* $\varepsilon>0$ *and* $C>1$ *be fixed. Let* J *and* H *be graphs satisfying Setup 17 and let* $u_1,\dots,u_h$ *be any* d_H*-degenerate ordering of the vertices of* H*. Suppose* $1<i\le h$ *and set* $r=\deg_{H_i}(u_i)$*. If* J *satisfies* $\mathrm{TUPLE}_\ell(\varepsilon,d_H)$ *and* H *is triangle-free, then*

$$|\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)|\le\varepsilon\ell C^{(i-1-r)D_H}p^{e(H_{i-1})}n^{i-1}.$$

In particular, if for a given $\eta>0$ *we set* $\varepsilon=\varepsilon'(\eta,C,H)=\eta/\ell C^{hD_H}$*, then*

$$|\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)|\le\eta p^{e(H_{i-1})}n^{i-1}$$

for all $1<i\le h$*.*

Proof. By definition, an embedding $f\in\mathcal{R}(H_{i-1},J;D_H,C)$ is ε-polluted if $f\big(N_{H_i}(u_i)\big)$ is stable and ε-untypical. Fix an r-tuple F such that $F^{\mathrm{set}}=N_{H_i}(u_i)$. Note that we have

$$\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)=\bigcup_X\mathcal{R}(H_{i-1},J,F,X;D_H,C),$$

where the union is taken over all stable and ε-untypical r-tuples X. Therefore

$$|\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)|\le\sum_X|\mathcal{R}(H_{i-1},J,F,X;D_H,C)|,\tag{28}$$

where the sum is over the same set of r-tuples X.

Since J satisfies $\mathrm{TUPLE}_\ell(\varepsilon,d_H)$, the number of r-tuples X that we are summing over in (28) is at most $\varepsilon\ell n^r$, where $r=\deg_{H_i}(u_i)\le d_H$. Observe also that $N_{H_i}(u_i)$ is a stable set in H_i, because $H_i\subset H$ is triangle-free. We now apply Lemma 18 to deduce from (28) that $|\mathcal{R}_{\mathrm{poll}}(H_{i-1},J;D_H,C)|$ is at most

$$\varepsilon\ell n^r\cdot C^{(i-1-r)D_H}p^{e(H_{i-1})}n^{i-1-r}=\varepsilon\ell C^{(i-1-r)D_H}p^{e(H_{i-1})}n^{i-1},$$

and our corollary follows. □

6.2. **Proof of Proposition 12.** Now we prove the Embedding Lemma.

Proof. Let H be any triangle-free, ℓ-partite graph with h vertices and e edges. We also fix any d_H-degenerate ordering $u_1, \dots, u_h$ of the vertices of H, and set $H_i = H[\{u_1, \dots, u_i\}]$ for every i, $1 \le i \le h$.

Throughout this proof, we suppose that $0 < \alpha$, $\eta \le 1$ and $C = 2/\alpha > 1$ are fixed constants. We shall prove by induction on i that

(**) for all $1 \le i \le h$ and all $\delta > 0$, there are $\varepsilon_i = \varepsilon_i(H, \alpha, \delta) > 0$, $\gamma_i = \gamma_i(H, \alpha, \delta) > 0$ such that for a given function $p = p(n) = o(1)$ satisfying $p^{d_H} n \gg 1$ there is $n(i) = n(i; H, \alpha, \delta, p)$ such that if
 (a) J is (ℓ, n, p)-partite and $n > n(i)$,
 (b) for all $U \subset V_j$ and $W \subset V_{j'}$, $j \ne j' \in [\ell]$, we have
$$e_J(U, W) \le \frac{p}{\alpha}|U||W| + \gamma_i \left(\frac{p}{\alpha}\right)^{\nu_H} n\sqrt{|U||W|},$$
 (c) J satisfies $\mathrm{TUPLE}_\ell(\varepsilon_i, d_H)$,
then
$$|\mathcal{R}(H_i, J; D_H, C)| = (1 \pm \delta) p^{e(H_h)} n^i. \tag{29}$$

Note that Proposition 12 follows from (**) by taking $\delta = \eta$, $\varepsilon = \varepsilon_h$, $\gamma = \gamma_h$, and $N_1(H, \alpha, \eta, p) = n(h; H, \alpha, \eta, p)$.

Clearly, when $h \ge i = 1$, (**) holds with $\varepsilon_1 = \delta$, $\gamma_1 = n(1) = 1$ for any $\delta > 0$ and any p. Suppose now that $1 < i \le h$ and that (**) holds for all smaller values of i.

For a given $\delta > 0$, set
$$\delta' = \min\left\{\frac{\delta}{6}, \frac{\delta}{2\ell^{D_H} C^{D_H}}\right\}$$
and let $\varepsilon_{i-1}(H, \alpha, \delta')$ and $\gamma_{i-1}(H, \alpha, \delta')$ be given by the induction hypothesis. Furthermore, let $\varepsilon'(\delta'/2, C, H)$ be guaranteed by Corollary 22. We now define

$$\varepsilon_i = \min\left\{\varepsilon_{i-1}(H, \alpha, \delta'), \varepsilon'\left(\tfrac{\delta'}{2}, C, H\right), \frac{\delta}{8}\right\}, \tag{30}$$

$$\gamma_i = \min\left\{\left(\frac{\varepsilon_i}{2^{i-1}\ell\left(\frac{C}{2}\right)^{2\nu_H - 2}}\right)^{1/2}, \gamma_{i-1}(H, \alpha, \delta')\right\}. \tag{31}$$

For any $p = p(n) = o(1)$ with $p^{d_H} n \gg 1$, let $n(i-1) = n(i-1; H, \alpha, \delta', p)$ be given by the induction hypothesis to guarantee that
$$|\mathcal{R}(H_{i-1}, J; D_H, C)| = (1 \pm \delta') p^{e(H_{i-1})} n^{i-1} \tag{32}$$
for and graph J satisfying (a)–(c) for $n > n(i-1)$.

Now Corollary 19 tells us that for $n > n_2(\delta'/2)$,
$$|\mathcal{R}_{\mathrm{ni}}(H_{i-1}, J; D_H, C)| \le \frac{\delta'}{2} p^{e(H_{i-1})} n^{i-1} \tag{33}$$
holds for any (ℓ, n, p)-partite graph J.

Finally, let n_3 be such that $p^{d_H}n \geq h/\varepsilon_i$ and $p < \alpha/2$ for $n > n_3$. We set

$$n(i) = \max\{n(i-1), n_2(\delta'/2), n_3\}, \tag{34}$$

and claim that this choice will do. Observe that we have

$$(1-2\delta')(1-3\varepsilon_i) \geq 1-\delta, \tag{35a}$$

$$(1+\delta')(1+\varepsilon_i) \leq 1+\frac{\delta}{2}, \tag{35b}$$

$$\delta'\ell^{D_H}C^{D_H} \leq \frac{\delta}{2}. \tag{35c}$$

Let J be a graph satisfying (a)–(c) for $n > n(i)$. Then, in addition to (32) and (33), the inequality

$$|\mathcal{R}_{\text{poll}}(H_{i-1}, J; D_H, C)| \leq \frac{\delta'}{2}p^{e(H_{i-1})}n^{i-1} \tag{36}$$

also holds because $\varepsilon_i \leq \varepsilon'(\delta'/2, C, H)$ (see Corollary 22).

We start by showing the lower bound on $|\mathcal{R}(H_i, J; D_H, C)|$. Let $r = \deg_{H_i}(u_i) \leq \min\{i-1, d_H\}$. Note that then $e(H_{i-1}) = e(H_i) - r$. By our choice of ε_i and $n(i)$, the number of embeddings in $\mathcal{R}(H_{i-1}, J; D_H, C)$ that are either ε_i-polluted or non-induced is at most

$$2\frac{\delta'}{2}p^{e(H_{i-1})}n^{i-1} = \delta' p^{e(H_{i-1})}n^{i-1} = \delta' p^{e(H_i)-r}n^{i-1}$$

(see (33) and (36)). Hence, by (32), the number $|\mathcal{R}_{\text{perf}}(H_{i-1}, J; D_H, C)|$ of ε_i-perfect embeddings of H_{i-1} in J is such that

$$(1-2\delta')p^{e(H_i)-r}n^{i-1} < \left|\mathcal{R}_{\text{perf}}(H_{i-1}, J; D_H, C)\right| < (1+\delta')p^{e(H_i)-r}n^{i-1}. \tag{37}$$

Given any such embedding $f' \in \mathcal{R}_{\text{perf}}(H_{i-1}, J; D_H, C)$, we estimate the number of embeddings $f \in \mathcal{R}(H_i, J; D_H, C)$ that extend f'. Let V_j be the vertex class into which we need to embed u_i.[3] Since f' is ε_i-clean, by Definition 21 we must have that either $f'(N_{H_i}(u_i))$ is not a stable set in J, or $f'(N_{H_i}(u_i))$ is not ε_i-untypical.

Since H is triangle-free, the set $N_{H_i}(u_i)$ is a stable set in H_i. Since f' is induced, the set $f'(N_{H_i}(u_i))$ is also a stable set. Hence, the second option must be true, and, consequently,

$$\big||N_J\big(f'(N_{H_i}(u_i))\big) \cap V_j| - p^r n\big| \leq \varepsilon_i p^r n. \tag{38}$$

Note that, to obtain an extension f of f' that belongs to $\mathcal{R}(H_i, J; D_H, C)$, we only need to select $f(u_i)$ in $\big(N_J(f'(N_{H_i}(u_i))) \cap V_j\big) \setminus f'(V(H_{i-1}))$ so that $f'(V(H_{i-1})) \cup \{f(u_i)\}$ does not contain a C-exceptionally neighborly s'-set for any $1 \leq s' \leq D_H$. We apply Corollary 15 with $S = f'(V(H_{i-1}))$ and obtain that at most

$$2^{i-1}\ell\gamma_i^2\left(\frac{C}{2}\right)^{2\nu_H-2}p^{2\nu_H-1-D_H}n \overset{(31)}{\leq} \varepsilon_i p^r n \tag{39}$$

[3]By definition, every embedding f must preserve the vertex classes of H, that is, if $u_i \in U_j$ then $f(u_i) \in V_j$.

vertices in V_j cannot be chosen as $f(u_i)$. The last inequality follows from the fact that $2\nu_H - 1 - D_H = 2(D_H + d_H + 1)/2 - 1 - D_H = d_H \geq r$ and from (31). The reader may check that this tight inequality for the exponents of p in (39) explains why we cannot reduce the exponent of q in the hypothesis of Proposition 12 that G should be $(q, \gamma q^{\nu_H} m)$-bi-jumbled.

From (38) it follows that the size of $\big(N_J(f'(N_{H_i}(u_i))) \cap V_j\big) \setminus f'(V(H_{i-1}))$ is at least

$$(1-\varepsilon_i)p^r n - (h-1) \geq (1-2\varepsilon_i)p^r n. \tag{40}$$

Consequently, every embedding $f' \in \mathcal{R}_{\mathrm{perf}}(H_{i-1}, J; D_H, C)$ can be extended to an embedding $f \in \mathcal{R}(H_i, J; D_H, C)$ in at least

$$\big|\big(N_J(f'(N_{H_i}(u_i))) \cap V_j\big) \setminus f'\big(V(H_{i-1})\big)\big| - \varepsilon_i p^r n \overset{(40)}{>} (1-3\varepsilon_i)p^r n \tag{41}$$

ways. Combining (37) and (41) yields

$$\big|\mathcal{R}(H_i, J; D_H, C)\big| > (1-2\delta')p^{e(H_i)-r}n^{i-1} \cdot (1-3\varepsilon_i)p^r n \overset{\text{(35a)}}{\geq} (1-\delta)p^{e(H_i)}n^i.$$

For the upper bound, we need to show that $|\mathcal{R}(H_i, J; D_H, C)| \leq (1 + \delta)p^{e(H_i)}n^i$. Fix an arbitrary $f' \in \mathcal{R}(H_{i-1}, J; D_H, C)$. The number of extensions of f' to embeddings of H_i in J is bounded from above by

$$\big|N_J\big(f'(N_{H_i}(u_i))\big)\big|. \tag{42}$$

If, furthermore, $f' \in \mathcal{R}_{\mathrm{perf}}(H_{i-1}, J; D_H, C)$, then we know that (38) holds and hence the quantity in (42) is bounded by $(1+\varepsilon_i)p^r n$. Combining this fact with the upper bound in (37), we obtain that the number of embeddings $f \in \mathcal{R}(H_i, J; D_H, C)$ whose restrictions to $V(H_{i-1})$ are in $\mathcal{R}_{\mathrm{perf}}(H_{i-1}, J; D_H, C)$ is at most

$$(1+\delta')p^{e(H_i)-r}n^{i-1} \cdot (1+\varepsilon_i)p^r n \overset{\text{(35b)}}{\leq} \left(1 + \frac{\delta}{2}\right)p^{e(H_i)}n^i. \tag{43}$$

We already know that (see (32) and (37))

$$\big|\mathcal{R}(H_{i-1}, J; D_H, C) \setminus \mathcal{R}_{\mathrm{perf}}(H_{i-1}, J; D_H, C)\big| \leq 3\delta' p^{e(H_i)-r}n^{i-1}.$$

Since $r = \deg_J(u_i) \leq d_H \leq D_H$ and f' is (D_H, C)-reasonable, each such embedding f' gives rise to at most $C^r p^r n$ embeddings $f \in \mathcal{R}(H_i, J; D_H, C)$. Therefore, the number of embeddings $f \in \mathcal{R}(H_i, J; D_H, C)$ whose restrictions to $V(H_{i-1})$ are not in $\mathcal{R}_{\mathrm{perf}}(H_{i-1}, J; D_H, C)$ is at most

$$3\delta' p^{e(H_i)-r}n^{i-1} \cdot C^r p^r n \overset{\text{(35c)}}{\leq} \frac{\delta}{2}p^{e(H_i)}n^i. \tag{44}$$

From (43) and (44) we deduce that $|\mathcal{R}(H_i, J; D_H, C)| \leq (1+\delta)p^{e(H_i)}n^i$, as required. $\square$

7. Proof of the Pair-to-Tuple Lemma (Proposition 11)

Recall first the statement we are proving: for given integers $d \geq 1$ and $\ell > 1$ and reals $0 < \alpha, \varepsilon \leq 1$, we need to find $\delta > 0$ and $\gamma > 0$ such that for any function $p = p(n)$ with $p^d n \gg 1$ there exists $N_0 > 0$ with the following property: any (ℓ, n, p)-partite graph J, where $n \geq N_0$, such that

(i) for all $U \subset V_i$ and $W \subset V_j$, $i \neq j \in [\ell]$, we have

$$e_J(U, W) \leq \frac{p}{\alpha}|U||W| + \gamma \left(\frac{p}{\alpha}\right)^{(d+3)/2} n\sqrt{|U||W|}, \tag{45}$$

(ii) J possesses $\mathrm{PAIR}_\ell(\delta)$

also satisfies $\mathrm{TUPLE}_\ell(\varepsilon, d)$.

Let d, ℓ, α, and ε be given. Without loss of generality we may assume $d \geq 3$ because (ii) implies $\mathrm{TUPLE}_\ell(\varepsilon, d)$ for $d = 1, 2$ and $\varepsilon \leq \delta$ (we do not need assumption (i) at all). Hence we must define δ and γ and, for a given $p = p(n)$, we must also define N_0 and then show that this choice is correct. Our proof uses a technique from [24, 28] (see Lemma 26 and the proof of Lemma 43 in [24]) and is based on the following lemma which is a well-known consequence of the Cauchy–Schwarz inequality.

Lemma 23. *For all $\varepsilon > 0$, there exists $0 < \varrho = \varrho(\varepsilon) < \varepsilon$ such that, for any family of real numbers $\{a_i \geq 0 \colon 1 \leq i \leq M\}$ satisfying the conditions*

(1) $\sum_{i=1}^M a_i \geq (1 - \varrho) M a$ *and*
(2) $\sum_{i=1}^M a_i^2 \leq (1 + \varrho) M a^2$

for some $a \geq 0$, we have

$$\left|\{i \colon |a_i - a| < \varepsilon a\}\right| > (1 - \varepsilon) M.$$

Our application of Lemma 23 will involve the sets $\mathcal{T}(I)$ defined at the beginning of Section 3.5. We first show that for any $\varrho > 0$ there is $\delta > 0$ so that if J possesses $\mathrm{PAIR}_\ell(\delta)$, then for every fixed multiset $I = \{i_1, \ldots, i_r\} \subset [\ell]$ with $3 \leq r \leq d$, and for every $j \in [\ell] \setminus I$, we can verify conditions (1) and (2) for the number $M = |\mathcal{T}(I)|$ of r-tuples $(x_1, \ldots, x_r) \in \mathcal{T}(I)$, $a = p^r n$, and each a_i corresponding to $|N_{V_j}(x_1, \ldots, x_r)|$ for some r-tuple $(x_1, \ldots, x_r) \in \mathcal{T}(I)$. This is formally done in the next fact. In what follows we denote by

$$\sum_{(x_\tau) \in \mathcal{T}(I)}$$

the sum over all $(x_1, \ldots, x_r) \in \mathcal{T}(I)$.

Fact 24. *For every $0 < \varrho \leq 1$ there exist $\delta = \delta(d, \ell, \alpha, \varrho)$ and $\gamma = \gamma(d, \ell, \alpha, \varrho)$ such that for every $p = p(n) = o(1)$ with $p^d n \gg 1$ there is an integer $N_4 = N_4(d, \ell, \alpha, \varrho, p)$ with the following property: If an (ℓ, n, p)-partite graph J satisfies conditions (i) and (ii) above for $n \geq N_4$, then for every multiset $I = \{i_1, \ldots, i_r\} \subset [\ell]$ with $3 \leq r \leq d$, and for every $j \in [\ell] \setminus I$, we have*

(1) $\sum_{(x_\tau)\in\mathcal{T}(I)} \left|N_{V_j}(x_1,\dots,x_r)\right| = (1\pm\varrho)n^r\cdot p^r n,$

(2) $\sum_{(x_\tau)\in\mathcal{T}(I)} \left|N_{V_j}(x_1,\dots,x_r)\right|^2 < (1+\varrho)n^r\cdot (p^r n)^2.$

Comparing Lemma 23 and Fact 24, one may expect that the upper bound in Fact 24(1) would not be necessary. However, it turns out that this upper bound is required in the proof of Fact 24(2).

Now we are ready to define δ and N_0: let $\varrho = \varrho(\varepsilon^2) < \varepsilon^2$ be the constant guaranteed by Lemma 23 and $\delta = \delta(d,\ell,\alpha,\varrho/3)$, $\gamma = \gamma(d,\ell,\alpha,\varrho/3)$, and $N_4 = N_4(d,\ell,\alpha,\varrho/3,p)$ be given by Fact 24. Let $N_5 > 0$ be such that for any $n \geq N_5$ and $3 \leq r \leq d$, we have

$$\frac{n^r}{1+\varrho/3} \leq (n-r)^r. \tag{46}$$

Set $N_0 = \max\{N_4, N_5\}$.

To prove that J satisfies $\mathrm{TUPLE}_\ell(\varepsilon, d)$, we fix an arbitrary multiset $I = \{i_1,\dots,i_r\} \subset [\ell]$ with $3 \leq r \leq d$ and any $j \in [\ell]\setminus I$, and show that

$$|N_{V_j}(x_1,\dots,x_r)| = (1\pm\varepsilon)p^r n$$

for all but at most εn^r r-tuples $(x_1,\dots,x_r) \in \mathcal{T}(I)$.

By (10) and (46), we have

$$\frac{n^r}{1+\varrho/3} \leq (n-r)^r \leq M = |\mathcal{T}(I)| \leq n^r. \tag{47}$$

By our choice of constants we can apply Fact 24 and obtain

$$\begin{aligned} \sum_{(x_\tau)\in\mathcal{T}(I)} \left|N_{V_j}(x_1,\dots,x_r)\right| \quad &> \quad \left(1-\frac{\varrho}{3}\right) n^r\cdot p^r n,\\ \sum_{(x_\tau)\in\mathcal{T}(I)} |N_{V_j}(x_1,\dots,x_r)|^2 \quad &< \quad \left(1+\frac{\varrho}{3}\right) n^r\cdot (p^r n)^2. \end{aligned}$$

It follows from (47) that

$$\sum_{(x_\tau)\in\mathcal{T}(I)} \left|N_{V_j}(x_1,\dots,x_r)\right| > (1-\varrho)M\cdot p^r n,$$

and

$$\sum_{(x_\tau)\in\mathcal{T}(I)} |N_{V_j}(x_1,\dots,x_r)|^2 < (1+\varrho)M\cdot (p^r n)^2.$$

We are now clearly in position to apply Lemma 23 with $a = p^r n$ and $M = |\mathcal{T}(I)|$. We deduce that

$$|N_{V_j}(x_1,\dots,x_r)| = (1\pm\varepsilon^2)p^r n = (1\pm\varepsilon)p^r n$$

holds for at least

$$(1-\varepsilon^2)M \overset{(47)}{\geq} \left(\frac{1+\varepsilon}{1+\varrho/3}\right)(1-\varepsilon)n^r > (1-\varepsilon)n^r$$

r-tuples $(x_1, \ldots, x_r) \in \mathcal{T}(I)$. The last inequality follows from the fact that $\varrho/3 < \varepsilon^2 \leq 1$. What remains to be proved is Fact 24.

Proof of Fact 24. Let $0 < \varrho < 1$, in addition to $d \geq 3$, $\ell \geq 2$, and $0 < \alpha \leq 1$, be given. We first set an auxiliary constant $C = 2/\alpha > 1$ and then we define

$$\delta = \delta(d, \ell, \alpha, \varrho) = \min\left\{\frac{\varrho}{\ell(d+2)}, \frac{\varrho}{2^{d+3}}, \frac{\varrho}{8\ell C^{2d}}\right\} \tag{48}$$

and

$$\gamma = \gamma(d, \ell, \alpha, \varrho) = \min\left\{\sqrt{\frac{\varrho\alpha^{d+1}}{2\ell}}, \sqrt{\frac{\varrho\alpha^d}{4\ell C^d}}.\right\} \tag{49}$$

For any $p = p(n) = o(1)$, $p^d n \gg 1$, let $N_4 = N_4(d, \ell, \alpha, \varrho, p)$ be such that

$$\frac{d}{(1-\delta)\delta} < pn \quad \text{and} \quad \frac{1+\varrho}{\varrho/8} < p^d n \tag{50}$$

and

$$p < \frac{\alpha}{2} \tag{51}$$

for every $n \geq N_4$.

Now let an (ℓ, n, p)-partite graph J satisfy conditions (i) and (ii) for $n \geq N_4$. We fix an arbitrary multiset $I = \{i_1, \ldots, i_r\} \subset [\ell]$, $3 \leq r \leq d$, and $j \in [\ell] \setminus I$.

To prove the lower bound in the first part of this fact, we observe

$$\sum_{(x_\tau) \in \mathcal{T}(I)} |N_{V_j}(x_1, \ldots, x_r)| \geq \sum_{y \in V_j} \prod_{k=1}^{r} \big(|N_{V_{i_k}}(y)| - r\big). \tag{52}$$

Since J satisfies $\mathrm{PAIR}_\ell(\delta)$, it follows that for all $j \in [\ell] \setminus I$ and any $i \in I$, at least $(1-\delta)n$ vertices $y \in V_j$ satisfy

$$\big|N_{V_i}(y) - pn\big| < \delta pn. \tag{53}$$

Since I contains at most ℓ distinct numbers, there are at least $(1-\ell\delta)n$ vertices $y \in V_j$ for which (53) holds simultaneously for all $i \in I$. Consequently, (52) yields

$$\sum_{(x_\tau) \in \mathcal{T}(I)} \big|N_{V_j}(x_1, \ldots, x_r)\big| > (1-\ell\delta)n\big((1-\delta)pn - r\big)^r. \tag{54}$$

Applying the fact that $(a-b)^r \geq a^r - ra^{r-1}b$ for $a > b \geq 0$ and $(1-\delta)pn > r$ by (50) to the right-hand side of (54), we obtain

$$\begin{aligned}(1-\ell\delta)n&\big((1-\delta)pn - r\big)^r \\ &\geq (1-\ell\delta)n\left[(1-\delta)^r(pn)^r - r(1-\delta)^{r-1}(pn)^{r-1}\right] \\ &\geq (1-\ell\delta)^{r+1}n(pn)^r\left(1 - \frac{r}{(1-\delta)pn}\right).\end{aligned} \tag{55}$$

Since $r/(1-\delta)pn \le d/(1-\delta)pn < \delta$ by (50), we deduce from (54) and (55) that

$$\sum_{(x_\tau)\in\mathcal{T}(I)} \big|N_{V_j}(x_1,\dots,x_r)\big| > (1-\ell\delta)^{r+2} n^r \cdot p^r n. \tag{56}$$

Since $\delta \le \varrho/\ell(d+2)$ and $r \le d$, we obtain

$$(1-\ell\delta)^{r+2} \ge 1-(r+2)\ell\delta \ge 1-\varrho.$$

Thus (56) becomes $\sum_{(x_\tau)\in\mathcal{T}(I)} \big|N_{V_j}(x_1,\dots,x_r)\big| > (1-\varrho)n^r \cdot p^r n.$

To prove the upper bound in (1) of Fact 24, we first observe that

$$\sum_{(x_\tau)\in\mathcal{T}(I)} \big|N_{V_j}(x_1,\dots,x_r)\big| \le \sum_{y\in V_j} \prod_{k=1}^{r} |N_{V_{i_k}}(y)|. \tag{57}$$

For each $y \in V_j$ such that inequality (53) is satisfied for all $i \in I$, we have

$$\prod_{k=1}^{r} |N_{V_{i_k}}(y)| \le (1+\delta)^r (pn)^r.$$

Since J satisfies $\mathrm{PAIR}_\ell(\delta)$, there are at most $\ell\delta n$ vertices $y \in V_j$ for which inequality (53) fails for some $i \in I$.

Define the set $A = \big\{y \in V_j \;:\; |N_{V_i}(y)| \le Cpn \;\forall i \in I\big\}$. Corollary 16(a)[4] and (49) imply $|A| > (1-\ell\gamma^2(1/\alpha)^{d+1}p^{d+1})n \ge (1-(\varrho/2)p^{d+1})n$. Note that for every $y \in A$ we have

$$\prod_{k=1}^{r} |N_{V_{i_k}}(y)| \le (Cpn)^r.$$

Finally, for every $y \notin A$ we have the trivial bound $\prod_{k=1}^{r} |N_{V_{i_k}}(y)| \le n^r$. From (57) we obtain

$$\begin{aligned}\sum_{(x_\tau)\in\mathcal{T}(I)} \big|N_{V_j}(x_1,\dots,x_r)\big| &\le n\cdot(1+\delta)^r(pn)^r + \ell\delta n\cdot(Cpn)^r + \frac{\varrho}{2}p^{d+1}n\cdot n^r\\ &\le \Big((1+\delta)^d + \ell\delta C^d + \frac{\varrho}{2}\Big)\, n^r \cdot p^r n\\ &\overset{(48)}{\le} \Big(1+\frac{\varrho}{8}+\frac{\varrho}{8}+\frac{\varrho}{2}\Big)\, n^r\cdot p^r n < (1+\varrho)n^r\cdot p^r n.\end{aligned}$$

This concludes the proof of the first part of Fact 24.

Now we prove the second part of Fact 24. By counting in two ways the pairs $\big((y_1,y_2),(x_1,\dots,x_r)\big)$ such that $(x_1,\dots,x_r) \in \mathcal{T}(I)$ and $y_1 \ne y_2 \in N_{V_j}(x_1,\dots,x_r)$, we obtain the following inequality:

$$\sum_{(x_\tau)\in\mathcal{T}(I)} \binom{\big|N_{V_j}(x_1,\dots,x_r)\big|}{2} 2! \le \sum_{(y_1,y_2)\in\mathcal{T}(\{j,j\})} \prod_{k=1}^{r} |N_{V_{i_k}}(y_1,y_2)|,$$

[4]Note that we need (45), $C = 2/\alpha$, and (51) to verify the assumptions of Corollary 16(a).

or, equivalently,

$$\sum_{(x_\tau)\in\mathcal{T}(I)} \big|N_{V_j}(x_1,\dots,x_r)\big|^2 \le \sum_{(y_1,y_2)\in\mathcal{T}(\{j,j\})} \prod_{k=1}^{r} |N_{V_{i_k}}(y_1,y_2)| + \sum_{(x_\tau)\in\mathcal{T}(I)} |N_{V_j}(x_1,\dots,x_r)|. \quad (58)$$

The second term on the right-hand side was already estimated in part (1):

$$\sum_{(x_\tau)\in\mathcal{T}(I)} |N_{V_j}(x_1,\dots,x_r)| < (1+\varrho)n^r p^r n \overset{(50)}{\le} \frac{\varrho}{8} n^r (p^r n)^2. \quad (59)$$

To estimate the first term in (58), we divide all pairs of vertices $(y_1,y_2)\in\mathcal{T}(\{j,j\})$ into four categories:

(a) pairs of vertices (y_1,y_2) such that

$$|N_{V_{i_k}}(y_1,y_2) - p^2 n| < \delta p^2 n$$

holds simultaneously for $k=1,\dots,r$. The contribution of these pairs to the sum $\sum_{(y_1,y_2)\in\mathcal{T}(\{j,j\})} \prod_{k=1}^{r} |N_{V_{i_k}}(y_1,y_2)|$ is bounded from above by

$$n^2\cdot\big((1+\delta)p^2n\big)^r \le (1+\delta)^d\cdot n^r\cdot(p^r n)^2 \overset{(48)}{\le} \Big(1+\frac{\varrho}{8}\Big)\cdot n^r\cdot(p^r n)^2. \quad (60)$$

(b) pairs of vertices (y_1,y_2) not included in **(a)** for which

$$|N_{V_{i_k}}(y_1,y_2)| \le C^2p^2n$$

holds simultaneously for all $k=1,\dots,r$. Since J satisfies condition $\mathrm{PAIR}_\ell(\delta)$ and I contains at most ℓ distinct values, there are at most $\ell\delta n^2$ pairs not in **(a)**. Their contribution to the sum $\sum_{(y_1,y_2)\in\mathcal{T}(\{j,j\})} \prod_{k=1}^{r} |N_{V_{i_k}}(y_1,y_2)|$ is bounded by

$$\ell\delta n^2\cdot\big(C^2p^2n\big)^r \le \ell\delta C^{2d}\cdot n^r\cdot(p^r n)^2 \overset{(48)}{\le} \frac{\varrho}{8}\cdot n^r\cdot(p^r n)^2. \quad (61)$$

(c) pairs of vertices (y_1,y_2) for which y_1 or $y_2\in A$ and

$$|N_{V_{i_k}}(y_1,y_2)| > C^2p^2n$$

holds for some $k\in\{1,\dots,r\}$. We estimate the number of these pairs as follows: Suppose $y_1\in A$ and define

$$B=\big\{y_2\in V_j:\ y_2\ne y_1, |N_{V_{i_k}}(y_1,y_2)|>C^2p^2n \text{ for some } k\in\{1,\dots,r\}\big\}.$$

Then, by Corollary 16(b),

$$|B| \le \frac{\ell\gamma^2}{2\alpha^d}p^d n \overset{(49)}{\le} \frac{\varrho}{8C^d}p^d n.$$

Thus, there are at most $2\cdot(\varrho/8C^d)p^d n\cdot n$ pairs in **(c)**, and, by the definition of A, for each pair we have $\prod_{k=1}^{r}|N_{V_{i_k}}(y_1,y_2)|\le(Cpn)^r$. Their contribution to (58) is bounded by

$$\frac{\varrho}{4C^d}p^d n^2\cdot(Cpn)^r\le\frac{\varrho}{4}\cdot n^r\cdot(p^r n)^2. \tag{62}$$

(d) pairs of vertices (y_1,y_2) for which $y_1\notin A$ and $y_2\notin A$. Since $|A|>(1-(\varrho/2)p^{d+1})n$, the number of such pairs is at most $\big((\varrho/2)p^{d+1}n\big)^2$. Their contribution to (58) is bounded by

$$\Big(\frac{\varrho}{2}p^{d+1}n\Big)^2\cdot n^r\le\frac{\varrho}{4}\cdot n^r\cdot(p^r n)^2. \tag{63}$$

Using (58)–(63) above, we see that

$$\sum_{(x_\tau)\in\mathcal{T}(I)}|N_{V_j}(x_1,\dots,x_r)|^2$$
$$\le\Big(1+\frac{\varrho}{8}+\frac{\varrho}{8}+\frac{\varrho}{8}+\frac{\varrho}{4}+\frac{\varrho}{4}\Big)n^r(p^r n)^2<(1+\varrho)n^r(p^r n)^2,$$

which completes the proof of Fact 24(2). □

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