Technical Report

TR-2006-009

Capacitated b-Edge Dominating Set and Related Problems

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Abstract In this paper, we discuss the approximability of the capacitated b-edge dominating set problem, which generalizes the edge dominating set problem by introducing capacities and demands on the edges. We present an approximation algorithm for this problem and show that it achieves a factor of 8/3 for general graphs and a factor of 2 for bipartite graphs. Moreover, we discuss the relationships of the edge dominating set problem and the vertex cover problem. The results show, that improving the approximation factor beyond 8/3 using our approach of adding valid inequalities to a natural linear programming relaxation is as hard as improving the approximation factor for vertex cover beyond 2. We also introduce some useful related problems and present approximation algorithms for them.

Keywords edge dominating set, approximation algorithm, integrality gap

1 Introduction

Let $\mathbb{Z}_+$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the sets of nonnegative integers, rational numbers and real numbers, respectively. Moreover, let $G = (V, E)$ be a simple undirected graph. We say that an edge $e = (u, v)$ *dominates* edges incident to u or v, and define an *edge dominating set* (EDS) to be a set F of edges such that each edge in E is dominated by at least one edge in F. Given a cost vector $w \in \mathbb{Q}_+^E$ together with G, the EDS problem asks to find an

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EDS with minimum cost. This problem is one of the fundamental covering problems such as the well-known vertex cover problem and has some useful applications [2,19]. The problem with a cost vector w with $w(e) = 1$ for all $e \in E$ is called the cardinality case; otherwise the problem is called the cost case.

The cardinality case is NP-hard even for some restricted classes of graphs such as planar or bipartite graphs of maximum degree 3 [10,19]. Moreover, it is proven that the cardinality case is hard to approximate within any constant factor smaller than 7/6 unless P=NP [6]. In contrast, some polynomially solvable cases are also found for the cardinality case [10,13,17].

For the cost case, the problem is approximable within factor of $2r$ if there is an r-approximation algorithm for the minimum cost vertex cover problem [5], where currently $r \leq 2$ is known. Furthermore, Carr et al. [5] presented a 2.1-approximation algorithm. Their algorithm constructs an instance of the *minimum cost edge cover problem* from the original instance and finds an optimal edge cover for the resulting instance. A key property for this method is that an edge cover in the resulting instance is also an EDS for the original instance and that its cost is at most 2.1 times the minimum cost of an EDS in the original instance. The property is proved based on a relation between the fractional edge dominating set polyhedron and the edge cover polyhedron. The former is a polyhedron containing all incidence vectors of EDSs, which may not be the convex hull of these vectors. In contrast, the edge cover polyhedron is the convex hull of all incidence vectors of edge covers, which is shown to be an integer polyhedron [16]. Afterwards, Fujito and Nagamochi [8], and Parekh [15] independently gave a 2-approximation algorithm by using a refined EDS polyhedron. In [5] it was also shown that the weighted vertex cover problem can be approximated as well as weighted EDS. Therefore, finding a constant approximation ratio of less than 2 for EDS is as unlikely as one for the vertex cover problem. Moreover, Könemann et al. proposed 3-approximation algorithms for the problem of finding a minimum cost EDS which forms a tree or a tour [11].

In this paper, we mainly discuss the approximability of the *capacitated b-edge dominating set* ((b,c)-EDS) problem. An instance of this problem consists of a graph $G = (V,E)$, a demand vector $b \in \mathbb{Z}_+^E$, a capacity vector $c \in \mathbb{Z}_+^E$ and a cost vector $w \in \mathbb{Q}_+^E$. A set F of edges in G is called a (b,c)-EDS if each $e \in E$ is adjacent to at least $b(e)$ edges in F, where we allow F to contain at most $c(e)$ multiple copies of edge e. The problem asks to find a minimum cost (b,c)-EDS. The (b,c)-EDS problem generalizes the EDS problem in much the same way that the set multicover problem generalizes the set cover problem [18] and that the b-vertex cover problem generalizes the vertex cover problem. If $b(e) = 1$ and $c(e) \geq 1$ for all $e \in E$, this problem is equivalent to the EDS problem. In the special case when all the capacities c are set to $+\infty$, we call the resulting problem the b-EDS problem and its feasible solutions b-EDSs.

A linear time 2-approximation for the cardinality b-EDS problem in general graphs and a linear time algorithm that optimally solves the cost case of the $\{0,1\}$-EDS problem (where $b_e \in \{0,1\}$ for all $e \in E$) in trees appears in [4]. In this paper we present an 8/3-approximation for the cost case of the

(b, c)-EDS problem in general graphs. This algorithm transform an instance of the (b, c)-EDS problem into that of a *capacitated d-edge cover* ((d, c)-edge cover) problem, which is a generalization of the edge cover problem, defined formally later. The analyses exploit the relation between two polytopes related to the above two problems as the analysis of the 2.1-approximation algorithm does. Moreover, we discuss the relationships of EDS problems and vertex cover problems, in particular how their linear programming formulations and their integrality gaps relate. We will also use these relationships, and a result by Arora et al. [3], to show that appropriate generalizations of the inequalities used for the 2-approximation for the EDS problem cannot improve the approximation ratio of our linear program beyond 8/3.

We also introduce the following useful problems related to the EDS problem or edge cover problem and give approximation algorithms.

The *graphic cover problem*
: As in the (b, c)-EDS problem, we are given a graph $G = (V, E)$, a demand vector b, a capacity vector c, and a cost vector w. However, all of b, c and w are defined to be in $\mathbb{Z}_+^{V \cup E}$ in this problem. Each vertex $v \in V \cup E$ *covers* the elements $\{v\} \cup \delta(v) \subseteq V \cup E$, where $\delta(v)$ denotes the set of edges adjacent to $v \in V$. Each edge $uv \in V \cup E$ *covers* the elements $\{u, v\} \cup \delta(u) \cup \delta(v) \subseteq V \cup E$. The graphic cover problem is that of finding a minimum cost multiset F of $V \cup E$ covering each element $x \in V \cup E$ $b(x)$ times, where F may contain at most $c(y)$ copies of any $y \in V \cup E$.

The *EDS problem in hypergraphs* (HEDS problem)
: This is an extension of the EDS problem to hypergraphs. We are given a hypergraph $H = (V, E)$, and a cost vector $w \in \mathbb{Q}_+^E$. The problem asks to find a minimum cost hyperedge set F such that each hyperedge $e \in E$ is either contained in F or adjacent to a hyperedge in F.

The (d, c)*-edge cover problem with degree constraints over subsets*
: This is an extension of the edge cover problem. We are given a graph $G = (V, E)$, a cost vector $w \in \mathbb{Q}_+^E$, a family $\mathcal{S} \subseteq 2^V$ of vertex sets, a demand vector $d \in \mathbb{Z}_+^{\mathcal{S}}$, and a capacity vector $c \in \mathbb{Z}_+^E$. The problem asks to find a minimum cost edge multiset F such that the sum of degrees in the graph (V, F) over $S \in \mathcal{S}$ is at least $d(S)$, where F may contain at most $c(e)$ copies of edge e.

The paper is organized as follows. Section 2 defines notation used in this paper. Section 3 introduces some polytopes for the (b, c)-EDS problem with a review of those used in the 2.1- and 2-approximation algorithms for the EDS problem. Section 4 describes and analyzes the approximation algorithm for the (b, c)-EDS problem, and Section 5 shows our hardness result. Section 6 considers other related problems.

2 Preliminaries

We denote by $\theta_k \in \mathbb{Q}_+$ the k-th harmonic number $\sum_{i=1}^{k} \frac{1}{i}$. Let $G = (V, E)$ denote a simple undirected graph with vertex set V and edge set E. An edge $e = (u, v) \in E$ in G is defined as a pair of distinct vertices u and v. Let $H = (V, E)$ denote a hypergraph, where an edge is defined by a set of two or more vertices and an edge in H may be called a hyperedge. For a vertex v, $\delta(v)$ denotes the set of edges incident to v. For an edge e, $\delta(e)$ denotes the set of edges incident to vertices contained in e, i.e., $\delta(e) = \{e' \in E \mid e \cap e' \neq \emptyset\}$. For a subset $S \subseteq V$, $\delta(S)$ denotes the set of edges $e = (u, v)$ with $u \in S$ and $v \in V - S$, and $E[S]$ denotes the set of edges contained in S, i.e., $E[S] = \{e \in E \mid e \subseteq S\}$. Let x be an $|E|$-dimensional nonnegative real vector, i.e., $x \in \mathbb{R}_+^E$. We indicate the entry in x corresponding to an edge e by $x(e)$. For a subset F of E, we denote $x(F) = \sum_{e \in F} x(e)$. For an edge set F such that each edge $e' \in F$ corresponds to an edge $e \in E$, $x_F \in \mathbb{R}_+^F$ denotes a projection of x to F, i.e., $x_F(e') = x(e)$ for all $e' \in F$.

3 LP relaxations for the (b, c)-EDS problem and the (d, c)-edge cover problem

For an instance $(G = (V, E), b, c, w)$, an integer program of the (b, c)-EDS problem is given as

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & x(e) \leq c(e) & \text{for each } e \in E, \\ & x(\delta(e)) \geq b(e) & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{1}$$

A vector $x \in \mathbb{Z}_+^E$ satisfying (1) is called a (b, c)-EDS.

Let us define a polytope $\mathrm{EDS}(G, b, c)$ as the set of vectors $x \in \mathbb{R}_+^E$ such that

$$\begin{array}{ll} \text{(a) } 0 \leq x(e) \leq c(e) & \text{for each } e \in E, \\ \text{(b) } x(\delta(e)) \geq b(e) & \text{for each } e \in E. \end{array}$$

This is the feasible region of an LP relaxation of problem (1). Thus the cost of an optimal solution in $\mathrm{EDS}(G, b, c)$ is a lower bound on the minimum cost of a given instance (G, b, c, w).

We now review some results on the (d, c)-edge cover problem, which is another important covering problem. This problem consists of a simple undirected graph $G = (V, E)$, a demand vector $d \in \mathbb{Z}_+^V$ defined on V, a capacity vector $c \in \mathbb{Z}_+^E$ and a cost vector $w \in \mathbb{Q}_+^E$. An integer vector $x \in \mathbb{Z}_+^E$ is called a (d, c)-*edge cover* if $x(\delta(v)) \geq d(v)$ for each $v \in V$ and $x(e) \leq c(e)$ for each $e \in E$. As in the (b, c)-EDS problem, we call the case when $c = +\infty$ the d-edge cover problem. The objective of the (d, c)-edge cover problem is to find a minimum cost (d, c)-edge cover, which is formulated as

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & x(e) \leq c(e) & \text{for each } e \in E, \\ & x(\delta(v)) \geq d(v) & \text{for each } v \in V, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{2}$$

There exists a polynomial time algorithm for this problem [14]. Furthermore, it is known [16] that this problem has an equivalent linear program formulation, where the convex hull of all feasible solutions is characterized by the following set of inequalities:

$$\begin{array}{lll} \text{(c)} & 0 \leq x(e) \leq c(e) & \text{for each } e \in E, \\ \text{(d)} & x(\delta(v)) \geq d(v) & \text{for each } v \in V, \\ \text{(e)} & x(E[U]) + x(\delta(U)) - x(F) \geq \left\lceil \frac{d(U)-c(F)}{2} \right\rceil & \text{for each } U \subseteq V,\ F \subseteq \delta(U) \\ & & \text{with odd } d(U)-c(F). \end{array}$$

Let $\mathrm{EC}(G, d, c)$ denote the polytope represented by these inequalities. If $c = +\infty$ and $F \neq \emptyset$, (e) is always satisfied because its right hand side equals to $-\infty$. Hence in $\mathrm{EC}(G, d, +\infty)$, (e) can be replaced by

$$x(E[U]) + x(\delta(U)) \geq \left\lceil \frac{d(U)}{2} \right\rceil \text{ for each } U \subseteq V \text{ with odd } d(U).$$

Carr et al. [5] derive the 2.1-approximation algorithm for the EDS problem by considering the relation between two relaxations $\mathrm{EDS}(G, 1, +\infty)$ and $\mathrm{EC}(G, \{0, 1\}, +\infty)$. Similarly, our algorithm described in Section 4 utilizes the relationship between $\mathrm{EDS}(G, b, c)$ and $\mathrm{EC}(G, d, c)$.

In the 2-approximation algorithm in [8], $\mathrm{EDS}(G, 1, +\infty)$ is replaced by a refined polyhedron whose region is defined by the following inequalities, which are valid for integral EDSs, together with (a) and (b):

$$2x(E[V(P)]) + x(\delta(V(P))) \geq \left\lceil \frac{|P|}{2} \right\rceil \text{ for each odd cycle } P. \tag{3}$$

The following inequalities are also valid for integral EDSs [15]:

$$x(E[U]) + x(\delta(U)) \geq \left\lceil \frac{|U|}{4} \right\rceil \begin{array}{l} \text{for each hypomatchable set } U \\ \text{with } |U| > 1, \end{array} \tag{4}$$

where a hypomatchable set is a set $U \subset V$ such that $G[U \setminus \{v\}]$ contains a perfect matching for all $v \in U$. Instead of directly augmenting $\mathrm{EDS}(G, 1, +\infty)$ with (4), the 2-approximation algorithm in [15] uses the relaxed inequalities obtained by replacing the right hand side of (4) with $\frac{1}{2}\lceil |U|/2 \rceil$; note that although (3) and (4) are incomparable, the aforementioned relaxed inequalities are implied by (3).

Although exponential in number, the inequalities (3) can either be separated in polynomial time [8] or replaced by polynomially many inequalities (see, for instance [16, Chapter 68]). Moreover, these inequalities may be rewritten by using variables, $y(v)$, corresponding to vertices:

$$\begin{array}{ll} x(\delta(v)) \geq y(v) & \text{for each } v \in V, \\ \sum_{v \in V(P)} y(v) \geq \left\lceil \frac{|P|}{2} \right\rceil & \text{for each odd cycle } P. \end{array}$$

When x is the incidence vector of an EDS, y can be chosen as the incidence vector of a vertex cover, and the above odd cycle inequalities are well-known valid polynomially separable inequalities for the vertex cover problem.

Thus it seems natural to consider an analogous approach for the (b, c)-EDS problem of adding valid (*b-vertex cover*) inequalities to $\mathrm{EDS}(G, b, c)$. However (4) do not seem to generalize to valid $\mathrm{EDS}(G, b, c)$ inequalities in a straightforward way, and we show in Section 5 that adding valid polynomial separable b-vertex cover inequalities on the vertex variables, $y(v)$ cannot improve the integrality gap of EDS(G, b, c) beyond 8/3 unless the vertex cover problem has a polynomially separable LP relaxation with integrality gap strictly less than 2. We provide both a matching upper bound on the integrality gap of EDS(G, b, c) and an approximation algorithm in the next section.

4 An approximation algorithm for the (b, c)-EDS problem

In this section, we present an approximation algorithm for the (b, c)-EDS problem. Given an instance (G, b, c, w) of the (b, c)-EDS problem, the algorithm first constructs an instance of the (d, c)-edge cover problem and then computes an optimal solution for it as an approximate solution to the input instance. A formal description is given in Algorithm 1. The algorithm needs a parameter $f > 0$. This parameter has no effect on the feasibility of solutions that the algorithm outputs. However, it must be set to an appropriate value for achieving a good approximation factor when $c(e)$ is finite for some $e \in E$ as described later.

Algorithm 1 DOMINATE(f)

Input: An instance (G, b, c, w) of the (b, c)-EDS problem and a real $f > 0$
Output: A (b, c)-EDS to the instance (G, b, c, w).

Step 1: Compute an optimal solution $x^* \in \mathbb{R}_+^E$ to the linear program that minimizes $\min w^T x$ subject to $x \in$ EDS(G, b, c). If it is infeasible, output "infeasible". Otherwise, let $E' := \emptyset$.
Step 2: For each $e \in E$ with $fx^*(e) > c(e)$, let $\bar{x}(e) := c(e)$, $E' := E' \cup \{e\}$ and set $b(e') := \max\{0, b(e') - c(e)\}$ for all $e' \in \delta(e)$.
Step 3: For each edge $e = (u, v) \in E$, let $b'_{x^*}(u, e) := b(e)$ and $b'_{x^*}(v, e) := 0$ if $x^*(\delta(u) - E') \geq x^*(\delta(v) - E')$, and let $b'_{x^*}(u, e) := 0$ and $b'_{x^*}(v, e) := b(e)$ otherwise.
Step 4: For each vertex $v \in V$, let $d_{x^*}(v) := \max_{e \in \delta(v)} b'_{x^*}(v, e)$.
Step 5: Set $\bar{x}_{E-E'}$ to a minimum cost (d_{x^*}, c')-edge cover for $G' = (V, E - E')$, $c' = c_{E-E'}$ and $w' = w_{E-E'}$. Then output $\bar{x}$ as a (b, c)-EDS to (G, b, c, w).

If the input instance is infeasible, then there exists an edge $e \in E$ with $c(\delta(e)) < b(e)$. Then, the LP relaxation to be solved in Step 1 is also infeasible. Hence DOMINATE(f) stops in Step 1 at that time.

We first show that $\bar{x}$ is a (b, c)-EDS. For an edge $e = (u, v) \in E$, let us suppose $x^*(\delta(u) - E') \geq x^*(\delta(v) - E')$. Then,

$$\bar{x}(\delta(u) - E') \geq d_{x^*}(u) \geq b(e) - c(\delta(e) \cap E').$$

The above first inequality holds since $\bar{x}_{E-E'}$ is a (d_{x^*}, c')-edge cover, and the second one holds by the definition of d_{x^*}. Since $\bar{x}(\delta(e) \cap E') = c(\delta(e) \cap E')$, it holds

$$\bar{x}(\delta(e)) \geq \bar{x}(\delta(u) - E') + \bar{x}(\delta(e) \cap E') \geq b(e).$$

We can easily check that $0 \le \bar{x}(e) \le c(e)$ also holds. Hence, $\bar{x}$ is a (b, c)-EDS and algorithm DOMINATE(f) outputs a feasible solution.

We now analyze the approximation factor of algorithm DOMINATE(f) by establishing a relation between $\mathrm{EDS}(G, b, c)$ and $\mathrm{EC}(G, d_{x^*}, c')$. In the following discussion, we suppose that $b(e) \ge 1$ for at least one edge $e \in E$, since if $b(e) = 0$ for all edges $e \in E$, DOMINATE(f) apparently outputs the optimal solution $\bar{x} = 0^E$. At first, we consider the b-EDS problem, i.e., $c = +\infty$. In this case, the parameter f makes no effect on the choice of E' in the algorithm and $E' = \emptyset$ always holds.

Lemma 1 *Let x be a vector in* $\mathrm{EDS}(G = (V, E), b, +\infty)$ *and $d_x \in \mathbb{Z}_+^V$ be the vector constructed from x by Step 4 of algorithm* DOMINATE(f). *Then vector $2x \in \mathbb{R}_+^E$ satisfies conditions* (c) *and* (d) *for* $\mathrm{EC}(G, d_x, +\infty)$.

Proof Let $x \in \mathrm{EDS}(G, b, +\infty)$. Then vector $2x$ satisfies condition (c) for $\mathrm{EC}(G, d_x, +\infty)$ because $x \in \mathbb{R}_+^E$ holds by (a) for $\mathrm{EDS}(G, b, +\infty)$. We now show that $2x$ satisfies (d), i.e., $2x(\delta(v)) \ge d_x(v)$ for all $v \in V$. Let v be a vertex in V. Then there is an edge $e = (u, v) \in E$ such that $d_x(v) = b'_x(v, e)$. If $b'_x(v, e) = 0$, then we have $2x(\delta(v)) \ge 0 = d_x(v)$ since $x \in \mathbb{R}_+^E$ holds. Therefore, let us assume $b'_x(v, e) > 0$. Then $b'_x(v, e) = b(e)$ and $x(\delta(v)) \ge x(\delta(u))$ hold. Now $x(\delta(e)) \ge b(e)$ holds by (b) for $\mathrm{EDS}(G, b, +\infty)$, which implies $x(\delta(v)) + x(\delta(u)) = x(\delta(e)) + x(e) \ge b(e) + x(e)$ holds. Then we have

$$2x(\delta(v)) \ge x(\delta(u)) + x(\delta(v)) \ge b(e) + x(e) \ge b(e) = b'_x(v, e) = d_x(v).$$

Therefore, (d) also holds for $2x$. $\square$

Lemma 2 *For a simple undirected graph $G = (V, E)$ and a demand vector $d \in \mathbb{Z}_+^V$, let $\beta = \min_{v \in V, d(v) \neq 0} d(v)$. Then, for any vector $x' \in \mathbb{R}_+^E$ satisfying conditions* (c) *and* (d) *for* $\mathrm{EC}(G, d, +\infty)$, *the vector*

$$y = \left(1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1}\right) x' \in \mathbb{R}_+^E$$

satisfies condition (e) *for* $\mathrm{EC}(G, d, +\infty)$.

Proof Let U be a subset of V such that $d(U)$ is odd. It suffices to show that (e) holds for $x = y$ and U. If U contains a vertex v such that $d(v) = 0$, then (e) follows inductively from $y(E[U']) + y(\delta(U')) \ge \lceil d(U')/2 \rceil$ for $U' = U - \{v\}$, since $y(E[U]) + y(\delta(U)) \ge y(E[U']) + y(\delta(U'))$ and $d(U) = d(U')$. Hence we assume without loss of generality that $d(v) \ge \beta$ for all $v \in U$. Moreover, if $|U| = 1$, then (e) is implied by (d) since for $U = \{v\}$, $y(E[U]) + y(\delta(U)) = y(\delta(v)) \ge x'(\delta(v)) \ge d(v) \ge \lceil d(v)/2 \rceil$. We now consider the case of $|U| = 2$. Let $U = \{v_1, v_2\}$. Since $d(U) = d(v_1) + d(v_2)$ is odd, $d(v_1) \neq d(v_2)$ holds, where we assume without loss of generality $d(v_1) > d(v_2)$. Then

$$\left\lceil \frac{d(U)}{2} \right\rceil = \left\lceil \frac{d(v_1) + d(v_2)}{2} \right\rceil \le d(v_1).$$

We have

$$x'(E[U]) + x'(\delta(U)) \ge x'(\delta(v_1))$$

because $E[U] \cup \delta(U) \supseteq \delta(v_1)$. Since x' satisfies $x'(\delta(v_1)) \geq d(v_1)$ by (d), we have

$$y(E[U]) + y(\delta(U)) \geq x'(E[U]) + x'(\delta(U)) \geq x'(\delta(v_1)) \geq d(v_1) \geq \left\lceil \frac{d(v_1)+d(v_2)}{2} \right\rceil = \left\lceil \frac{d(U)}{2} \right\rceil .$$

In what follows, we assume that $|U| \geq 3$ and $d(v) \geq \beta$ for all $v \in U$. Since $x'(\delta(v)) \geq d(v)$ holds for all $v \in U$ by (d) for $\mathrm{EC}(G, d, +\infty)$, we have

$$2x'(E[U]) + x'(\delta(U)) = \sum_{v \in U} x'(\delta(v)) \geq d(U).$$

Therefore

$$x'(E[U]) + x'(\delta(U)) \geq \frac{d(U) + x'(\delta(U))}{2} \geq \frac{d(U)}{2}.$$

To show (e), we only have to prove that

$$\frac{\lceil d(U)/2 \rceil}{d(U)/2} = 1 + \frac{1}{d(U)} \leq 1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1},$$

or equivalently

$$d(U) \geq 2\lfloor 3\beta/2 \rfloor + 1. \tag{5}$$

From the assumption, $d(U) \geq 3\beta$ holds. Moreover, since $d(U)$ is odd, $d(U) \geq 3\beta + 1$ if 3β is even. This implies (5). □

Theorem 1 *Let $\beta = \min_{e \in E, b(e) \neq 0} b(e)$. Algorithm* DOMINATE$(f)$ *delivers an approximate solution of a cost within a factor of*

$$\rho = 2\left(1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1}\right) \left(\leq \frac{8}{3}\right)$$

*to the b-*EDS *problem.*

Proof Let $\bar{x} \in \mathbb{Z}_+^E$ be a vector obtained by algorithm DOMINATE(f). We have already observed that $\bar{x}$ is a $(b, +\infty)$-EDS to instance $(G, b, +\infty, w)$. We show that $\bar{x}$ is a ρ-approximate solution. We denote by OPT the minimum cost of a $(b, +\infty)$-EDS for $(G, b, +\infty, w)$. Let $x^* \in \mathbb{R}_+^E$ be the vector computed in Step 1 of DOMINATE(f). Since $\mathrm{EDS}(G, b, +\infty)$ contains a minimum cost $(b, +\infty)$-EDS, it holds $w^T x^* \leq \mathrm{OPT}$. By Lemma 1, vector $2x^*$ satisfies conditions (c) and (d) for $\mathrm{EC}(G, d_{x^*}, +\infty)$. Since $b(e) \geq \beta$ for all $e \in E$ such that $b(e) \neq 0$, we see that $d_{x^*}(v) \geq \beta$ or $d_{x^*}(v) = 0$ holds for each $v \in V$. Therefore, from Lemma 2, we have $\rho x \in \mathrm{EC}(G, d_{x^*}, +\infty)$. Since algorithm DOMINATE(f) outputs a solution $\bar{x}$ of minimum cost over all vectors in $\mathrm{EC}(G, d_{x^*}, +\infty)$, we have $w^T \bar{x} \leq \rho w^T x^*$, from which $w^T \bar{x} \leq \rho\mathrm{OPT}$ follows, as required. □

In addition, algorithm DOMINATE(f) achieves a better approximation factor in some special cases. We introduce some results.

Theorem 2 *For a demand vector* $b \in \mathbb{Z}_+^E$ *such that* $\beta = \min_{e \in E} b(e) \geq 1$, *algorithm* DOMINATE($f$) *delivers an approximate solution of a cost within a factor of*

$$\rho = 2\left(1 + \frac{1}{4\beta + 1}\right)\left(\leq \frac{12}{5}\right)$$

to the b-EDS *problem.*

Proof Let $x \in \mathrm{EDS}(G, b, +\infty)$ and U be a subset of V such that $|U| \geq 3$ and $d_x(U) < 4\beta + 1$, where $d_x \in \mathbb{Z}_+^E$ is the vector constructed from x in Step 4 of DOMINATE(f). Below we show that the vector $y = 2x$ satisfies (e) for $\mathrm{EC}(G, d_x, +\infty)$ and U. From this fact, we can assume without loss of generality that $b(U) \geq 4\beta + 1$. Combined with Lemma 1 and the discussion in the proof of Lemma 2, this proves the theorem.

Let $e \in E[U]$. Then it holds $x(E[U]) + x(\delta(U)) \geq x(E[U]) \geq x(\delta(e)) \geq b(e) \geq \beta$. Therefore, $y(E[U]) + y(\delta(U)) \geq 2\beta$. On the other hand, we have $\left\lceil \frac{d_x(U)}{2} \right\rceil \leq 2\beta$ from the assumption. Combining these inequalities leads to $y(E[U]) + y(\delta(U)) \geq \left\lceil \frac{d_x(U)}{2} \right\rceil$ as required. □

Theorem 3 DOMINATE(f) *is a* 2*-approximation algorithm for the* b*-EDS problem in bipartite graphs.*

Proof For bipartite graphs, the edge cover polytopes are determined by only inequalities (a) and (b) [16]. Hence the theorem follows from Lemma 1. □

When b takes the same value for all edges, a better guarantee can be derived as follows.

Lemma 3 *Let* $x \in \mathbb{R}_+^E$ *be a vector in* $\mathrm{EDS}(G, b, +\infty)$*. If* $b(e) = \beta \geq 1$ *for all* $e \in E$*, then* ρx *belongs to* $\mathrm{EC}(G, d_x, +\infty)$*, where* $\rho = 2.1$ *for* $\beta = 1$ *and* $\rho = 2$ *for* $\beta \geq 2$.

Proof Lemma 1 shows that $2x$ satisfies (c) and (d) for $\mathrm{EC}(G, d_x, +\infty)$. Therefore, it suffices to prove that ρx satisfies (e) for $\mathrm{EC}(G, d_x, +\infty)$. Let U be a subset of V such that $d_x(U)$ is odd. As in the proof of Lemma 2, we can assume that $|U| \geq 3$ and $d_x(v) \geq \beta$ holds for all $v \in U$.

Let $x' = 2x$. From the inequalities (d) for $\mathrm{EC}(G, d_x, +\infty)$ and (b) for $\mathrm{EDS}(G, b, +\infty)$ we get that

$$x'(\delta(u)) + x'(\delta(v)) \geq \begin{cases} 2b(e) + x'(e) & e = (u, v) \in E, \\ d_x(u) + d_x(v) & \text{otherwise.} \end{cases}$$

By summing up the above inequalities over all pairs of distinct u and v in $U \times U$, we get

$$\begin{aligned}(|U| - 1)\sum_{u \in U} x'(\delta(u)) &\geq 2b(E[U]) + x'(E[U]) + \sum_{\substack{u,v \in U \\ (u,v) \notin E}} (d_x(u) + d_x(v)), \\ &= 2b(E[U]) + x'(E[U]) + (|U| - 1)d_x(U) - \sum_{\substack{u,v \in U \\ (u,v) \in E}} (d_x(u) + d_x(v)).\end{aligned}$$

Now, $b(e) = \beta$ for all $e \in E$. Hence $d_x(v) \le \beta$ for each $v \in V$. This leads to $2b(e) \ge d_x(u) + d_x(v)$ for each $e = (u, v) \in E$, which implies

$$2b(E[U]) - \sum_{\substack{u,v \in U \\ (u,v) \in E}} (d_x(u) + d_x(v)) \ge 0.$$

Therefore, we have

$$(|U| - 1) \sum_{u \in U} x'(\delta(u)) \ge x'(E[U]) + (|U| - 1) d_x(U).$$

Recall that $|U| \ge 3$ is assumed. Note that $\sum_{u \in U} x'(\delta(u)) = x'(\delta(U)) + 2x'(E[U])$. Hence

$$x'(E[U]) + x'(\delta(U)) \ge \frac{(|U| - 2) x'(\delta(U)) + (|U| - 1) d_x(U)}{2|U| - 3} \ge \frac{(|U| - 1) d_x(U)}{2|U| - 3}.$$

Therefore, we have

$$\begin{aligned} \frac{\lceil d_x(U)/2 \rceil}{x'(E[U]) + x'(\delta(U))} &= \frac{(d_x(U) + 1)/2}{(|U| - 1) d_x(U)/(2|U| - 3)} \\ &= \left(1 + \frac{1}{d_x(U)}\right) \cdot \frac{2|U| - 3}{2|U| - 2}. \end{aligned} \tag{6}$$

We analyze the maximum value of the right hand size of (6). Since we consider the case where $d_x(v) \ge \beta$ holds for all $v \in U$, $d_x(U) \ge \beta |U|$ holds. Therefore we have

$$\left(1 + \frac{1}{d_x(U)}\right) \cdot \frac{2|U| - 3}{2|U| - 2} \le \left(1 + \frac{1}{\beta |U|}\right) \cdot \frac{2|U| - 3}{2|U| - 2}. \tag{7}$$

For $\beta = 1$, the right hand side of (7) takes the maximum value $\frac{21}{20}$ when $|U| = 5$. On the other hand, if $\beta \ge 2$, then the right hand side of (6) is at most 1. Therefore, ρx satisfies (e) for $\mathrm{EC}(G, d_x)$. □

Lemma 3 directly implies the following theorem.

Theorem 4 *Suppose that $b(e) = \beta$ for all $e \in E$. Then algorithm* DOMINATE(f) *delivers an approximate solution of a cost within a factor of* 2.1 *if $\beta = 1$ or a factor of* 2 *if $\beta \ge 2$ to the b-EDS problem.*

We now analyse the approximation factor of DOMINATE(f) for the general (b, c)-EDS problem, i.e., when c takes finite values for some edges. In this case, we need to set f to an appropriate value. Let $\beta = \min\{d_x(U) - c(F) \mid U \subseteq V, F \subseteq \delta(U) - E', d_x(U) - c(F)$ is odd and $\ge 3\}$ and $\rho = 2(1 + 1/\beta)$ be the factor. If $f \ge \rho$, we can prove that $\rho x_{E-E'} \in \mathrm{EC}(G' = (V, E - E'), d_x, c)$, where $x \in \mathrm{EDS}(G, b, c)$ (the proof is similar with that of Lemmas 1 and 2). Then, algorithm DOMINATE(f) achieves the approximation factor of f because of the following reasons. The cost of output edges in E' is bounded as

$$w_{E'}^T \bar{x}_{E'} \le w_{E'}^T c_{E'} < f w_{E'}^T x_{E'}^*.$$

With regard to edges in $E - E'$, it holds that

$$w_{E-E'}^T \bar{x}_{E-E'} \le \rho w_{E-E'}^T x_{E-E'}^*$$

from the above-mentioned relation. Hence, it holds

$$w^T \bar{x} = w_{E'}^T \bar{x}_{E'} + w_{E-E'}^T \bar{x}_{E-E'} < f w^T x^* \le f\mathrm{OPT},$$

where OPT denotes the cost of the optimal solution. Notice that ρ depends on f because f decides which edges are added to E'. As we make f smaller with keeping $f \ge \rho$, we can obtain a better approximation factor. Especially, DOMINATE(8/3) is a 8/3-approximation algorithm.

Theorem 5 DOMINATE(8/3) *is an* 8/3*-approximation algorithm for the* (b, c)*-EDS problem.*

We also obtain the same results described in Theorems 3.

Theorem 6 DOMINATE(2) *is a* 2*-approximation algorithm for the* (b, c)*-EDS problem in bipartite graphs.*

5 Hardness

As hinted in Section 3, we may reflect the fact that an EDS is a edge cover of a vertex cover by augmenting EDS$(G, 1, +\infty)$ with variables, $y(v)$ for all $v \in V$. Extending this idea to the (b, c)-EDS problem yields the following relaxation, which we call EDS$_y(G, b, c)$:

$$\begin{aligned} x(\delta(v)) &\ge y(v) && \text{for each } v \in V, \\ y(u) + y(v) &\ge b(uv) + x(uv) && \text{for each } uv \in E, \\ y(v) &\ge 0 && \text{for each } v \in V, \\ c(e) \ge x(e) &\ge 0 && \text{for each } e \in E. \end{aligned}$$

In an integral solution to the above, the x variables correspond to a (b, c)-EDS while the y variables correspond to a b-vertex cover. It is not difficult to establish that the projection of EDS$_y(G, b, c)$ onto the x variables is equivalent to EDS(G, b, c). In the sequel we do not necessarily explicitly say "the projection of" and refer to EDS$_y(G, b, c)$ and EDS(G, b, c) interchangeably.

For the special case of the EDS problem the integrality gap of the relaxation EDS$_y(G, 1, +\infty)$ is 2.1 [5] and is reduced to 2 [8] by adding the odd-cycle inequalities:

$$\sum_{v \in V(C)} y(v) \ge \left\lceil \frac{|C|}{2} \right\rceil \text{ for each odd cycle } C, \qquad (8)$$

which are valid for vertex covers. By the results of the previous section, the integrality gap of EDS$_y(G, b, c)$ is at most 8/3, and this gap is tight even for instances of the $\{0, 1\}$-EDS problem: For some positive integer k, consider a complete graph on $3k$ vertices, where $b(e) = 0$ and $w(e) = 1$ for the edges of

k vertex-disjoint triangles, and $b(e) = 1$ and $w(e) = +\infty$ for all other edges; an optimal fractional solution need only set each edge of the triangles to a value of $1/4$, for a total cost of $3k/4$, while an integral solution must pick two edges for all but one triangle for a total cost of $2k - 1$.

One may naturally wonder if the odd-cycle inequalities may be generalized for $\mathrm{EDS}_y(G, b, c)$; indeed, they may:

$$\sum_{v \in V(C)} y(v) \geq \left\lceil \frac{b(C)}{2} \right\rceil \text{ for each cycle } C \text{ with } b(C) \text{ odd,} \tag{9}$$

where we abbreviate $b(E[C])$ as $b(C)$. Although we may hope or even expect that (9) reduces the gap of $\mathrm{EDS}_y(G, b, c)$, the, perhaps surprising, main result of this section is that (9) does not improve the integrality gap of $\mathrm{EDS}_y(G, b, c)$; moreover, we show that no polynomially separable class of valid b-vertex cover inequalities on the y variables is likely to reduce this gap beyond $8/3$.

In particular our result applies to the special case, $(\{0, 1\}, +\infty)$-EDS and the corresponding relaxation $\mathrm{EDS}_y(G, \{0, 1\}, +\infty)$, which with respect to approximability and integrality gap respectively, we conjecture to be the hardest (b, c)-EDS instances. Given an instance of $(\{0, 1\}, +\infty)$-EDS, we find it convenient to refer to $D = \{e \in E \mid b(e) = 1\}$ and use D and $(\{0, 1\}, +\infty)$ interchangeably.

We assume we are given a class of valid inequalities for the vertex cover problem. To ensure full generality we suppose that the inequalities are specified by an oracle $\mathcal{O} = \mathcal{O}(G)$ which, given a query vector $y \in \mathbb{R}^V$, either certifies that y satisfies the vertex cover inequalities for G implicitly represented by $\mathcal{O}$ or otherwise returns some inequality that is violated by y. We assume without loss of generality that $\mathcal{O}$ contains the inequalities $y(u) + y(v) \geq 1$ for all $uv \in E$, since these may be checked in linear time. The polyhedron denoted by $\mathcal{P}_{\mathcal{O}} \subseteq \mathbb{R}^V$ consists of the points feasible for the inequalities of $\mathcal{O}$, and we let

$$\mathrm{EDS}^{\mathcal{O}}_y(G, D) = \{(x, y) \in \mathbb{R}^{E \cup V} \mid (x, y) \in \mathrm{EDS}_y(G, D) \text{ and } y \in \mathcal{P}_{\mathcal{O}((V,D))}\}.$$

Note that since $\mathcal{O}$ represents valid vertex cover inequalities, we have that $y \in \mathcal{P}_{\mathcal{O}}$ when y is the incidence vector of a vertex cover.

We are almost in a position to state our main theorem; however, first we must make precise the notion of integrality gaps for the polyhedra under study. Given a D-EDS instance $\mathcal{G} = (G, D, w)$, we let $\mathrm{OPT}^{int}_{D-\mathrm{EDS}}(\mathcal{G})$ denote the cost of an optimal integral D-EDS, whereas $\mathrm{OPT}^{frac}_{D-\mathrm{EDS}}(\mathcal{G}, \mathcal{O}) = \min\{w^T x \mid x \in \mathbb{R}^E \text{ and } \exists y \in \mathbb{R}^V \text{ s.t. } (x, y) \in \mathrm{EDS}^{\mathcal{O}}_y(G, D)\}$. Thus we have that the integrality gap,

$$\mathrm{GAP}^{\mathcal{O}}_{D-\mathrm{EDS}} = \sup_{\mathcal{G}} \frac{\mathrm{OPT}^{int}_{D-\mathrm{EDS}}(\mathcal{G})}{\mathrm{OPT}^{frac}_{D-\mathrm{EDS}}(\mathcal{G}, \mathcal{O})}.$$

Analogously, for an instance $\mathcal{G} = (G, w)$ of the weighted vertex cover problem, we let $\mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G})$ denote the cost of an optimal integral vertex cover, whereas

$\mathrm{OPT}^{frac}_{\mathrm{VC}}(\mathcal{G},\mathcal{O}) = \min\{w^T y \mid y \in \mathcal{P}_{\mathcal{O}(G)}\}$. Not surprisingly, we let

$$\mathrm{GAP}^{\mathcal{O}}_{VC} = \sup_{\mathcal{G}} \frac{\mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G})}{\mathrm{OPT}^{frac}_{\mathrm{VC}}(\mathcal{G},\mathcal{O})}.$$

We are now in a position to state the main result of the section.

Theorem 7 *Assume there exists a vertex cover oracle $\mathcal{O}$ and a constant $\varepsilon > 0$ such that*

$$GAP^{\mathcal{O}}_{D-EDS} \leq \frac{8}{3} - \varepsilon.$$

Then there exists another vertex cover oracle $\mathcal{O}'$ and a constant $\varepsilon' > 0$ such that

$$GAP^{\mathcal{O}'}_{VC} \leq 2 - \varepsilon'.$$

Moreover, if a query to $\mathcal{O}(V,E)$ takes time $t(|V|,|E|)$, then a query to the oracle $\mathcal{O}'(V',E')$ takes time $t(3|V'|, 9|E'| + 3|V'|) + O(|V'| + |E'|)$.

Before proceeding with a proof of the theorem, we note that no vertex cover oracle with a polynomial query time and a gap of $2-\varepsilon$, for a fixed $\varepsilon > 0$, is conjectured to exist. In fact Arora, Bollobás, and Lovász [3] give general classes of inequalities which definitely do not improve the gap beyond $2-\varepsilon$.

Proof (Theorem 7) We first describe a construction that generates a $D-$EDS instance from a given vertex cover instance. This transformation will be used in constructing the oracle $\mathcal{O}'$ using the oracle $\mathcal{O}$. Let $\mathcal{G}' = (G' = (V',E'), w')$ be a vertex cover instance, and let $y' \in \mathbb{R}^{V'}$. We will describe a construction, represented by the overloaded map f, which gives a $D-$EDS instance $\mathcal{G} = f(\mathcal{G}') = (G = (V,E), D, w, y)$ and a vector $y = f(y') \in \mathbb{R}^V$. We define $V = \{v^j : v \in V', 1 \leq j \leq 3\}$. Furthermore $E = E_1 \cup E_2$, where $E_1 = \{v^1v^2, v^2v^3, v^3v^1 \mid v \in V'\}$ and $E_2 = \{u^j v^{j'} \mid 1 \leq j, j' \leq 3, uv \in E'\}$. In other words, G is a graph with $3|V'|$ vertices, where each vertex of G' is represented by a triangle in G; if uv is an edge in E' then G has all the nine edges between the two sets of three vertices of G representing u and v, respectively. We also let $D = E_2$ and define $w(e) = w'(v)$ if $e = v^j v^{j'} \in E_1$ $(j \neq j')$, and $w(e) = +\infty$ if $e \in E_2$. Moreover, we let $y(v^j) = y'(v)$ for every $v \in V'$ and $1 \leq j \leq 3$.

We now suppose that $\mathcal{O}$ is an oracle as given by the statement of the theorem and that we are given an instance $\mathcal{G}'$ of vertex cover along with a vector $y' \in \mathbb{R}^{V'}$. We shall construct $\mathcal{O}'$ by simply applying $\mathcal{O}(f(\mathcal{G}'))$ on the point $f(y')$. See Algorithm 2 for details. The time bound claimed in the theorem follows since the oracle $\mathcal{O}'$ needs to construct a graph with $3|V'|$ vertices and $|E| = 9|E'| + 3|V'|$ edges.

The correctness of Steps 3 and 4 of Algorithm 2 follows from the fact that by the definition of f we have,

$$\sum_{v \in V'} \sum_{j=1}^{3} a(v^j) y(v^j) = \sum_{v \in V'} \left(\sum_{j=1}^{3} a(v^j) \right) y'(v), \tag{10}$$

Algorithm 2 Separation algorithm for the oracle O'

Input: A vertex cover instance $\mathcal{G}' = (G' = (V', E'))$, $y' \in \mathbb{R}^{V'}$.
Output: "feasible" or an inequality violated by y'.

Step 1: Construct $\mathcal{G} = f(\mathcal{G}')$ and $y = f(y')$.
Step 2: Run the separation algorithm for the oracle $\mathcal{O}$ with input $\mathcal{G}$ and y.
Step 3: If y is feasible for $\mathcal{G}$, then $\mathcal{O}'$ returns "feasible."
Step 4: If y violates an inequality $\displaystyle\sum_{v \in V'} \sum_{j=1}^{3} a(v^j) y(v^j) \geq a_0$, then $\mathcal{O}'$ returns that y' violates the inequality $\displaystyle\sum_{v \in V'} \left(\sum_{j=1}^{3} a(v^j) \right) y'(v) \geq a_0$.

hence we need only show that each inequality $\sum_{v \in V'} (\sum_{j=1}^{3} a(v^j)) y'(v) \geq a_0$ is satisfied when y' is the incidence vector of an integral vertex cover. Note that since y' is a vertex cover in G', y is a vertex cover in G. Moreover, since $\mathcal{O}$ is a valid oracle, we must have $\sum_{v \in V'} \sum_{j=1}^{3} a(v^j) y(v^j) \geq a_0$, hence the result follows from (10).

Next we address the relationship between the gaps induced by $\mathcal{O}$ and $\mathcal{O}'$ by showing that for a vertex cover instance $\mathcal{G}'$:

(i) $\mathrm{OPT}^{int}_{D-\mathrm{EDS}}(f(\mathcal{G}')) = 2 \cdot \mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G}')$, and
(ii) $\mathrm{OPT}^{frac}_{D-\mathrm{EDS}}(f(\mathcal{G}'), \mathcal{O}) \leq 3/2 \cdot \mathrm{OPT}^{frac}_{\mathrm{VC}}(\mathcal{G}', \mathcal{O}')$.

For any $\delta > 0$, we may select a vertex cover instance $\mathcal{G}'$ such that:

$$\begin{aligned}
\mathrm{GAP}^{\mathcal{O}'}_{\mathrm{VC}} &\leq \frac{\mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G}')}{\mathrm{OPT}^{frac}_{\mathrm{VC}}(\mathcal{G}', \mathcal{O}')} + \delta \\
&\overset{(i),(ii)}{\leq} \frac{\frac{1}{2} \cdot \mathrm{OPT}^{int}_{D-\mathrm{EDS}}(f(\mathcal{G}'))}{\frac{2}{3} \cdot \mathrm{OPT}^{frac}_{D-\mathrm{EDS}}(f(\mathcal{G}'), \mathcal{O})} + \delta \\
&\leq \frac{3}{4} \mathrm{GAP}^{\mathcal{O}}_{D-\mathrm{EDS}} + \delta \\
&\leq 2 - \frac{3}{4} \cdot \varepsilon + \delta,
\end{aligned}$$

which gives the theorem.

Proof of Claim (i): Let $V^* \subseteq V'$ be an optimal integer vertex cover of $\mathcal{G}'$, i.e. $w'(V^*) = \mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G}')$. Then $E^* = \{v^1v^2, v^2v^3 \mid v \in V^*\}$ is an integral $D-\mathrm{EDS}$ for G of cost $2 \cdot \mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G}')$. Therefore $\mathrm{OPT}^{int}_{D-\mathrm{EDS}}(f(\mathcal{G}')) \leq 2 \cdot \mathrm{OPT}^{int}_{\mathrm{VC}}(\mathcal{G}')$.

Now let $E^* \subseteq E$ be an optimal integral $D-\mathrm{EDS}$ of the instance $f(\mathcal{G}')$. Then $E^* \subseteq E_1$ since edges in E_2 have infinite cost. We claim that E^* contains either 0 or 2 edges from each triangle representing a vertex v of G'. If E^* contained all three edges v^1v^2, v^2v^3 and v^3v^1, then just two of these edges cover the same set of edges of G and E^* would not be optimal.

Suppose w.l.o.g. E^* contains v^1v^2 but not v^2v^3 and v^3v^1, then E^* must cover the edges u^jv^3 for all $u \in V'$ such that $uv \in E'$ and $1 \leq j \leq 3$. Hence E^* contains at least two edges from each triangle representing a vertex u which is adjacent to v in G'. But then all edges in D which are incident with v^1 or v^2 are covered even if we remove the edge v^1v^2 from E^*, contradicting the optimality of E^*. Therefore E^* contains either 0 or 2 edges from each triangle representing a vertex v of G'. Thus $V^* = \{v \in V \mid \exists j, j' \text{ s.t. } x_{v^jv^{j'}} = 1\}$ is a vertex cover of G'. Moreover, we have $c'(V^*) = \frac{1}{2}c(E^*)$ and therefore

$$\mathrm{OPT}_{\mathrm{VC}}^{int}(\mathcal{G}') \leq \frac{1}{2}\mathrm{OPT}_{D-\mathrm{EDS}}^{int}(f(\mathcal{G}')),$$

proving Claim (i).

Proof of Claim (ii): Let y^* be a minimizer of $\min\{w'^T y \mid y \in \mathcal{P}_{\mathcal{O}'(G')}\}$, i.e. $w'^T y^* = \mathrm{OPT}_{\mathrm{VC}}^{frac}(\mathcal{G}', \mathcal{O}')$. We define a fractional $D-\mathrm{EDS}$ in $f(\mathcal{G}')$ by letting $x(e) = 0$ for all $e \in E_2$ and $x(v^jv^{j'}) = \frac{1}{2}y^*(v)$ for all $v \in V'$, $1 \leq j < j' \leq 3$. Certainly $w^T x = \frac{3}{2}w'^T y^* = \frac{3}{2}\mathrm{OPT}_{\mathrm{VC}}^{frac}(\mathcal{G}', \mathcal{O}')$.

We let $y = f(y^*)$. Since $y^* \in \mathcal{P}_{\mathcal{O}'(G')}$ we have $y \in \mathcal{P}_{\mathcal{O}(V', E_2)}$, and consequently $y(u^i) + y(v^j) \geq 1$ for all $u^iv^j \in E_2$. Moreover, $x(\delta(v^j)) = y(v^j)$ for every $v^j \in V$, which when combined with the preceding inequality yields,

$$y(u^i) + y(v^j) \geq b(u^iv^j) + x(u^iv^j) \text{ for every } u^iv^j \in E.$$

Thus $(x, y) \in \mathrm{EDS}_y^{\mathcal{O}}(G, E_2)$, and

$$\mathrm{OPT}_{D-\mathrm{EDS}}^{frac}(\mathcal{G}, \mathcal{O}) \leq w^T x = 3/2 \cdot \mathrm{OPT}_{\mathrm{VC}}^{frac}(\mathcal{G}', \mathcal{O}').$$

□

In fact, Arora et al. [3, Section 4] show that the odd cycle inequalities (8) do not improve the integrality gap of vertex cover to a factor of $2-\varepsilon$ for any constant $\varepsilon > 0$. However, even by Theorem 7, this does not immediately imply that (9) cannot improve the integrality gap of $\mathrm{EDS}(G, D)$ to $8/3-\varepsilon$. This is because even if we set $\mathcal{O}$ in Theorem 7 to an oracle for (9), it is not quite clear exactly which inequalities $\mathcal{O}'$ corresponds to. It is not difficult to show that $\mathcal{O}'$ contains the inequalities in $\mathcal{O}$; however, the converse may not always be true as the inequalities derived for $\mathcal{O}'$ (see Algorithm 2) may have coefficients of magnitude thrice as large as those for inequalities $\mathcal{O}$. Fortunately, when $\mathcal{O}$ corresponds to the specialization of (9) for $\mathrm{EDS}(G, \{0,1\}, +\infty)$, one can show that $\mathcal{O}'$ corresponds precisely to the standard odd cycle inequalities (8), yielding:

Corollary 1 *Let $\mathcal{O}$ be an oracle for the odd cycle inequalities* (9). *For any $\varepsilon > 0$, $GAP_{D-EDS}^{\mathcal{O}} > \frac{8}{3} - \varepsilon$.*

6 Some related problems

6.1 Graphic cover

In this section we show that a general covering problem in the vein of the problem of finding a minimum cost *total cover* (see [1]) reduces to the (b, c)-EDS problem. As in the (b, c)-EDS problem, we are given a graph $G = (V, E)$, a demand vector b, a capacity vector c, and a cost vector w. However, all of b, c and w are defined to be in $\mathbb{Z}_+^{V+E}$ in this problem. Each vertex $v \in V \cup E$ *covers* the elements $\{v\} \cup \delta(v) \subseteq V \cup E$, and each edge $uv \in V \cup E$ *covers* the elements $\{u, v\} \cup \delta(u) \cup \delta(v) \subseteq V \cup E$. The graphic cover problem consists of finding a minimum cost multiset F of $V \cup E$ covering each element $x \in V \cup E$ $b(x)$ times, where F may contain at most $c(y)$ copies of any $y \in V \cup E$.

Proposition 1 *Graphic cover is equivalent to (b, c)-EDS.*

Proof Suppose we are given an instance of graphic cover as described above. A slight modification to G yields an instance G' of (b, c)-EDS. For each vertex $v \in V$, we simply augment G with a vertex v' and an edge vv', setting $w_{vv'}$, $b_{vv'}$, and $c_{vv'}$ to the corresponding parameters for v. Since $\delta_{G'}(v) = \delta_G(v) \cup vv'$ we have the desired result.

6.2 Hyperedge dominating set

In this section we discuss the hypergraph version of the edge dominating set problem. Given a hypergraph $H = (V, E)$ and a cost vector $w \in \mathbb{Q}_+^E$, the problem is formulated as

$$\begin{array}{ll} \text{minimize} & w^T x \\ \text{subject to} & x(\delta(e)) \geq 1 \text{ for each } e \in E, \\ & x \in \mathbb{Z}_+^E. \end{array} \tag{11}$$

An HEDS is defined as a vector $x \in \mathbb{Z}_+^E$ such that $x(\delta(e)) \geq 1$ for each $e \in E$. In addition, we call the LP relaxation of the HEDS problem *the fractional HEDS problem* and its feasible solution a *fractional HEDS*.

To obtain an approximate solution to the HEDS problem, we transform a given instance of the HEDS problem to an instance of the *set cover problem*. The set cover problem is considered as a hypergraph version of the edge cover problem. A hyperedge set $F \subseteq E$ is called a *set cover* of a hypergraph $H = (V, E)$ if $\cup_{e \in F} e = V$ and the set cover problem asks to find a minimum cost set cover. Given a hypergraph $H = (V, E)$ and a cost vector $w \in \mathbb{Q}_+^E$, the formulation of the set cover problem is given as follows.

$$\begin{array}{ll} \text{minimize} & w^T x \\ \text{subject to} & x(\delta(v)) \geq 1 \text{ for each } v \in V, \\ & x \in \mathbb{Z}_+^E. \end{array} \tag{12}$$

Note that this problem is proven to be NP-hard [9]. Moreover, the LP relaxation of the set cover problem is called the *fractional set cover problem* and its feasible solutions are called *fractional set covers*. It is known that a simple greedy algorithm finds an approximate solution for the set cover problem, and that the cost of the solution is bounded in terms of the minimum cost of a fractional set cover, as described in the following theorem.

Theorem 8 [7,12] *Let $w \in \mathbb{Q}_+^E$ be a given cost vector, $\hat{x}$ be a minimum cost fractional set cover for a hypergraph $H = (V, E)$, and k be the maximum size of a hyperedge in H. Then a set cover whose cost is at most $\theta_k w^T \hat{x}$ can be obtained in polynomial time.*

Since the HEDS problem is a special case of the set cover problem, the HEDS problem can be reduced to the set cover problem directly. Let $(H = (V, E), w)$ be a given instance of the HEDS problem. Construct a hypergraph $H' = (V', E')$ such that its vertex set V' consists of vertices v'_e corresponding to its edges $e \in E$ and edge set E' consists of $e'_e = \{v'_{e''} \mid e'' \in \delta(e)\}$ corresponding to $\delta(e)$. A component of the cost vector $w'(e'_e)$ is set to be $w(e)$. Then, it is easy to see that a set cover for (H', w') gives an HEDS for (H, w) of same cost and vice versa. Let d be the maximum size of a hyperedge in E', i.e., the maximum size of $\delta(e)$ for all $e \in E$, where $d = O(|V|^k)$ holds for the maximum size k of a hyperedge in H. By Theorem 8, this direct reduction gives a θ_d-approximation algorithm for the HEDS problem. Note that $\theta_d = O(k \log |V|)$.

In algorithm HYPER described in Algorithm 3, an instance of the HEDS problem is transformed into an instance of the set cover problem. To prove that the approximation factor of HYPER is $k\theta_k$ by using Theorem 8, we show that for the vector x^* obtained in Step 1, the vector kx^* is a fractional set cover of (H_{x^*}, w').

Algorithm 3 HYPER

Input: A hypergraph $H = (V, E)$ and a cost vector $w \in \mathbb{Q}_+^E$.
Output: An HEDS for H.

Step 1: Find a minimum cost solution $x^* \in \mathbb{R}^E$ to the fractional HEDS problem for H and w.
Step 2: Let $V' := \{v \in V \mid x^*(\delta(v)) = \max_{u \in e} x^*(\delta(u))$ for some $e \in E\}$, $E' := \{e \cap V' \mid e \in E\}$, and $w' := w_{E'} \in \mathbb{Q}_+^{E'}$.
Step 3: Find a set cover $\bar{x}$ for the hypergraph $H_{x^*} = (V', E')$ such that $w'^T \bar{x}$ is at most θ_k times the minimum cost of a fractional set cover, and output $\bar{x}_E$ as an HEDS for H.

Lemma 4 *Let $x \in \mathbb{R}^E$ be a fractional* HEDS *for a hypergraph $H = (V, E)$, $H_x = (V', E')$ be the hypergraph obtained in Step 3 of* HYPER *from x and k be the maximum hyperedge size of H. Then the vector $kx_{E'} \in \mathbb{R}^{E'}$ is a fractional set cover for H_x.*

Proof Suppose that $v \in V'$ is a vertex in a hyperedge $e \in E'$ such that $x(\delta(v)) \geq x(\delta(u))$ for all $u \in e$. Since $\sum_{u \in e} x(\delta(u)) \geq x(\delta(e)) \geq 1$, we have

$x(\delta(v)) \geq 1/k$. Therefore $kx\langle E'\rangle(\delta(v)) \geq 1$, which means that $kx_{E'} \in \mathbb{R}^{E'}$ is a fractional set cover for H_x. ◻

Theorem 9 *The algorithm* HYPER *achieves an approximation factor of* $k\theta_k$ *for the* HEDS *problem, where* k *is the maximum size of a hyperedge.*

Proof Let $w'(F)$ be the minimum weight of fractional set covers of H_{x^*}. As described in the algorithm, $w'^T\bar{x} \leq \theta_k w'(F)$. Moreover, Lemma 4 implies that $w'(F) \leq kw'^T x^*_{E'}$. Hence

$$w'^T\bar{x} \leq \theta_k w'(F) \leq k\theta_k w'^T x^*_{E'} = k\theta_k w^T x^*.$$

Since $w'^T\bar{x}$ is the cost of solution algorithm outputs and $w^T x^*$ is a lower bound of the optimal cost, it completes the proof. ◻

Note that the approximation factor $k\theta_k = O(k\log k)$ of algorithm HYPER is superior to that of the algorithm obtained from the direct reduction if $k\theta_k < \theta_d$, i.e., if H is a dense hypergraph such that $d = \Omega(|V|^k)$.

6.3 (d,c)-edge cover with degree constraints over subsets

Given an undirected graph $G = (V, E)$, a cost vector $w \in \mathbb{Q}_+^E$, a family $\mathcal{S} \subseteq 2^V$ of subsets of V, a demand vector $d \in \mathbb{Z}_+^{\mathcal{S}}$ and capacity vector $c \in \mathbb{Z}_+^E$, the (d,c)-edge cover with degree constraints over subsets is formulated by

$$\begin{array}{lll} \text{minimize} & w^T x & \\ \text{subject to} & \sum_{v\in S} x(\delta(v)) \geq d(S) & \text{for each } S \in \mathcal{S}, \\ & x(e) \leq c(e) & \text{for each } e \in E, \\ & x \in \mathbb{Z}_+^E. & \end{array} \tag{13}$$

Note that if $\mathcal{S} = \{\{v\} \mid v \in V\}$, then problem (13) is equivalent to the (d,c)-edge cover problem (2). If $\mathcal{S} = \{\{u,v\} \mid (u,v) \in E\}$, then problem (13) seems similar to the (b,c)-EDS problem, but its first constraint $x(\delta(u)) + x(\delta(v)) \geq b(e)$ on each $e = (u,v) \in E$ is different from the constraint $x(\delta(e)) \geq b(e)$ for the (b,c)-EDS. Let $\mathrm{DC}(G, \mathcal{S}, d, c)$ denote the set of all vectors $x \in \mathbb{R}_+^E$ satisfying the inequalities in (13), i.e., the relaxation of the covering problem. We show that problem (13) is approximable by algorithm COVER(f) described in Algorithm 4.

For each $S \in \mathcal{S}$, the vertex $v = \arg\max_{u\in S} x^*(\delta(u) - E')$ satisfies

$$\sum_{u\in S} \bar{x}(\delta(u) - E') \geq \bar{x}(\delta(v) - E') \geq \tilde{d}_{x^*}(v) \geq d(S) - 2c(E[S] \cap E') - c(\delta(S) \cap E').$$

Since

$$\sum_{u\in S} \bar{x}(\delta(u)) \geq \sum_{u\in S} \bar{x}(\delta(u) - E') + 2\bar{x}(E[S] \cap E') + \bar{x}(\delta(S) \cap E') \geq d(S),$$

we can see that $\bar{x}$ is a feasible solution for problem (13). In which follows, we discuss the approximation factor of COVER(f). It can be derived analogously to that of algorithm DOMINATE(f). First, let us consider the case of $c(e) = +\infty$. Notice that $E' = \emptyset$ for any f in this case.

Algorithm 4 COVER(f)

Input: A simple undirected graph $G = (V, E)$, a cost vector $w \in \mathbb{Q}_+^E$, a family $\mathcal{S} \subseteq 2^V$ of subsets of V, a demand vector $d \in \mathbb{Z}_+^{\mathcal{S}}$, a capacity vector $c \in \mathbb{Z}_+^E$, and a real $f > 0$.
Output: A vector $x \in \mathbb{Z}_+^E$ feasible to the covering problem (13).

Step 1: Let $E' := \emptyset$. Moreover, find a minimum cost solution x^* of $\mathrm{DC}(G, \mathcal{S}, d, c)$. If $\mathrm{DC}(G, \mathcal{S}, d, c) = \emptyset$, output "infeasible".
Step 2: For each $e \in E$ with $fx^*(e) > c(e)$, let $\bar{x}(e) := c(e)$, $E' := E' \cup \{e\}$, $d(S) := \max\{0, d(S) - 2c(e)\}$ for each $S \in \mathcal{S}$ with $e \in E[S]$, and $d(S) := \max\{0, d(S) - c(e)\}$ for each $S \in \mathcal{S}$ with $e \in \delta(S)$.
Step 3: For each $S \in \mathcal{S}$, $d'_{x^*}(v, S) := d(S)$ if $x^*(\delta(v) - E') \geq x^*(\delta(u) - E')$ for all $u \in S$ and $d'_{x^*}(v, S) := 0$ otherwise.
Step 4: For each $v \in V$, $\tilde{d}_{x^*}(v) := \max_{S \in \mathcal{S}: v \in S} d'_{x^*}(v, S)$.
Step 4: Compute a minimum cost $(\tilde{d}_{x^*}, c)$-edge cover $\bar{x}_{E-E'}$ for $G' = (V, E - E')$ and $w_{E-E'}$, and output $\bar{x}$ as a solution to (13).

Lemma 5 *Let $x \in \mathrm{DC}(G, \mathcal{S}, d, +\infty)$ and $h = \max_{S \in \mathcal{S}} |S|$. Then the vector hx satisfies conditions* (c) *and* (d) *for $\mathrm{EC}(G, \tilde{d}_x, +\infty)$, where $\tilde{d}_x \in \mathbb{Z}_+^V$ is the vector obtained from x in Step 4 of* COVER(f).

Proof Since $x \in \mathbb{R}_+^{E-E'}$, vector hx satisfies (b) for $\mathrm{EC}(G, \tilde{d}_x, +\infty)$. We show that hx satisfies (d), i.e., $hx(\delta(v)) \geq \tilde{d}_x(v)$ for each $v \in V$. Let v be a vertex in V. If $\tilde{d}_x(v) = 0$, then $hx(\delta(v)) \geq 0 = \tilde{d}_x(v)$ holds. Now assume $\tilde{d}_x(v) > 0$. There exists a subset $S \in \mathcal{S}$ such that $x(\delta(v)) \geq x(\delta(u))$ holds for all $u \in S$ and $\tilde{d}_x(v) = d'_x(v, S) = d(S)$. From this inequality and the condition $\sum_{u \in S} x(\delta(u)) \geq d(S)$ for $\mathrm{DC}(G, \mathcal{S}, d, +\infty)$, we have

$$hx(\delta(v)) \geq |S| x(\delta(v)) \geq \sum_{u \in S} x(\delta(u)) \geq d(S) = d'_x(v, S) = \tilde{d}_x(v).$$

This implies that hx satisfies (d) for $\mathrm{EC}(G', \tilde{d}_x, +\infty)$. ◻

Lemmas 2 and 5 now imply the following theorem.

Theorem 10 *Algorithm* COVER(f) *achieves an approximation factor of*

$$h \cdot \left(1 + \frac{1}{2\lfloor 3\beta/2 \rfloor + 1}\right) \left(\leq \frac{4}{3} h\right)$$

for problem (13) *with $c(e) = +\infty$, where $h = \max\{|S| \mid S \in \mathcal{S}\}$ and $\beta = \min_{S \in \mathcal{S}, b(S) \neq 0} b(S)$.*

Proof Let $y = h \cdot \left(1 + (2\lfloor 3\beta/2 \rfloor + 1)^{-1}\right) x^*$. By Lemmas 2 and 5, it holds that $y \in \mathrm{EC}(G, \tilde{d}_x, +\infty)$, which implies that $w^T y$ is at least the minimum cost over $\mathrm{EC}(G, \tilde{d}_x, +\infty)$. Since the algorithm outputs a vector of minimum cost over $\mathrm{EC}(G, \tilde{d}_x, +\infty)$ and $w^T x^*$ is a lower bound of the optimal cost, the proof is completed. ◻

For the case where $c(e)$ is finite, we can also derive an approximation factor as in the (b, c)-EDS problem, if the parameter f is set appropriately. Particularly, COVER$(4h/3)$ achieves the factor of $4h/3$.

7 Conclusion

We introduced the (b, c)-EDS problem and proposed an approximation algorithm, which achieves a factor of 8/3 for general graphs by utilizing the relationship between the polytopes EDS(G, b, c) and EC(G, d, c). Moreover, we showed that no polynomially separable class of valid b-vertex cover inequalities on the y variables is likely to reduce the integrality gap of EDS(G, b, c) beyond 8/3. In fact, our result and a result by Arora et al. [3] show that adding a generalization of odd-cycle inequalities does not improve the gap. We also introduced some useful related problems and proposed approximation algorithms for them.

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