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Retracts of Posets: The Chain Gap Property and the Selection Property are Independent

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RETRACTS OF POSETS: THE CHAIN-GAP PROPERTY AND THE SELECTION PROPERTY ARE INDEPENDENT

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Abstract. Posets which are retract of products of chains are characterized by means of two properties: *the chain-gap property* and *the selection property* (Rival and Wille, 1981 [8]). Examples of posets with the selection property and not the chain-gap property are easy to find. To date, the Boolean lattice $\mathcal{P}(\omega_1)/Fin$ was the sole example of lattice without the selection property [8]. We prove that it does not have the chain-gap property. We provide an example of a lattice which has the chain-gap property but not the selection property. This answer questions raised in [8].

1. Introduction

A poset P is a *retract* of a poset Q if there are two order-preserving maps $f : P \to Q$ and $g : Q \to P$ such that $g \circ f = 1_P$; these maps being respectively called a *coretraction* and a *retraction*. The first author and I. Rival [2] have defined *an order variety* to be a class of posets closed under direct products and retracts. I. Rival and R. Wille[8] characterized members of the order variety generated by the class of chains as posets satisfying two properties: *the chain-gap property* and *the selection property*. And, they raised the question of their relationship. They gave examples of lattices with the selection property for which the chain-gap property fails. They showed that $\mathcal{P}(\omega_1)/Fin$, the quotient of the power set of ω_1 by the ideal Fin of the finite sets, does not have the selection property. They asked if it has the gap-property and we answer this by the negative.

Theorem 1.1. *If E is infinite, $\mathcal{P}(E)/Fin$ does not have the chain-gap property.*

They also asked if there is a lattice with the gap property but without the selection property, and we answer this question positively. Our example is a distributive lattice of size $\aleph_1$ which does not embed the ordinal ω_1. It is built from a Sierpinskization of a subchain $\mathbb{S}$ of the real line $\mathbb{R}$ which is $\aleph_1$-dense, that is, $|]a, b[\cap \mathbb{S}| \geq \aleph_1$ for every $a < b$ in $\mathbb{S}$, has no end points and has size $\aleph_1$ (the existence of such chains is well-known and easily proved). Let $\leq_{\omega_1}$ be an ordering on $\mathbb{S}$ such that the chain $(\mathbb{S}, \leq_{\omega_1})$ has order type ω_1, let $\leq_{\mathbb{R}}$ be the usual ordering on the reals. The Sierpinskization of $\mathbb{S}$ is the poset $(\mathbb{S}, \leq)$ where $\leq$ is the ordering on $\mathbb{S}$ defined by $x \leq y$ iff $x \leq_{\omega_1} y$ and $x \leq_{\mathbb{R}} y$. Let $L(\mathbb{S}, \leq)$ be the lattice generated within the lattice of subsets of $\mathbb{S}$ by the principal initial segments of $(\mathbb{S}, \leq)$. So $L(\mathbb{S}, \leq)$ consists of all the finite unions of finite intersections of sets of the form $\downarrow x$ for $x \in \mathbb{S}$, where $\downarrow x := \{y \in \mathbb{S} \text{ and } y \leq x\}$. With this construction in mind we show:

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Theorem 1.2. *$L(\mathbb{S}, \leq)$ has the chain-gap property, but not the selection property.*

2. Preliminaries

If C is a subset of a poset (or quasiorder) P, then let $C^* := \{x \in P : y \leq x \text{ for all } y \in C\}$ denote the set of upper bounds of C and $C_* := \{x \in P : x \leq y \text{ for all } y \in C\}$ denotes the set of lower bounds of C. A pair (A, B) of subsets of P is a *pregap* of P if $A \subseteq B_*$ or, equivalently $B \subseteq A^*$. A pregap (A, B) is called *separable* if $A^* \cap B_*$ is non-empty, otherwise this is a *gap* of P. We denote by $B(P)$ the set of separable pregaps of P. Pregaps are quasiordered as follows: $(A, B) \leq (A', B')$ if $A \leq A'$ and $B' \leq^d B$, where $A \leq A'$ means that every $a \in A$ is majorized by some $a' \in A'$, and $B' \leq^d B$ means that every $b \in B$ majorizes some $b' \in B'$.

The *cardinality* of a pair (A, B) of subsets of a poset P is the pair $(|A|, |B|)$. We call the pair *regular* if $|A|$ and $|B|$ are both regular, or one is regular and the other is zero. Say that (A', B') is a *subpair* of (A, B) if $A' \subseteq A$ and $B' \subseteq B$; if both pairs are gaps, call (A', B') a *subgap* of (A, B). A gap (A, B) of P is said to be *minimal* if all subgaps have the same cardinality as (A, B). Call a gap (A, B) *irreducible* if for all subpairs (A', B'), (A', B') is a gap if and only if it has the same cardinality as (A, B). It is straightforward to show that every gap has a subgap which is minimal. On an other hand, irreducible gaps are just minimal gaps all of whose subpairs, of its cardinality, are gaps.

We now come to the main concepts of this paper.

Definition 2.1. (1) *The poset P has the selection property (the strong selection property in the terminology of Nevermann, Wille* [6]*) if there is an order-preserving map φ from $B(P)$ to P which associates to every pair $(A, B) \in B(P)$ an element of $A^* \cap B_*$.*

(2) *An order-preserving map g from P into a poset Q* preserves *a gap (A, B) of P if $(g[A], g[B])$ is a gap of Q. If g preserves all gaps of P, it is* gap-preserving. *A poset Q* preserves *a gap (A, B) of P if there is an order-preserving map $g : P \to Q$ which preserves (A, B). The poset P is said to have the* chain-gap property *if each gap of P is preserved by some chain.*

The relationship between the chain-gap property and regular irreducible gaps is given by the following result by Duffus and Pouzet.

Theorem 2.2. [1] *An ordered set P has the chain-gap property if and only if every gap of P contains a regular, irreducible gap.*

In presence of the selection property, they have proved a bit more.

Proposition 2.3. [1] *Let (A, B) be a minimal gap of P with $\lambda := |A|$ and $\mu := |B|$ both infinite. If P has the selection property then there are two chains C and D of type respectively $cf(\lambda)$ and $cf(\nu)^*$ such that $A \leq C$, $D \leq^d B$ and (C, D) is a gap. Moreover, if (A, B) is an irreducible gap then ordinal sum $C \oplus D$ is a retract of P which preserves (A, B).*

We illustrate how the above notions relate to a central problem in the study of retracts of posets, namely to find conditions that a map $f : P \to Q$ must satisfy in order to be a coretraction. Posets P for which maps satisfying these conditions are coretractions are called *absolute retracts w.r.t. these conditions.* For example, each coretraction must be an order-embedding. As it is well-known, *the absolute retracts w.r.t. order-embeddings* are the complete lattices. Moreover, these are *injective w.r.t. order-embedding* (that is, every order-preserving map from a poset Q to P extends to an order-preserving map to every poset Q' in which Q order-embeds) and there are enough of them in the sense that every poset order-embeds into a complete lattice, that is, one of them. Every coretraction must be gap-preserving. A somewhat similar situation to the case of order-embeddings was observed by Duffus and Pouzet [1], and by Nevermann and Rival [5]:

> A poset P is an *absolute retract w.r.t. gap-preserving maps* if and only if it has the selection property. Moreover, absolute retracts coincide with *injective objects w.r.t. gap-preserving maps* and there are enough of them.

The class of absolute retracts is preserved under retraction and products (Rival and Wille[8]), it contains the chains (Duffus, Rival and Simonovits [3]) hence the variety generated by the class of chains. According to Rival and Wille [8]:

> A poset P embeds by a gap-preserving map into a product of chains iff P has the chain-gap property.

The chain-gap property implies that P is a lattice. Every countable lattice belongs to the variety generated by the class of chains [7], hence satisfies the chain-gap property. However, there are many lattices for which the chain gap property fails (see [2], [8]).

We conclude this section with some notation and remarks necessary for the proof of Theorems 1.1 and 1.2.

For E any set, let $\mathcal{P}(E)$ be the Boolean algebra made of all subsets of E and let $\mathcal{P}(E)/Fin$ be the quotient of $\mathcal{P}(E)$ by the ideal Fin of finite subsets of E. Define $p : \mathcal{P}(E) \to \mathcal{P}(E)/Fin$ to be the canonical projection. For $X, Y \in \mathcal{P}(E)$, we set $X \leq_{Fin} Y$ if $X \setminus Y \in Fin$; this defines a quasiorder on $\mathcal{P}(E)$, its image by p is the order on $\mathcal{P}(E)/Fin$. Since $\mathcal{P}(E)/Fin$ is a lattice, there are no gaps of cardinality (λ, μ) where either λ or μ is finite. Moreover, by a countable diagonalization argument as first observed by Hadamard [4], there are no gaps of cardinality (ω, ω) either.

To avoid trivialities, let us assume that E is infinite in what follows. Gaps of $\mathcal{P}(E)$ under the above quasiorder correspond under p to gaps in the poset $\mathcal{P}(E)/Fin$, so for notational simplicity all our discussion regarding gaps in $\mathcal{P}(E)$ can be translated in the latter structure if necessary.

We also recall that the usual Hausdorff topology on $\mathcal{P}(E)$ is obtained by identifying each subset of E with its characteristic function and giving the resulting space $\{0,1\}^E$ the product topology. A basis of open sets consists of subsets of the form $O(F, G) := \{X \in \mathcal{P}(E) : F \subseteq X \text{ and } G \cap X = \emptyset\}$, where F, G are finite subsets of E. Endowed with this topology $\mathcal{P}(E)$ is compact and Hausdorff, therefore a Baire space (i.e. any countable union of closed sets with empty interior has empty interior).

Now toward Theorem 1.2, we recall that in a poset P the *initial segment generated by a subset* A of P is $\downarrow A := \{x \in P : x \leq y \text{ for some } y \in P\}$. A subset A is *cofinal* in P if $\downarrow A = P$. The *cofinality* of P, $cf(P)$, is the least cardinality of a cofinal subset. The notions of *final segment generated by a subset* A and of *coinitiality* of P are defined dually and denoted respectively $\uparrow A$ and $ci(P)$. For a singleton $x \in P$, we use the notation $\downarrow x$ instead of $\downarrow\{x\}$. If the reference to P is needed, particularly in case of several orders on the same ground set, we use the notation $\downarrow_P A$ instead of $\downarrow A$.

3. Proof of Theorem 1.1

Consider $E = T_2$ the binary tree of finite sequences of 0 and 1, and $T_2(n)$ those sequences of length at most n. We denote by $()$ the empty sequence and by $s.(i)$ the sequence obtained by adding $i \in \{0,1\}$ to the sequence s. As mentioned above for notational simplicity we will consider the quasiorder $\leq_{Fin}$ on $\mathcal{P}(E)$ as opposed to the poset $\mathcal{P}(E)/Fin$ itself.

For $B \subseteq \mathcal{P}(E)$ set $B^c := \{E \setminus X : X \in B\}$. We will be particularly interested in the set $\mathcal{B}$ of maximal branches of T_2, a closed subset of $\mathcal{P}(E)$ with no isolated points. Notice that for $(A, B) \in \mathcal{P}(\mathcal{B}) \times \mathcal{P}(\mathcal{B})$, (A, B^c) is a pregap if and only if A and B are disjoint.

Proposition 3.1. *Let A and B be disjoint subsets of $\mathcal{B}$. Then (A, B^c) is separable if and only if A and B are covered under inclusion by disjoint F_σ sets.*

Proof. If $X \in A^* \cap B^c_*$ separates (A, B^c), then we can simply let $A' = \bigcup_n \{Y \in \mathcal{P}(E) : Y \setminus X \subseteq T_2(n)\}$ and $B' = \bigcup_n \{Y \in \mathcal{P}(E) : Y \setminus (E \setminus X) \subseteq T_2(n)\}$, two disjoint F_σ sets covering A and B as required.

Conversely let A', B' be disjoint F_σ sets covering A and $B \subseteq \mathcal{B}$. We may assume without loss of generality that $A' = \bigcup A'_n$ and $B' = \bigcup B'_n$ are increasing chains of closed sets in $\mathcal{B}$.
For any fixed n, we claim that there must be an integer k_n such that any $s \in X \cap Y$ has length at most k_n for any $X \in A'_n$ and $Y \in B'_n$. Indeed otherwise for infinitely many k we could find $s_k \in X_k \cap Y_k$ of length at least k for some $X_k \in A'_n$ and $Y_k \in B'_n$. But then we could find a subsequence of $\{s_m : m \in \mathbb{N}\}$ converging to a maximal branch which by closure would be in $A'_n \cap B'_n$, a contradiction.
We can also assume that the produced sequence $\{k_n : n \in \mathbb{N}\}$ is strictly increasing, and we conclude that $X = \bigcup_n \{A_n \cap T_2(k_{n+1}) \setminus T_2(k_n)\} \in A^* \cap B^c_*$, and therefore (A, B^c) is separable. □

By considering A to be the single branch, and $B = \mathcal{B} \setminus A$, one concludes that the above result cannot be strengthened to a covering by disjoint closed sets.

Although the first part of the proof does generalize to any separable pregap in $\mathcal{P}(E)$, it is interesting that the converse is not true as is shown by an example given by Todorcevic [10]. Indeed $A' = \{\{s \in E : s.0 \in b\} : b \in \mathcal{B}\}$ and $B' = \{\{s \in E : s.1 \in b\} : b \in \mathcal{B}\}$ are two disjoint closed sets in $\mathcal{P}(E)$ forming a (Luzin) gap in $\mathcal{P}(E)$.

Since as mentioned above $\mathcal{P}(E)$ is a Baire space, we further have:

Corollary 3.2. *If $A \subseteq \mathcal{B}$ and $B = \mathcal{B} \setminus A$ are both dense then the pair (A, B^c) is a gap in $\mathcal{P}(E)$.*

We finally arrive at the main reason for considering this structure.

Proposition 3.3. *Let A and B be two disjoint subsets of $\mathcal{B}$. If (A, B^c) is a gap, then it does not contain a regular irreducible gap.*

Proof. For $s \in E$ and $D \subseteq \mathcal{P}(E)$, we set $D(s) := \{X \in D : s \in X\}$ and $\check{D} := \{s \in E : |D(s)| = |D|\}$.

Now observe that for an infinite $D \subseteq \mathcal{B}$, the least element of T_2, namely the empty sequence $()$, belongs to $\check{D}$. Moreover if $s \in \check{D}$, then either $s.(0)$ or $s.(1)$ belongs to

$\check{D}$, so we conclude that $\check{D}$ contains a branch and so certainly is infinite. Moreover, if $|D|$ is regular and uncountable, then $\check{D}$ must contain more than a chain and is therefore itself is not a chain.

With this, suppose for contradiction that (A, B^c) contains a regular irreducible gap of size (λ, μ) in $\mathcal{P}(E)$. This means that there is a pair (A', B') such that $A' \subseteq A$, $|A'| = \lambda$, $B' \subseteq B$, $|B'| = \mu$ such that (A', B'^c) is an irreducible gap.

As noticed in a previous remark, λ and μ must be infinite and one of them uncountable. With no loss of generality, we may suppose that this is λ. According to the above observation, $\check{A}'$ is not a chain and $\check{B}'$ is infinite, hence there are $s \in \check{A}'$, $t \in \check{B}'$ which are incomparable with respect to the order on T_2. Let $A'' := A'(s)$ and $B'' := B'(t)$. We have $A'' \subseteq A'$, $|A''| = |A'| = \lambda$, $B'' \subseteq B'$, $|B''| = |B'| = \mu$, and therefore (A'', B''^c) must be a gap by the irreducibility assumption. On the other hand for $Z := \bigcup A''$, we have $X \leq_{Fin} Z \leq_{Fin} Y$ for every $X \in A''$ and $Y \in B''^c$, a contradiction. □

With this in hand, the proof of Theorem 1.1 breaks into two cases.

Case 1. E is denumerable. We deduce Theorem 1.1 as follows. We identify E by T_2, and choose $A \subseteq \mathcal{B}$ and $B := \mathcal{B} \setminus A$ both dense in $\mathcal{B}$. According to Corollary 3.2, (A, B^c) is a gap of $\mathcal{P}(E)$, and according to Proposition 3.3, it does not contain a regular irreducible gap. According to Theorem 2.2, $\mathcal{P}(E)/Fin$ does not have the chain-gap property.

Case 2. E is uncountable. Let E' be a denumerable subset of E. The identity map $1_{E'}$ on E' extends to a map φ from $\mathcal{P}(E')/Fin$ into $\mathcal{P}(E)/Fin$. This map is gap-preserving. Thus, if a gap (A, B) in $\mathcal{P}(E')/Fin$ is not preserved by a chain, its image $(\varphi[A], \varphi[B])$ cannot be preserved by a chain. Since $\mathcal{P}(E')/Fin$ contains such gaps, $\mathcal{P}(E)/Fin$ does, too. Thus, it does not have the chain-gap property. □

4. Proof of Theorem 1.2

The proof naturally breaks into two main parts.

Part 1: $L(\mathbb{S}, \leq)$ does not have the selection property.

It suffices to prove the following.

Proposition 4.1. (1) ω_1 *does not embed into* $L(\mathbb{S}, \leq)$.
(2) $L(\mathbb{S}, \leq)$ *has a minimal gap* $(A, \emptyset)$ *of size* $(\aleph_1, 0)$.

Indeed to see how Theorem 1.2 follows, let $(a_\alpha)_{\alpha<\omega_1}$ be an enumeration of the elements of A. Set $A_\alpha := \{a_\beta : \beta < \alpha\}$. If the selection property holds, then to every pair $(A_\alpha, \emptyset)$ we can associate an element $x_\alpha \in A^*_\alpha \cap \emptyset_* = A^*_\alpha$ such that $(A_\alpha, \emptyset) \leq (A_{\alpha'}, \emptyset)$ implies $x_\alpha \leq x_{\alpha'}$. In particular, for $\alpha \leq \alpha'$ we must have $x_\alpha \leq x_{\alpha'}$. If ω_1 does not embed into $L(\mathbb{S}, \leq)$ then the sequence x_α must be stationary, and in particular has an upper bound. If u is such an upper bound, then $u \in A^*_\alpha$ for every α, thus $u \in A^*$. This is impossible since A is unbounded.

Proof (of Proposition 4.1). We first prove that (2) holds.

Lemma 4.2. *Fix* $r \in \mathbb{S}$ *arbitrary and let* $A := \{\downarrow x : x \in \mathbb{S}$ *and* $x \leq_{\mathbb{R}} r\}$. *Then* $(A, \emptyset)$ *is a minimal gap in* $L(\mathbb{S}, \leq)$ *of size* $(\aleph_1, 0)$.

Proof (of Lemma 4.2). The proof will follow after these two claims.

Claim 4.3. $(\mathbb{S}, \leq)$ *is up directed.*

Proof (of Claim 4.3). Let $x, y \in \mathbb{S}$. The set $X := \{z \in \mathbb{S} : z \leq_{\omega_1} x \text{ or } z \leq_{\omega_1} y\}$ is countable, but on the other hand the set $Y := \{z \in \mathbb{S} : x, y \leq_{\mathbb{R}} z\}$ is uncountable. Thus, $Y \setminus X$ is non empty, and every $z \in Y \setminus X$ majorizes x and y in $(\mathbb{S}, \leq)$, proving our claim. □

Claim 4.4. *A subset B of $L(\mathbb{S}, \leq)$ has an upper bound if and only if $\bigcup B$ has an upper bound in $(\mathbb{S}, \leq)$.*

Proof (of Claim 4.4). If B has an upper bound in $L(\mathbb{S}, \leq)$, then there is some member X of $L(\mathbb{S}, \leq)$ which includes every element of B, hence $\bigcup B \subseteq X$. This set X is a finite union of finite intersections of principal initial segments of $(\mathbb{S}, \leq)$; hence B is a subset of a finite union $\downarrow x_1 \cup \cdots \cup \downarrow x_k$ of principal initial segments of $(\mathbb{S}, \leq)$. Since $(\mathbb{S}, \leq)$ is up-directed by Claim 4.3, these is some x which majorizes $x_1, \ldots, x_k$, and such an x majorizes $\bigcup B$. The converse is trivial and the claim is verified. □

With these claims, the proof of Lemma 4.2 goes as follows. From the fact that $\mathbb{S}$ is $\aleph_1$-dense of size $\aleph_1$, A has size $\aleph_1$. Next, let's see that $(A, \emptyset)$ is a gap.

Now if A was bounded, then Claim 4.4 would imply that $\bigcup A \subseteq \downarrow z$ for some $z \in \mathbb{S}$. In particular, the uncountable initial segment of $\mathbb{S}$ below r under $\leq_{\mathbb{R}}$ from $\mathbb{S}$ would be a subset of the countable initial segment of $\mathbb{S}$ below z under $\leq_{\omega_1}$, a contradiction. A is therefore unbounded in $L(\mathbb{S}, \leq)$ and $(A, \emptyset)$ is a gap.

Finally, to show that $(A, \emptyset)$ is a minimal gap amounts to show that every countable subset A' of A is bounded. Indeed let $\check{A}' := \{x \in \mathbb{S} : \downarrow x \in A'\}$. Then $\check{A}'$ is countable thus is bounded in $\leq_{\omega_1}$. If y is such a bound, then according to Claim 4.3 there is some $z \in \mathbb{S}$ such that $y \leq z$ and $r \leq z$. Then $\downarrow z$ is a bound of A'. □

This concludes the verification of statement (2) of the Proposition, and we now turn our attention to statement (1).

Claim 4.5. *Let $a_1, \ldots, a_n \in (\mathbb{S}, \leq)$ and $A := \downarrow a_1 \cap \ldots \cap \downarrow a_n$. Then there are $i, j \leq n$ such that $A = \downarrow a_i \cap \downarrow a_j$.*

Proof (of Claim 4.5). Let i such that $a_i \leq_{\omega_1} a_k$ for all k, $1 \leq k \leq n$, and let j such that $a_j \leq_{\mathbb{R}} a_k$ for all k, $1 \leq k \leq n$. Then $A = \downarrow a_i \cap \downarrow a_j$. □

From this we immediately get:

Claim 4.6. *Every finite intersection of principal initial segments of $(\mathbb{S}, \leq)$ is of the form $\downarrow x \cap \downarrow y$ with $x \leq_{\mathbb{R}} y$ and $y \leq_{\omega_1} x$.*

Now toward a proof that ω_1 does not embed in $L(\mathbb{S}, \leq)$, let $(A_\alpha)_{\alpha < \omega_1}$ be an ω_1-sequence of elements of $L(\mathbb{S}, \leq)$. According to Claim 4.6, for each $\alpha < \omega_1$, we may write $A_\alpha := \bigcup\{A_{\alpha,i} : i \in I_\alpha\}$ where I_α is a finite set and $A_{\alpha,i} := \downarrow x_{\alpha,i} \cap \downarrow y_{\alpha,i}$ with $x_{\alpha,i} \leq_{\mathbb{R}} y_{\alpha,i}$ and $y_{\alpha,i} \leq_{\omega_1} x_{\alpha,i}$. Set $X_\alpha := \{x_{\alpha,i} : i \in I_\alpha\}$, $Y_\alpha := \{y_{\alpha,i} : i \in I_\alpha\}$ and $Z_\alpha := X_\alpha \cup Y_\alpha$.

Claim 4.7. *If $(A_\alpha)_{\alpha < \omega_1}$ is strictly increasing then the sets Z_α's cannot be pairwise disjoint.*

Proof (of Claim 4.7). Let $\alpha < \omega_1$. Set $x_\alpha = \max_{\mathbb{R}}(X_\alpha)$, $\downarrow_{\mathbb{R}} x_\alpha = \{z \in \mathbb{S} : z \leq_{\mathbb{R}} x_\alpha\}$, $\downarrow_{\mathbb{R}} A_\alpha := \{z \in \mathbb{S} : z \leq_{\mathbb{R}} x \text{ for some } x \in A_\alpha\}$. Since $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ we have $\downarrow_{\mathbb{R}} A_\alpha \subseteq \downarrow_{\mathbb{R}} A_\beta$. Since further ω_1 does not embed into the chain $\mathbb{S}$, it does not embed

into the chain of initial segments of $(\mathbb{S}, \leq_{\mathbb{R}})$, hence the ω_1-sequence $(\downarrow_{\mathbb{R}} A_\alpha)_{\alpha<\omega_1}$ is eventually constant. Let α_0 such that $\downarrow_{\mathbb{R}} A_\alpha = \downarrow_{\mathbb{R}} A_{\alpha_0}$ for $\alpha_0 \leq \alpha < \omega_1$ and define $A := \downarrow_{\mathbb{R}} A_{\alpha_0}$. Since $A_\alpha \subseteq \downarrow X_\alpha$, we have $\downarrow_{\mathbb{R}} A_\alpha \subseteq \downarrow_{\mathbb{R}} x_\alpha$.
Suppose now that that the Z_α's are pairwise disjoint. Then in particular all the X_α's are pairwise disjoint, and therefore there is at most one α such that $A = \downarrow_{\mathbb{R}} x_\alpha$ and we may without loss of generality assume that $x_\alpha \in \mathbb{S} \setminus A$ for all $\alpha \geq \alpha_0$. Since ω_1^* does not embed into $\mathbb{S}$, there is some $x \in \mathbb{S} \setminus A$ for which $X_x := \{\alpha : \alpha_0 < \alpha < \omega_1 \text{ and } x <_{\mathbb{R}} x_\alpha\}$ is uncountable. Since on the other hand the Y_α are also assumed to be pairwise disjoints, all $y_{\alpha,i}$'s are therefore distincts, and since $\{z \in \mathbb{S} : z \leq_{\omega_1} x\}$ is countable, there is some $\alpha \in X_x$ such that $x \leq_{\omega_1} x_\alpha$ and $x \leq_{\omega_1} y_{\alpha,i}$ for all $i \in I_\alpha$. But now for such an α $x <_{\mathbb{R}} x_\alpha \leq_{\mathbb{R}} y_{\alpha,i}$ where i is such that $x_\alpha = x_{\alpha,i}$, then this implies that $x \in A_\alpha$. Since $A_\alpha \subseteq \downarrow_{\mathbb{R}} A_\alpha = A$, we get $x \in A$ contradicting our definition of A. □

Claim 4.8. *If there is a strictly increasing ω_1-sequence of elements of $L(\mathbb{S}, \leq)$ then there an $\aleph_1$-dense subchain $\mathbb{S}'$ of $\mathbb{R}$ and a strictly increasing ω_1-sequence of elements of $L(\mathbb{S}', \leq')$ for which all Z_α's are pairwise disjoint.*

Proof (of Claim 4.8). Start with a strictly increasing ω_1-sequence $(A_\alpha)_{\alpha<\omega_1}$ of members of $L(\mathbb{S}, \leq)$. Since the Z_α's are finite, there is an uncountable subset U of ω_1 and a finite subset F of $\mathbb{S}$ such that for all $\alpha, \beta \in U$, F is an initial segment of Z_α w.r.t. $(\mathbb{S}, \leq_{\omega_1})$ and $Z_\alpha \cap Z_\beta = F$. That is $(Z_\alpha)_{\alpha \in U}$ forms an uncountable Δ-system.

Let $x \in \mathbb{S}$ such that $F \subseteq \downarrow_{\omega_1} x$ and write $X = \downarrow_{\omega_1} x$. Set $\mathbb{S}' := \mathbb{S} \setminus X$, $A'_\alpha := A_\alpha \setminus X$, $A'_{\alpha,i} := A_{\alpha,i} \setminus X$, $I'_\alpha := \{i \in I_\alpha : A'_{\alpha,i} \neq \emptyset\}$.

Since X is countable, $\mathbb{S}'$ is again an $\aleph_1$ dense chain with no end-points and the well-ordering induced has order type ω_1. The intersection order $\leq'$ is the order induced by $\leq$ on $\mathbb{S}'$.

The ω_1-sequence $(A'_\alpha)_{\alpha<\omega_1}$ is increasing and since X is countable, it contains a strictly increasing subsequence $(A'_\alpha)_{\alpha \in U'}$, for some uncountable subset U' of U. Let $\alpha \in U' \setminus \min(U')$. Then $A'_\alpha \neq \emptyset$, hence $A' = \cup\{A'_{\alpha,i} : i \in I'_\alpha\}$. Since X is an initial segment of $(\mathbb{S}, \leq)$ it follows that $A'_{\alpha,i} = \downarrow_{(\mathbb{S}',\leq')} x_{\alpha,i} \cap \downarrow_{(\mathbb{S}',\leq')} y_{\alpha,i}$. Hence $X'_\alpha = \{x_{\alpha,i} : i \in I'_\alpha\}$, $Y'_\alpha = \{y_{\alpha,i} : i \in I_\alpha\}$, and $Z'_\alpha := X'_\alpha \cup Y'_\alpha$. Thus the Z'_α for $\alpha \in U \setminus \min(U)$ are pairwise disjoint. □

From Claim 4.7 and Claim 4.8, there is no strictly increasing ω_1-sequence of elements of $L(\mathbb{S}, \leq)$. The proof of Proposition 4.1 is complete. □

Part 2: $L(\mathbb{S}, \leq)$ has the chain-gap property.

Let (A, B) be a gap in $L(\mathbb{S}, \leq)$.

Lemma 4.9. *There is a partition of $L(\mathbb{S}, \leq)$ into a prime ideal I and a prime filter F such that $(A, \emptyset)$ is a gap of I and $(\emptyset, B)$ is a gap of F*

Proof. This just follows from the fact that $L(\mathbb{S}, \leq)$ is a distributive lattice (see Pouzet, Rival [7]). □

Lemma 4.10. *The gap $(\emptyset, B)$ of F can be separated by a chain.*

Proof. According to Pouzet-Rival [7], it suffices to show that the coinitiality of F is countable.

Let $K := \{x \in \mathbb{S} : \downarrow x \in F\}$. As a a subset of $\mathbb{R}$, K has a countable coinitiality, so we can select a countable subset D coinitial in K w.r.t the order $\leq_{\mathbb{R}}$. Let $U := \{x \in$

$K : x \leq_{\omega_1} y$ for some $y \in D\}$. Since D is countable, U is countable. Moreover, U is coinitial in K. Indeed, let $x \in K$, then there is $x_1 \in D$ such that $x_1 \leq_{\mathbb{R}} x$. Now, either $x_1 \leq_{\omega_1} x$, in which case $x_1 \leq x$, or $x <_{\omega_1} x_1$ but then by definition of U, $x \in U$. So in both cases, x majorizes an element of U. Let $\check{U} = \{\downarrow x \cap \downarrow y : x \in U, y \in U\}$. Since U is countable, this set is countable. Moreover it is coinitial in F. Indeed, let $a \in F$, then a is of the form $a = a_1 \cup \ldots \cup a_n$ where $a_i := \downarrow x_i \cap \downarrow y_i$. Since F is a prime filter, some $a_i \in F$ and since F is a filter, x_i, y_i belong to K. Now U being coinitial in K it follows that there are $x'_i, y'_i \in U$ such that $x'_i \leq x_i$ and $y'_i \leq y_i$ hence $\downarrow x'_i \cap \downarrow y' \subseteq a_i \subseteq a$ proving that $\check{U}$ is coinitial in F. □

Lemma 4.11. *The gap $(A, \emptyset)$ in I can be separated by a chain.*

Proof. Elements of I are of the form $a = a_1 \cup a_2 \ldots \cup a_n$ where $a_i = \downarrow x_i \cap \downarrow y_i$. Since I is a prime ideal, for every a_i one of the sets $\downarrow x_i, \downarrow y_i$ belongs to I. Consequently the set of finite unions of members of I of the form $\downarrow x$ is cofinal in I. For a subset X of I let $\check{X} := \{x \in \mathbb{S} : x \in u \text{ for some } u \in X\}$, in other words $\check{X} = \bigcup X$.

Claim 4.12. *Let A be a subset of I. Then $(A, \emptyset)$ is a gap in I iff $\check{A}$ is not contained into a finitely generated initial segment of $\check{I}$.*

Proof. If $(A, \emptyset)$ is not a gap in I then from the above observation there are $x_1, \ldots x_k \in \mathbb{S}$ such that for $a := \downarrow x_1 \cup \ldots \cup \downarrow x_k$ we have $a \in I$ and a dominates A. This implies $\check{A} \subseteq a$, and proves our claim. The converse is obvious. □

Claim 4.13. *Let A be a subset of I, then $(A, \emptyset)$ is a gap in I iff $(\bar{A}, \emptyset)$, where $\bar{A} := \{\downarrow x$ such that $\downarrow x \subseteq a$, for some $a \in A\}$, is a gap.*

Proof. Observe that $\check{A} = \check{\bar{A}}$ and apply Claim 4.12. □

Let $(A, \emptyset)$ be a gap in I. We may suppose that $(A, \emptyset)$ is minimal. Since every elements of A majorize some element of $\bar{A}$, in order to separate $(A, \emptyset)$ by a chain it is enough to separate $\bar{A}, \emptyset)$. Let $\leq_*$ one of the two orderings $\leq_{\omega_1}, \leq_{\mathbb{R}}$ restricted to $\check{I}$ and let $\check{I}_* := (\check{I}, \leq_*)$. We consider two cases:

(1) $\check{A}$ is an unbounded subset of $\check{I}_*$ for some $\leq_*$.
(2) $\check{A}$ is a bounded subset of $\check{I}_*$ for the two possible orderings $\leq_*$.

Case (1). The ideal $\check{I}$ is unbounded, hence $\check{I}_*$ is unbounded too. Let $(c_\alpha)_{\alpha<\mu}$ ($\mu = \omega_1$ or ω) be an increasing cofinal sequence of elements of $\check{I}_*$. For $\alpha < \mu$, let $I_\alpha := \{u \in I$ such that $u \subseteq \downarrow_{\check{I}_*} c_\alpha\}$. Clearly, the sequence $(I_\alpha)_{\alpha<\mu}$ is increasing. Next $I = \bigcup_{\alpha<\mu} I_\alpha$. Indeed, let $u \in I$. There are $x_1, \ldots, x_k \in \check{I}$ such that $u \subseteq \downarrow x_1 \cup \ldots \cup \downarrow x_k$. Hence, there is some c_α such that $x_1, \ldots, x_k \leq_* c_\alpha$. Since for each $i, 1 \leq i \leq k$, $\downarrow x_i \subseteq \downarrow_{\check{I}_*} x_i \subseteq \downarrow_{\check{I}_*} c_\alpha$ we have $u \subseteq \downarrow_{\check{I}_*} c_\alpha$ thus $u \in I_\alpha$. Finally, for every α, $A \setminus I_\alpha$ is non empty. Indeed, since $\check{A}$ is unbounded in $\check{I}_*$, there is some $x \in \check{A}$ such that $c_{\alpha+1} \leq_* x$. By definition of $\check{A}$, $x \in a$ for some $a \in A$. But then $a \notin I_\alpha$. Pick $a_\alpha \in A \setminus I$ for each $\alpha < \mu$. Let $A' := \{a_\alpha : \alpha < \mu\}$. Since $(A, \emptyset)$ is a minimal gap and μ is regular, $(A', \emptyset)$ is a regular irreducible gap.

Case (2). Note that in this case $\check{I}$ is not an ideal of $(\mathbb{S}, \leq)$ (otherwise $\check{A}$ would be bounded in $\check{I}$ and $(A, \emptyset)$ would no be a gap in I).

Since $\check{A}$ is a bounded subset of $\check{I}$ w.r.t. ω_1, it is countable. Hence, there is a least element b of $(\mathbb{S}, \leq_{\omega_1})$ for which $\check{A} \cap \downarrow_{(\mathbb{S}, \leq_{\omega_1})} b$ contains a subset B which is not contained into a finitely generated initial segment of $\check{I}$. Let $\widetilde{B} := \{\downarrow x : x \in B\}$. The pair $(\widetilde{B},\emptyset)$ is a gap and in fact a subgap of $(A,\emptyset)$ too. Hence it suffices to show that $(\widetilde{B},\emptyset)$ is preserved by a chain.

Let $G := \{z \in \check{I} : B \subseteq\downarrow_{(\mathbb{S},\leq_{\omega_1})} z\}$. Clearly b is a lower-bound of G w.r.t. $\leq_{\omega_1}$. Let $B_1 := B\cap \downarrow_{\mathbb{S}} G$ and let $B_2 := B \setminus B_1$. Since B is not contained into a finitely generated initial segment of $\check{I}$ there is some $i \in \{1,2\}$ such that B_i has the same property.

Subcase 1. $i = 2$. In this case, due to the choice of b, B_2 is cofinal into $\downarrow_{(\mathbb{S},\leq_{\omega_1})} b$ thus into B. Let B' be a cofinal subset of B_2 w.r.t. $\leq_{\omega_1}$ having order type ω. We claim that no countable subset of B' can be contained into a finitely generated initial segment of $\check{I}$. Indeed, if there is one, then there is one, say B'', which is contained into some set of the form $\downarrow z$, with $z \in \check{I}$. But, since w.r.t. the order $\leq_{\omega_1}$, B'' is cofinal into B', B' is cofinal into B_2 and B_2 is cofinal into B, B'' is cofinal into B, hence from $B'' \subseteq\downarrow_{(\mathbb{S},\omega_1)} z$ we get $z \in G$. With the fact that $B'' \subseteq\downarrow_{(\mathbb{S},\leq_{\mathbb{R}})} z$ this implies $B'' \subseteq B_1$, contradiction. Let $\widetilde{B'} := \{\downarrow x : x \in B'\}$. The property above says that $(\widetilde{B'},\emptyset)$ is a gap and every countable subset too. This gap is regular and irreducible; since $(\widetilde{B},\emptyset)$ contains this gap, it can be preserved by a chain.

Subcase 2. Subcase 1 does not hold. Hence $i = 1$. Again, due to the choice of b, B_1 is cofinal into $\downarrow_{(\mathbb{S},\leq_{\omega_1})} b$ thus into B. Select a cofinal sequence into B_1 with type ω, say $x_0 <_{\omega_1} x_1 <_{\omega_1} \ldots <_{\omega_1} x_n <_{\omega_1} \ldots$. Observes that G has no largest element w.r.t. the order $\leq_{\mathbb{R}}$ (otherwise, if u is the largest element, then we have both $B_1 \subseteq\downarrow_{(\mathbb{S},\leq_{\omega_1})} u$ and $B_1 \subseteq\downarrow_{(\mathbb{S},\leq_{\mathbb{R}})} u$, thus $B_1 \subseteq\downarrow u$, contradicting the unboundedness of B_1). Hence, the cofinality of G w.r.t. $\leq_{\mathbb{R}}$ is denumerable and we may select $u_0 <_{\mathbb{R}} u_1 <_{\mathbb{R}} \ldots <_{\mathbb{R}} u_n \ldots$ into G forming a cofinal sequence w.r.t. the order $\leq_{\mathbb{R}}$.

Claim 4.14. *There is a sequence $y_0 < y_1 < \ldots < y_n < \ldots$ of elements of B_1 such that $D := \{y_n : n < \omega\}$ is cofinal in $(B_1, \leq_{\omega_1})$ and in $(G, \leq_{\mathbb{R}})$.*

Proof. First, we define y_0. Since $x_0 <_{\omega_1} b$, $B_1 \bigcap \downarrow_{(\mathbb{S},\leq_{\omega_1})} x_0$ is contained into a finitely generated initial segment of $\check{I}$. Hence $B_1 \bigcap \uparrow_{(\mathbb{S},\leq_{\omega_1})} x_0$ is not contained into a finitely generated initial segment of $\check{I}$. In particular,

$$B_1 \bigcap \uparrow_{(\mathbb{S},\leq_{\omega_1})} x_0 \not\subseteq\downarrow u_0 \tag{1}$$

Since $u_0 \in G$ we have $B \subseteq\downarrow_{(\mathbb{S},\leq_{\omega_1})} u_0$. From (1) we get

$$B_1 \bigcap \uparrow_{(\mathbb{S},\leq_{\omega_1})} x_0 \not\subseteq\downarrow_{\leq_{\mathbb{R}}} u_0 \tag{2}$$

From (2) there is some $y_0 \in B_1$ such that $y_0 \geq_{\omega_1} x_0$ and $y_0 \geq_{\mathbb{R}} u_0$.

Suppose $y_0 < y_1 \ldots < y_n$ be defined with $x_i \leq_{\omega_1} y_i, u_i \leq_{\mathbb{R}} y_i$. In order to define y_{n+1} select x_{n_1} and u_{n_1} such that:

$$y_n \leq_{\omega_1} x_{n_1}, x_{n+1} <_{\omega_1} x_{n_1} \text{ and } y_n \leq_{\mathbb{R}} u_{n_1}, u_{n+1} <_{\mathbb{R}} u_{n_1}$$

As above, since $x_{n_1} <_{\omega_1} b$, $B_1 \bigcap \uparrow_{(\mathbb{S},\leq_{\omega_1})} x_{n_1}$ is not contained into a finitely generated initial segment of $\check{I}$ so $B_1 \bigcap \uparrow_{(\mathbb{S},\leq_{\omega_1})} x_{n_1} \not\subseteq\downarrow_{\leq_{\mathbb{R}}} u_0$ and thus there is an element, say y_{n+1} such that $x_{n_1} \leq_{\omega_1} y_{n+1}$ and $u_{n_1} \leq_{\mathbb{R}} y_{n+1}$. Clearly, $y_n < y_{n+1}$, $x_{n+1} \leq_{\omega_1} y_{n+1}$ and $u_{n+1} \leq_{\mathbb{R}} y_{n+1}$. From our construction, D is cofinal in $(B_1, \leq_{\omega_1})$ and in $(G, \leq_{\mathbb{R}})$. □

Since D is cofinal in in $(B_1, \leq_{\omega_1})$ and in $(G, \leq_{\mathbb{R}})$, D is unbounded in $\check{I}$. But since $\widetilde{D}$ is a chain, it is unbounded in I, hence $(\widetilde{D}, \emptyset)$ is a regular irreducible gap in I. Since $(\widetilde{B}, \emptyset)$ contains this gap, it can be preserved by a chain.

With this, the proof of Lemma 4.11 is complete. □

Problem 4.15. *Let κ be such that $\omega < \kappa \leq 2^{\aleph_0}$, $\mathbb{S}$ be a κ-dense subchain of $\mathbb{R}$ of size κ and $L(\mathbb{S}, \leq)$ be the distributive lattice associated with a Sierpinskization of $\mathbb{S}$. Does $L(\mathbb{S}, \leq)$ have the chain-gap property?*

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