Math 346, HW9 Solution

5.5.3

In the linear program:

$$\begin{array}{ll} \text{Maximize} & 11x_1 + 4x_2 + x_3 + 15x_4 \\ \text{subject to} & 3x_1 + x_2 + 2x_3 + 4x_4 \leq 28 \\ & 8x_1 + 2x_2 - x_3 + 7x_4 \leq 50 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

By simplex method, the final tableau looks like (from Page 184):

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	-2	0	5	1	2	-1	6
x_2	11	1	-18	0	-7	4	4
	3	0	2	0	2	1	106

Note that by taking the fourth column and second column in the original LP, the matrix

$$B = \begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}.$$

We can compute its inverse, which gives:

$$B^{-1} = \begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}.$$

Suppose we change 28 to $28+\lambda$, note that this only changes $\vec{b}$, the new $B^{-1}\vec{b}$ is equal to

$$\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} \cdot \begin{bmatrix} 28+\lambda \\ 50 \end{bmatrix} = \begin{bmatrix} 6+2\lambda \\ 4-7\lambda \end{bmatrix}.$$

If $B^{-1}\vec{b} \geq \vec{0}$, then this remains the optimal solution, and the basis remains to be $\{x_4, x_2\}$. Solving the inequality gives $-3 \leq \lambda \leq 4/7$. This optimal solution gives a maximum that is equal to

$$11x_1 + 4x_3 + x_3 + 15x_4 = 4(4-7\lambda) + 15(6+2\lambda) = 2\lambda + 106.$$

5.6.1

In the step 2 of the dual simplex algorithm, suppose there exists r such that $b_r < 0$, and $a_{rj} \geq 0$ for all j, then if we consider the r-th constraint, it looks like:

$$a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = b_r.$$

However, if all $a_{rj} \geq 0$, then the left hand side of this equality is nonnegative for a feasible solution (since all x_i's need to be nonnnegative in a feasible solution), while the right hand side is equal to b_r which is strictly negative, contradiction. Therefore this system of linear constraints has no feasible solution.

6.2.5

Figure 6.3:

$$\begin{aligned} \text{subject to} \quad & 5x_1 + 4x_2 - 20 \leq M_1(1 - y_1) \\ & 3x_1 + 8x_2 - 24 \leq M_2(1 - y_2) \\ & y_1 + y_2 \geq 1 \\ & x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\} \end{aligned}$$

Now let's determine the value of M_1, M_2. Note that for any solution (x_1, x_2) in the feasible region, one always have $0 \leq x_1 \leq 8$, and $0 \leq x_2 \leq 5$. Therefore

$$5x_1 + 4x_2 - 20 \leq 5 \times 8 + 4 \times 5 - 20 = 40.$$

One can choose M_1 to be any real number greater or equal to 40.

For M_2, note that

$$3x_1 + 8x_2 - 24 \leq 3 \times 8 + 8 \times 5 - 24 = 40.$$

So M_2 can be chosen as any real number at least 40.

Figure 6.4:

$$\begin{aligned} \text{subject to} \quad & x_1 + x_2 - 1 \leq M_1(1 - y_1) \\ & 3x_1 + x_2 - 3 \geq M_2(1 - y_2) \\ & 3x_1 + 4x_2 - 12 \leq M_3(1 - y_2) \\ & y_1 + y_2 \geq 1 \\ & x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\} \end{aligned}$$

Note that any feasible solution (x_1, x_2) satisfy $0 \leq x_1 \leq 4$, and $0 \leq x_2 \leq 3$. Therefore

$$x_1 + x_2 - 1 \leq 4 + 3 - 1 = 6$$

$$3x_1 + x_2 - 3 \geq 3 \times 0 + 0 - 3 = -3$$

$$3x_1 + 4x_2 - 12 \leq 3 \times 4 + 4 \times 3 - 12 = 12$$

Therefore one can choose M_1 to be any real at least 6, M_2 to be any real at most -3, M_3 to be any real at least 12. Note that the constraint $y_1 + y_2 \geq 1$ can also be changed to $y_1 + y_2 = 1$, since the two shaded triangles only intersect trivially.

Figure 6.5:

$$\begin{array}{ll} \text{subject to} & 2x_1 + 5x_2 - 10 \leq M_1(1 - y_1) \\ & x_1 - 3 \leq M_2(1 - y_2) \\ & x_1 - x_2 \geq M_3(1 - y_2) \\ & y_1 + y_2 \geq 1 \\ & x_1, x_2 \geq 0; y_1, y_2 \in \{0, 1\} \end{array}$$

Note that any feasible solution (x_1, x_2) satisfy $0 \leq x_1 \leq 5$, and $0 \leq x_2 \leq 3$. Therefore

$$2x_1 + 5x_2 - 10 \leq 2 \times 5 + 5 \times 3 - 10 = 15$$

$$x_1 - 3 \leq 5 - 3 = 2$$

$$x_1 - x_2 \geq 0 - 3 = -3.$$

So we can choose M_1 to be any real at least 15, M_2 be any real at least 2, and M_3 be any real at most -3.

6.2.16

We let $y_i = 1$ if the i-th constraint is satisfied, and let $y_i = 0$ if the i-th constraint is not satisfied. Now we can write down the integer programming (we will pick the constant M_i's later):

$$\begin{array}{ll} \text{Maximize} & 9x_1 + 8x_2 + 7x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 500 \\ & 3x_1 - 3x_2 + 4x_3 - 1000 \leq M_1(1 - y_1) \\ & x_1 - 2x_3 - 200 \geq -M_2(1 - y_2) \\ & x_1 + x_2 - 300 \leq M_3(1 - y_3) \\ & x_1 + x_2 - 300 \geq -M_4(1 - y_3) \\ & y_1 + y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \leq 1 \\ & x_1, x_2, x_3, y_1, y_2, y_3 \geq 0; y_1, y_2, y_3 \text{ integral.} \end{array}$$

Next we will decide the value for M_i's. For example, we know that for all x_i, they lie in the interval $[0, 500]$ from the first inequality. Therefore,

$$3x_1 - 3x_2 + 4x_3 - 1000 \leq 3 \times 500 - 3 \times 0 + 4 \times 500 - 1000 = 2500,$$

which means that if we pick $M_1 = 2501$, then the first inequality is automatically satisfied in the case $y_1 = 0$ (we want no extra constraint in this case). Similarly since $x_1 - 2x_3 - 200 \geq 0 - 2 \times 500 - 200 = -1700$, we can pick $M_2 = 1800$, and determine the value for M_3 and M_4 as well.

6.4.1(a)

We would like to solve the following integer program using the Branch-and-Bound algorithm:

$$
\begin{array}{lll}
(IP1) & \text{Maximize} & z = 5x_1 + 2x_2 \\
 & \text{subject to} & 6x_1 + 2x_2 \leq 13 \\
 & & -6x_1 + 7x_2 \leq 14 \\
 & & x_1, x_2 \geq 0 \text{ and integral.}
\end{array}
$$

Using simplex method, we know that the corresponding LP has an optimal solution $(7/6, 3)$ that gives $z = 71/6$. Now we consider two new integer programs by taking $x_1 \leq 1$, and $x_1 \geq 2$, respectively. For the first IP:

$$
(IP2a) \quad
\begin{array}{ll}
\text{Maximize} & z = 5x_1 + 2x_2 \\
\text{subject to} & 6x_1 + 2x_2 \leq 13 \\
 & -6x_1 + 7x_2 \leq 14 \\
 & x_1 \leq 1 \\
 & x_1, x_2 \geq 0 \text{ and integral.}
\end{array}
$$

The corresponding LP has an optimal solution $(1, 20/7)$ that gives $z = 75/7$. The second IP:

$$
(IP2b) \quad
\begin{array}{ll}
\text{Maximize} & z = 5x_1 + 2x_2 \\
\text{subject to} & 6x_1 + 2x_2 \leq 13 \\
 & -6x_1 + 7x_2 \leq 14 \\
 & x_1 \geq 2 \\
 & x_1, x_2 \geq 0 \text{ and integral.}
\end{array}
$$

corresponds to a LP which has an optimal solution $(2, 1/2)$ that gives $z = 11$.

Now we continue to branch from these two IPs, IP2a has two branches using $x_2 \leq 2$ and $x_2 \geq 3$ respectively:

$$
(IP3a) \quad
\begin{array}{ll}
\text{Maximize} & z = 5x_1 + 2x_2 \\
\text{subject to} & 6x_1 + 2x_2 \leq 13 \\
 & -6x_1 + 7x_2 \leq 14 \\
 & x_1 \leq 1 \\
 & x_2 \leq 2 \\
 & x_1, x_2 \geq 0 \text{ and integral.}
\end{array}
$$

Its corresponding LP has an optimal solution $(1, 2)$ that gives $z = 9$, for this branch the algorithm stops here since we already arrive at an integral optimal solution.

The second branch of IP2a is:

$$
(IP3b) \quad
\begin{array}{ll}
\text{Maximize} & z = 5x_1 + 2x_2 \\
\text{subject to} & 6x_1 + 2x_2 \leq 13 \\
 & -6x_1 + 7x_2 \leq 14 \\
 & x_1 \leq 1 \\
 & x_2 \geq 3 \\
 & x_1, x_2 \geq 0 \text{ and integral.}
\end{array}
$$

Its corresponding LP is infeasible.

Now for IP2b, we also have two branches according to $x_2 \le 0$ (meaning that $x_2 = 0$), or $x_2 \ge 1$:

$$(IP3c) \quad \begin{array}{ll} \text{Maximize} & z = 5x_1 + 2x_2 \\ \text{subject to} & 6x_1 + 2x_2 \le 13 \\ & -6x_1 + 7x_2 \le 14 \\ & x_1 \ge 2 \\ & x_2 = 0 \\ & x_1, x_2 \ge 0 \text{ and integral.} \end{array}$$

The corresponding LP has an optimal solution $(13/6, 0)$ that gives $65/6$. The second branch:

$$(IP3d) \quad \begin{array}{ll} \text{Maximize} & z = 5x_1 + 2x_2 \\ \text{subject to} & 6x_1 + 2x_2 \le 13 \\ & -6x_1 + 7x_2 \le 14 \\ & x_1 \ge 2 \\ & x_2 \ge 1 \\ & x_1, x_2 \ge 0 \text{ and integral.} \end{array}$$

corresponds to a LP that is infeasible.

Now we start the branching process from IP3c, by setting $x_1 \le 2$ (since already $x_1 \ge 2$, we have $x_1 = 2$) and $x_1 \ge 3$:

$$(IP4a) \quad \begin{array}{ll} \text{Maximize} & z = 5x_1 + 2x_2 \\ \text{subject to} & 6x_1 + 2x_2 \le 13 \\ & -6x_1 + 7x_2 \le 14 \\ & x_1 = 2 \\ & x_2 = 0 \\ & x_1, x_2 \ge 0 \text{ and integral.} \end{array}$$

The LP has a unique optimal solution $(2, 0)$ that gives $z = 10$ (which also updates the current best value 9). For

$$(IP4b) \quad \begin{array}{ll} \text{Maximize} & z = 5x_1 + 2x_2 \\ \text{subject to} & 6x_1 + 2x_2 \le 13 \\ & -6x_1 + 7x_2 \le 14 \\ & x_1 \ge 3 \\ & x_2 = 0 \\ & x_1, x_2 \ge 0 \text{ and integral.} \end{array}$$

Its corresponding LP is infeasible. Therefore we finish the branch-and-bound algorithm and conclude that the optimal solution is $(x_1, x_2) = (2, 0)$, which gives optimum value $z = 10$ (same with what graphical method gives in problem 6.1.1(b)).