Math 346, HW8 Solution

5.1.2

(a,b) We need to verify that for $\lambda \geq 0$ up to certain upper limit, the linear program:

$$\begin{array}{ll} \text{Minimize} & 16x_1 + 14x_2 \\ \text{subject to} & 10x_1 + 4x_2 \geq 124 + 2\lambda \\ & 3x_1 + 5x_2 \geq 60 - \lambda \\ & x_1, x_2 \geq 0. \end{array}$$

has the same optimal solution with the LP when $\lambda = 0$.

From the textbook, we know that $(x_1, x_2) = (10, 6)$ is the optimal solution with optimum value 244, therefore the final simplex tableau looks like this (x_3, x_4 are the slack variables, so B is formed by taking the two coluns $(10, 3)$ and $(4, 5)$, for which we can compute B^{-1}):

$$\left[\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & \\ \hline x_1 & 1 & 0 & 5/38 & -2/19 & 10 \\ x_2 & 0 & 1 & -3/38 & 5/19 & 6 \\ \hline & 0 & 0 & c_3 & c_4 & -244 \end{array}\right]$$

When we change $b = (124, 60)$ to $(124 + 2\lambda, 60 - \lambda)$, the $B^{-1}b$ is equal to

$$\begin{bmatrix} 10 \\ 6 \end{bmatrix} + \begin{bmatrix} 5/38 & -2/19 \\ -3/38 & 5/19 \end{bmatrix} \begin{bmatrix} 2\lambda \\ -\lambda \end{bmatrix} = \begin{bmatrix} 10 + 7/19 \cdot \lambda \\ 6 - 8/19 \cdot \lambda \end{bmatrix}$$

As long as it is nonnegative, the daily minimum cost is equal to $16(10 + 7/19 \cdot \lambda) + 14(6 - 8/19 \cdot \lambda) = 244$ and remains unchanged, solving $6 - 8/19 \cdot \lambda \geq 0$ gives $\lambda \leq 57/4$.

5.1.3

We consider the linear program (suppose the daily requirement for nutritional element A increases by λ_1, that of B increases by λ_2)

$$\begin{array}{ll} \text{Minimize} & 16x_1 + 14x_2 \\ \text{subject to} & 10x_1 + 4x_2 \geq 124 + \lambda_1 \\ & 3x_1 + 5x_2 \geq 60 + \lambda_2 \\ & x_1, x_2 \geq 0. \end{array}$$

Similarly as before, $B^{-1}b$ is equal to $(10 + 5/38 \cdot \lambda_1 - 2/19 \cdot \lambda_2, 6 - 3/38 \cdot \lambda_1 + 5/19 \cdot \lambda_2)$. The objective function is equal to

$$16(10 + 5/38 \cdot \lambda_1 - 2/19 \cdot \lambda_2) + 14(6 - 3/38 \cdot \lambda_1 + 5/19 \cdot \lambda_2) = 244 + \lambda_1 + 2\lambda_2.$$

If $\lambda_2 > 0$ (we increase the requirement for element B), then the cost increases by $2\lambda_1$. So for each 10-unit increase, the cost increases by 20 cents which is more than the increase of the value (15 cents), so this won't do any good. But if $\lambda_1 > 0$, then the cost increases by only 10 cents while the value of the stock increases by 15 cents.

To determine the upper limit for increasing λ_1, we set $\lambda_2 = 0$ and let $B^{-1}b \geq 0$, we have $10 + 5/38 \cdot \lambda_1 \geq 0$, and $6 - 3/38 \cdot \lambda_1 \geq 0$, solving them gives $\lambda_1 \leq 76$. So this works for up to 76 units of increment for the element A. Another way is to solve the following LP (the objective function is cost minus increase in value)

$$\begin{array}{ll} \text{Minimize} & 16x_1 + 14x_2 - 1.5\lambda_1 \\ \text{subject to} & 10x_1 + 4x_2 \geq 124 + \lambda_1 \\ & 3x_1 + 5x_2 \geq 60 \\ & x_1, x_2, \lambda_1 \geq 0. \end{array}$$

The solution is $(x_1, x_2, \lambda_1) = (20, 0, 76)$, which means that the producer can increase the requirement for element A by at most 76, and during this procedure the value of stock minus the cost increases.

5.2.2

(a) For the LP with the artificial variables (3.6.5), the basic variables in the final tableau are x_1 and x_3, and

$$B = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix},$$

(b) In the final tableau for LP without artificial variable, the basic variables are x_1 and x_2, thus

$$B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix},$$

and

$$B^{-1} = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix},$$

(c) It suggests two things, first if one changes the basis to $\{x_1, x_2\}$ in the LP with artificial variables, the last two columns (of artificial variables) would become the B^{-1} in (b). Moreover, it suggests that the B^{-1} could be different for LP's with or without artificial variables. When doing the sensitivity analysis one should calculate B^{-1} on the LP without aritifical variables.

5.3.6

The final tableau for the dual LP is:

$$\left[\begin{array}{c|ccccc|c} & y_1 & y_2 & y_3 & y_4 & y_5 & \\ \hline y_2 & 0 & 1 & -3/2 & 1/4 & -3/8 & 1 \\ y_1 & 1 & 0 & 9/2 & -1/4 & 7/8 & 1 \\ \hline & 0 & 0 & 72 & 6 & 21 & 144 \end{array}\right]$$

From the tableau, we know that the optimal solution is $(y_1, y_2, y_3) = (1, 1, 0)$, with y_1, y_2 being the basic variable. Now we would like to fix the nutritional requirement for B and C, and check for which range of nutritional requirement for A, the optimal solution of the dual remains the same. In other words, we change the vector b in the dual:

$$\begin{array}{ll} \text{Maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0, \end{array}$$

and we hope to keep the optimal solution. Note that after we change 60 to λ, the last row of the tableau becomes

$$-[\lambda, 84, 72, 0, 0] + [84, \lambda] \begin{bmatrix} 0 & 1 & -3/2 & 1/4 & -3/8 \\ 1 & 0 & 9/2 & -1/4 & 7/8 \end{bmatrix} = [0, 0, 9\lambda/2 - 198, 21 - \lambda/4, 7\lambda/8 - 63/2]$$

This vector is nonnegative when $44 \leq \lambda \leq 84$.

Now suppose we fix requirements for A and C and change requirement for B from 84 to λ, then we have the last row being:

$$-[60, \lambda, 72, 0, 0] + [\lambda, 60] \begin{bmatrix} 0 & 1 & -3/2 & 1/4 & -3/8 \\ 1 & 0 & 9/2 & -1/4 & 7/8 \end{bmatrix} \geq 0$$

We can solve $60 \leq \lambda \leq 132$.

Again, if we fix A and B and change the requirement for C from 72 to λ, we have the last row being:

$$-[60, 84, \lambda, 0, 0] + [84, 60] \begin{bmatrix} 0 & 1 & -3/2 & 1/4 & -3/8 \\ 1 & 0 & 9/2 & -1/4 & 7/8 \end{bmatrix} \geq 0$$

Solving this we get $\lambda \leq 144$.