

## Math 346, HW7 Solution

### 4.4.5

Using the weak duality, we need to show that  $(0, 3, 15/7, 0, 9/7)$  is feasible to the original linear program (I skip the verification of feasibility here), and  $(1/3, 1, 2/3)$  is feasible to the dual LP, which is:

$$\begin{array}{ll} \text{Minimize} & 6y_1 + 12y_2 + 6y_3 \\ \text{subject to} & 4y_1 + 3y_2 + y_3 \geq 3 \\ & y_1 + 3y_2 - 2y_3 \geq 2 \\ & -y_1 + 2y_2 + 5y_3 \geq 5 \\ & 2y_1 - y_2 - y_3 \geq -2 \\ & 4y_1 - y_2 + y_3 \geq 1 \\ & y_1, y_2, y_3 \geq 0. \end{array}$$

Moreover, we need to show that they give the same values for their objective function. For the primal,  $3x_1 + 2x_2 + 5x_3 - 2x_4 + x_5 = 2 \times 3 + 5 \times 15/7 + 9/7 = 18$ . For the dual we have  $6y_1 + 12y_2 + 6y_3 = 6 \times 1/3 + 12 \times 1 + 6 \times 2/3 = 18$ .

### 4.5.3 (d)

It is easy to check that for the solution  $(1, 0, 1, 0)$ , both constraints in the primal LP is binding. Since the first and third coordinate is strictly positive, it means that if it is the optimal solution, then the first and third constraint in the dual must be binding from the complementary slackness. The dual is

$$\begin{array}{ll} \text{Maximize} & 3y_2 \\ \text{subject to} & y_1 + 2y_2 \leq 5 \\ & 2y_1 + 3y_2 \leq 8 \\ & -y_1 + y_2 \leq 4 \\ & y_1 - y_2 \leq 2 \\ & y_1, y_2 \geq 0. \end{array}$$

So we have  $y_1 + 2y_2 = 5$  and  $-y_1 + y_2 = 4$ , solving this gives  $y_1 = -1$ ,  $y_2 = 3$ , which contradicts that  $y_i \geq 0$ . So the  $(1, 0, 1, 0)$  cannot be an optimal solution to the primal LP.

### 4.5.3 (h)

From the primal LP, we know that the first and third coordinate of the vector  $(1, 0, 3)$  is strictly positive. So if it is optimal, by the complementary slackness, the first and third constraints in the dual LP are binding. The dual LP is as follows:

$$\begin{aligned} \text{Minimize} \quad & 9y_1 + 4y_2 \\ \text{subject to} \quad & 3y_1 + y_2 \geq 6 \\ & 3y_1 + 2y_2 \geq 9 \\ & 2y_1 + y_2 \geq 5 \\ & y_1, y_2 \geq 0. \end{aligned}$$

So  $3y_1 + y_2 = 6$  and  $2y_1 + y_2 = 5$ , which gives  $y_1 = 1$  and  $y_2 = 3$ . Note that both  $(1, 0, 3)$  and  $(1, 3)$  are feasible to their LPs and they both give value 21 for the objective functions. By the weak duality, we know both solutions are optimal.

### Network flow problem

First we can check that the flow listed in the bottom picture is indeed a flow (by verifying that it satisfies the capacity constraint, and for every vertex other than the source and sink, the incoming flow is equal to the outgoing flow). This part is easy and I will leave it to you.

What is more important is to show that the flow is indeed max flow. By the max-flow min-cut theorem, we only need to find a cut of size equal to the size of this flow (which is 23). For example, the cut  $S = \{s, a, c, d\}$ ,  $T = \{b, t\}$  gives size  $c(S, T) = c_{ab} + c_{db} + c_{dt} = 12 + 7 + 4 = 23$ .

An alternative way is to use the augmenting path algorithm I mentioned in the class: first create an auxiliary directed graph, then show that there is no directed path from  $s$  to  $t$ . This also leads to the min cut.