Math 346, HW6 Solution

4.2.1 (e)

The dual of the linear program is:

$$\begin{array}{ll} \text{minimize} & 20y_1 + 40y_2 + 60y_3 \\ \text{subject to} & 8y_2 \geq 1 \\ & 2y_1 + y_3 = -7 \\ & 5y_1 - 3y_2 + 4y_3 \geq 3 \\ & y_1, y_2 \text{ unrestricted}, y_3 \leq 0. \end{array}$$

4.2.3

In the linear programming problem of Example 4.2.1,

$$\begin{array}{ll} \text{Maximize} & 6x_1 + x_2 + 4x_3 \\ \text{subject to} & 3x_1 + 7x_2 + x_3 \leq 15 \\ & x_1 - 2x_2 + 3x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

the dual is equal to

$$\begin{array}{ll} \text{Minimize} & 15y_1 + 20y_2 \\ \text{subject to} & 3y_1 + y_2 \geq 6 \\ & 7y_1 - 2y_2 \geq 1 \\ & y_1 + 3y_2 \geq 4 \\ & y_1, y_2 \geq 0. \end{array}$$

(a) To show that the dual is bounded from below, note that $15y_1 + 20y_2 \geq 3y_1 + y_2 \geq 6$, the first inequality is because both y_1 and y_2 are nonnegative, the second inequality comes from the first constraint.

(b) (the sketching skipped) the optimal solution is $(y_1, y_2) = (7/4, 3/4)$, the optimum value of LP is $165/4$.

(c) We use the simplex algorithm in the tableau form. Note that the slack variables can serve as initial basis. We also convert it into the minimization of $-6x_1 - x_2 - 4x_3$.

	x_1	x_2	x_3	x_4	x_5	
x_4	3	7	1	1	0	15
x_5	1	−2	3	0	1	20
	−6	−1	−4	0	0	0

	x_1	x_2	x_3	x_4	x_5	
x_1	1	7/3	1/3	1/3	0	5
x_5	0	−13/3	8/3	−1/3	1	15
	0	13	−2	2	0	30

	x_1	x_2	x_3	x_4	x_5	
x_1	1	23/8	0	3/8	−1/8	25/8
x_3	0	−13/8	1	−1/8	3/8	45/8
	0	39/4	0	7/4	3/4	165/4

So the optimal value (for the maximization problem) is also 165/4, attained by $(x_1, x_2, x_3) = (25/8, 0, 45/8)$.

(d) Note that the coefficients of x_4 and x_5 in the last row of the final tableau gives the optimal solution of the dual. (Please see the proof of Theorem 4.4.2 on pages 140–142 why this is always true).

4.4.1

From the weak duality, we know that if x_0 is a feasible solution to the maximization problem and y_0 is a feasible solution to its dual, then $c^T x_0 \leq b^T y_0$. So suppose the dual minimization problem is feasible, then for all feasible x_0, the maximization problem is bounded from above by $b^T y_0$, which contradicts its unboundedness. Therefore if the maximization problem is not bounded from above, then the dual is infeasible. Similarly one can show the second part of the statement.

4.4.2

The primal LP is not feasible, because if $x_1 - x_2 \leq 1$ and $-x_1 + x_2 \leq -2$, then we have $2 \leq x_1 - x_2 \leq 1$, then $2 \leq 1$, contradiction.

Similarly the dual LP is as follows:

$$\begin{array}{ll} \text{Minimize} & y_1 - 2y_2 \\ \text{subject to} & y_1 - y_2 \geq 1 \\ & -y_1 + y_2 \geq 0 \\ & y_1, y_2 \geq 0. \end{array}$$

Then a feasible solution must have $1 \leq y_1 - y_2 \leq 0$, which gives $1 \leq 0$, again contradiction. So both the primal and the dual LPs are infeasible.

4.5.2

(a) The dual LP is:

$$\begin{array}{ll} \text{Minimize} & y_1 + y_2 + 3y_3 \\ \text{subject to} & y_1 + y_3 \geq 2 \\ & y_2 + y_3 \geq 2 \\ & y_1 + y_2 + 2y_3 \geq 0 \\ & y_1 - y_2 \geq 0 \\ & y_1, y_2, y_3 \geq 0. \end{array}$$

(b) It is easy to check that $X^* = (1, 1, 0, 0)$ satisfies all the constraints in the primal (the first two are binding), and $Y^* = (1, 1, 1)$ satisfies all the constraints of the dual (first, second and fourth constraint are binding).

(c) Note that X_j^* is strictly positive for $j = 1, 2$, and the first and second constraint of the dual problem is binding (meaning that the slack is equal to zero).

(d) Y^* is not an optimal solution, for example taking $Y = (2, 2, 0)$, it is feasible to the dual, and the value of the objective functino is 4 which is smaller than 5 that $(1, 1, 1)$ gives.

(e) This does not contradict the complementary slackness theorem, for the reason that $Y_3^* = 1 > 0$ and the third constraint in the primal is also non-binding.