Math 346, HW3 Solution

3.2.2

Pivoting on the term x_1 in the first equation gives (basically, adding 2 times the first equation to the second one)

$$\begin{aligned} &x_1 + x_2 - 3x_3 = 7 \\ &3x_2 - x_3 = 16 \\ &3x_2 - x_3 = 16 \end{aligned}$$

Therefore this system of linear equations is redundant. We might eliminate the last equation, and then pivot on $3x_2$, this gives

$$\begin{aligned} &x_1 - \tfrac{8}{3}x_3 = \tfrac{5}{3} \\ &x_2 - \tfrac{1}{3}x_3 = \tfrac{16}{3} \end{aligned}$$

Note that choosing $\{x_1, x_2\}$ as basic variables gives a basic feasible solution $(5/3, 16/3, 0)$, and the system above is in the canonical form with respect to $\{x_1, x_2\}$.

3.2.6

(a) In order to determine which choices of basis leads to a basic feasible solution, geometrically we have to decide which two columns of the matrix A from the LP could span a cone containing the vector $(2, 2)$. (For more explanations, check Example 3.2.3). The four column vectors of matrix A are $(1, -2)$, $(1, 0)$, $(-2, 1)$ and $(3, 0)$. It is not hard to see that $(2, 2)$ is only contained in the cone spanned by $(-2, 1)$ and $(1, 0)$, and the cone spanned by $(-2, 1)$ and $(3, 0)$. Therefore there are only two basic feasible solutions.

(b) These two basic feasible solutions can be computed in the following way. Choosing $\{x_2, x_3\}$ as the basis gives $x_1 = x_4 = 0$, this leads to the equations $x_2 - 2x_3 = 2$ and $x_3 = 2$. Solving this system gives $x_2 = 6$ and $x_3 = 2$, therefore $(x_1, x_2, x_3, x_4) = (0, 6, 2, 0)$ is a basic feasible solution. The other BFS can be obtained by taking $\{x_3, x_4\}$ as the basis, so $x_1 = x_2 = 0$, and solving $-2x_3 + 3x_4 = 2$ and $x_3 = 2$ gives $(x_1, x_2, x_3, x_4) = (0, 0, 2, 2)$.

(c) There are a couple of ways to show that the objective function is bounded from below (meaning that there exists a constant C, such that the objective function is always greater or equal to C). One easy way is to the following inequality:

$$5x_1 + 2x_2 + 3x_3 + x_4 \geq -2x_1 + x_3 = 2.$$

The inequality is because all x_i's are nonnegative. Even easier, one can just use all the variables are nonnegative to conclude that the objective function is at least 0. Alternatively you might compute the extreme directions for this LP, and show that the inner product of the vector $(5, 2, 3, 1)$ with all the extreme vectors are nonnegative.

(d) From this assumption that the minimal value is attained at a basic feasible solution, we just check the value of the objective function at these two BFS we computed in (b). $(0, 6, 2, 0)$ gives 18 and $(0, 0, 2, 2)$ gives 8. So the optimal solution is $(0, 0, 2, 2)$ which gives the smaller value 8.

LP in Exercise 3.3.2 on Page 76

The linear program is already in the standard form. To solve it, we may use the general representation theorem. First we compute the extreme points of this linear program (that is, the basic feasible solutions). This could be done by choosing all the $\binom{4}{2}$ pairs of variables as basis. We start from choosing x_1, x_2 as the basis, therefore $x_3 = x_4 = 0$, this gives $x_1 = 6$, $x_2 = 2$, which gives a basic solution: $(6, 2, 0, 0)$. Similarly choosing other possible bases gives us the following basic solutions: $(3, 0, 1, 0)$, $(7, 0, 0, \frac{1}{2})$, $(0, -2, 2, 0)$, $(0, 14, 0, -3)$, $(0, 0, \frac{7}{4}, -\frac{3}{8})$. Among them, only the followings are BFS: $(6, 2, 0, 0)$, $(3, 0, 1, 0)$, $(7, 0, 0, \frac{1}{2})$.

Now we compute the extreme directions, suppose (d_1, d_2, d_3, d_4) is a extreme direction, then it must be a BFS of the following linear program:

$$\begin{aligned}
&d_1 + d_2 + 5d_3 + 2d_4 = 0 \\
&2d_1 + d_2 + 8d_3 = 0 \\
&d_1 + d_2 + d_3 + d_4 = 1 \\
&d_1, d_2, d_3, d_4 \geq 0
\end{aligned}$$

Note that all d_i are nonnegative and their positive linear combination $d_1 + d_2 + 5d_3 + 2d_4 = 0$. This implies that $d_1 = d_2 = d_3 = d_4 = 0$, wich contradicts the third constraint. Therefore there is no recession direction. From the representation theorem, any feasible solution to the original LP is the convex combination of the extreme points. By plugging the basic feasible solutions to the objective function. We get the optimal solution is $(3, 0, 1, 0)$, with optimum value 7.

The intersection of a finite number of convex sets is convex

Suppose $S_1, \cdots, S_n$ are the convex sets considered. We would like to prove that $\bigcap_{i=1}^n S_i$ is also a convex set. It suffices to show that for any two points $x, y \in \bigcap_{i=1}^n S_i$, and an arbitrary $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in \bigcap_{i=1}^n S_i$. Note that for every $i = 1, \cdots, n$, both x and y is contained in S_i. Since S_i is convex,

$\lambda x + (1 - \lambda)y$ is still contained in S_i. Note that this holds for all i, so clearly $\lambda x+(1-\lambda)y$ is contained in the intersection of all S_i. And the proof is complete.