Class 5 - Rates of change: average vs. instantaneous - §2.1

1. **Warm-up.** In 2013, Diana Nyad became the first person to swim the 110-mile Florida Straits unaided. Other swimmers questioned whether her swim was truly unaided; one of the things they looked at was her speed along the swim.(1) In response, Nyad released GPS data about her 53-hour swim. Here's a very small sample of the data:

Time (hours)	0	5	7	15	31	32	40
Distance (miles)	0	6.67	8.90	18.72	54.85	58.72	82.88

Find Nyad's average speed between 5 and 7 hours into her swim. Also find her average speed between 31 and 32 hours into her swim. Why might these numbers have raised suspicion?

2. A rocket is launched straight up into the air. The function $f(t) = 10(t+2)^3 - 80$ describes the rocket's height in meters t seconds after liftoff.

(a) Without using a calculator, sketch the graph of $f(t)$ for t between 0 and 7.

(b) Find the average velocity of the rocket between $t = 1$ and $t = 3$ (be sure to include units). How would you represent this velocity graphically on your graph from (a)?

Suppose you'd now like to know the rocket's *instantaneous* velocity 5 seconds after liftoff. That is, you'd like to know how fast the rocket is going right at the instant $t = 5$.

(c) Would it work to find the rocket's average velocity between $t = 5$ and $t = 5$? Why or why not?

(1) See `http://www.nytimes.com/2013/09/09/sports/questions-and-doubt-after-a-record-swim-from-cuba-to-florida.html`

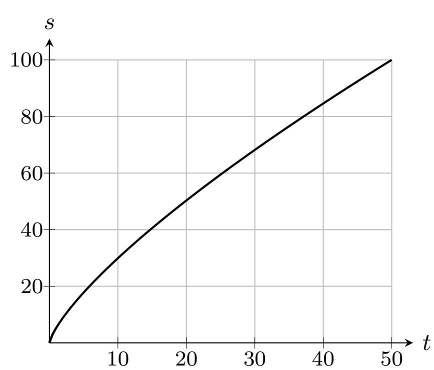
Continued: $f(t) = 10(t+2)^3 - 80$ is the rocket's height in meters t seconds after liftoff.

(d) How could you use average velocities to *approximate* the rocket's instantaneous velocity at $t = 5$? Use your idea to give a good estimate of the instantaneous velocity at $t = 5$.

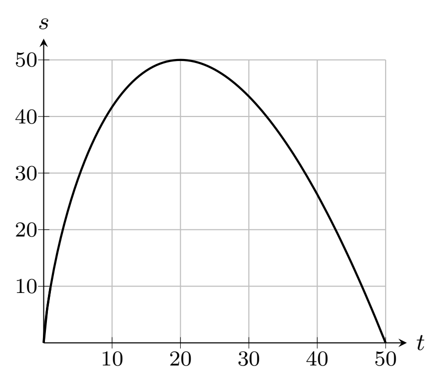
(e) Based on your answer to (d), what do you think the rocket's instantaneous velocity 5 seconds after liftoff is? How would you visualize this instantaneous velocity on the graph you drew in (a)?

3. *Average speed vs. average velocity.* A swimmer is swimming a 100 m long race, which is one lap in a 50 m long pool. Let $s(t)$ be his distance from the starting position t seconds after the start of the race.

(a) Which of the following is a more reasonable graph for $s(t)$? Why?



(A)



(B)

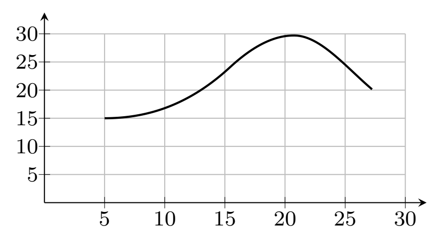
(b) According to the graph you chose, what was the swimmer's average speed for the race? Average velocity for the race?

(c) What was the swimmer's average speed over the first 20 seconds of the race? Average velocity?

(d) What was the swimmer's average speed over the last 50 m of the race? Average velocity?

(e) What is the difference between velocity and speed?

4. The maximum swimming speed of salmon depends on the water temperature. Suppose $S(T)$ is the maximum swimming speed (in cm/s) of salmon when the water temperature is T degrees Celsius; the graph of S is shown at right. What does the average rate of change of S on the interval $[15, 20]$ represent? What are its units? Graphically, how would you represent this average rate of change?



5. So far, we have described concavity only qualitatively. We can give a more precise definition using tangent lines. Draw a few examples of functions that are concave up on an interval $[a, b]$, as well as a few examples of functions that are concave down on an interval $[a, b]$. Then, use your graphs to fill in the following blanks with either "above" or "below".

- When a curve is concave up, its tangent lines lie ______________ the curve.
- When a curve is concave down, its tangent lines lie ______________ the curve.

Class 5 - Rates of change: average vs. instantaneous - §2.1

1. **Warm-up.** In 2013, Diana Nyad became the first person to swim the 110-mile Florida Straits unaided. Other swimmers questioned whether her swim was truly unaided; one of the things they looked at was her speed along the swim.[1] In response, Nyad released GPS data about her 53-hour swim. Here's a very small sample of the data:

Time (hours)	0	5	7	15	31	32	40
Distance (miles)	0	6.67	8.90	18.72	54.85	58.72	82.88

Find Nyad's average speed between 5 and 7 hours into her swim. Also find her average speed between 31 and 32 hours into her swim. Why might these numbers have raised suspicion?

2. A rocket is launched straight up into the air. The function $f(t) = 10(t+2)^3 - 80$ describes the rocket's height in meters t seconds after liftoff.

   (a) *Without using a calculator, sketch the graph of $f(t)$ for t between 0 and 7.*

   (b) *Find the average velocity of the rocket between $t = 1$ and $t = 3$ (be sure to include units). How would you represent this velocity graphically on your graph from (a)?*

   Suppose you'd now like to know the rocket's *instantaneous* velocity 5 seconds after liftoff. That is, you'd like to know how fast the rocket is going right at the instant $t = 5$.

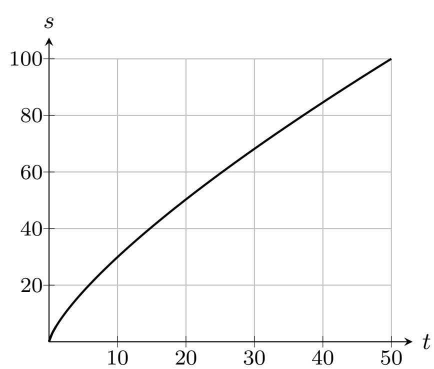
   (c) *Would it work to find the rocket's average velocity between $t = 5$ and $t = 5$? Why or why not?*

   (d) *How could you use average velocities to approximate the rocket's instantaneous velocity at $t = 5$? Use your idea to give a good estimate of the instantaneous velocity at $t = 5$.*

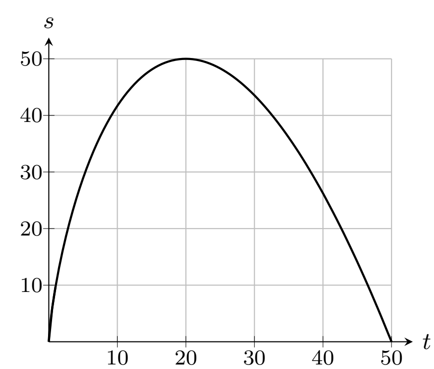
   (e) *Based on your answer to (d), what do you think the rocket's instantaneous velocity 5 seconds after liftoff is? How would you visualize this instantaneous velocity on the graph you drew in (a)?*

3. *Average speed vs. average velocity.* A swimmer is swimming a 100 m long race, which is one lap in a 50 m long pool. Let $s(t)$ be his distance from the starting position t seconds after the start of the race.

   (a) *Which of the following is a more reasonable graph for $s(t)$? Why?*



(A)



(B)

[1] See http://www.nytimes.com/2013/09/09/sports/questions-and-doubt-after-a-record-swim-from-cuba-to-florida.html

Solution. Choice (B) is the reasonable one: when the swimmer reaches the opposite end of the pool, he is 50 m from the starting point. At the end of the race, he is back at the starting point, so he is 0 m from the starting point. (Choice (A) would say that, at the end of the race, the swimmer is 100 m from his starting point.)

(b) *According to the graph you chose, what was the swimmer's average speed for the race? Average velocity for the race?*

Solution. First, remember that average speed $= \frac{\text{change in distance traveled}}{\text{change in time}}$, while average velocity $= \frac{\text{change in position}}{\text{change in time}}$.

When the swimmer starts the race, he has traveled 0 m; when the swimmer ends, he has traveled 100 m; therefore, the change in his distance traveled is 100 m. The race takes 50 seconds, so his average speed for the race is $\frac{100 \text{ m}}{50 \text{ s}} = 2$ m/s.

The swimmer's change in position for the race is 0 because he ends in the same place he started, so his average velocity for the race is $\frac{0 \text{ m}}{50 \text{ s}} = 0$ m/s.

(c) *What was the swimmer's average speed over the first 20 seconds of the race? Average velocity?*

Solution. In the first 20 seconds, the swimmer swam a distance of 50 m. So, his average speed on this interval was $\frac{50 \text{ m}}{20 \text{ s}} = 2.5$ m/s.

The swimmer's average velocity over the first 20 seconds is $\frac{\text{change in position}}{\text{change in time}} = \frac{s(20)-s(0)}{20-0} = \frac{50 \text{ m}}{20 \text{ s}} =$ 2.5 m/s.

(d) *What was the swimmer's average speed over the last 50 m of the race? Average velocity?*

Solution. The swimmer swam the last 50 m in 30 seconds, so his average speed was $\frac{50 \text{ m}}{30 \text{ s}} = \frac{5}{3}$ m/s.

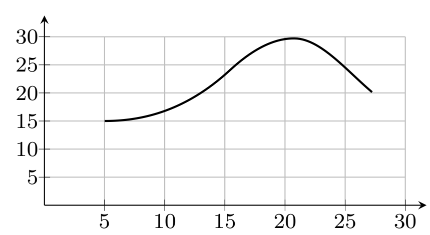
His average velocity on this interval was $\frac{s(50)-s(20)}{50-20} = \frac{-50 \text{ m}}{30 \text{ s}} = -\frac{5}{3}$ m/s.

(e) *What is the difference between velocity and speed?*

Solution. As we said already, average velocity over an interval is $\frac{\text{change in position}}{\text{change in time}}$, while average speed is $\frac{\text{change in distance traveled}}{\text{change in time}}$. Just knowing the value of one of these quantities does not tell us the other; for example, in (b), knowing that the average speed for the race is 2 m/s doesn't tell us what the average velocity is, and knowing that the average velocity is 0 m/s doesn't tell us the swimmer's average speed.

On the other hand, instantaneous speed is simply the absolute value of instantaneous velocity.

4. *The maximum swimming speed of salmon depends on the water temperature. Suppose $S(T)$ is the maximum swimming speed (in cm/s) of salmon when the water temperature is T degrees Celsius; the graph of S is shown at right. What does the average rate of change of S on the interval $[15, 20]$ represent? What are its units? Graphically, how would you represent this average rate of change?*



Solution. The average rate of change of S on the interval $[15, 20]$ is $\frac{S(20)-S(15)}{20-15}$; the numerator has units of cm/s, while the denominator has units of °C, so the average rate of change has units of $\boxed{\text{cm/s per °C}}$.

From the graph, we can see that $S(20) \approx 30$ and $S(15) \approx 23$, so this average rate of change is $\approx \frac{7}{5}$, or ≈ 1.4 cm/s per °C. What this means is that, when the temperature is around 15 to 20°C, we expect each increase of 1°C to cause the salmons' maximum speed to increase by about 1.4 cm/s. (For example, if the temperature were to increase from 16°C to 16.5°C, we'd expect the salmons' maximum swimming speed to increase by about (1.4 cm/s per °C)(0.5°C) = 0.7 cm/s.) Graphically, we can visualize this average rate of change as the $\boxed{\text{slope of the secant line between } (15, S(15)) \text{ and } (20, S(20))}$.

5. *So far, we have described concavity only qualitatively. We can give a more precise definition using tangent lines. Draw a few examples of functions that are concave up on an interval $[a, b]$, as well as a few examples of functions that are concave down on an interval $[a, b]$. Then, use your graphs to fill in the following blanks with either "above" or "below".*

   - *When a curve is concave up, its tangent lines lie* ***below*** *the curve.*
   - *When a curve is concave down, its tangent lines lie* ***above*** *the curve.*

   Solution. This is easy to see if you draw a few pictures: