# Class 12 - Differentiation Rules - §3.2

In Chapter 2 we were only able to use the limit definition of the derivative to compute derivatives. Last class we finally saw some quick ways of computing derivatives. Today we'll continue developing some new rules, to find even more derivatives! All of these come from the limit definition though!

**Differentiation Rules to Remember!**

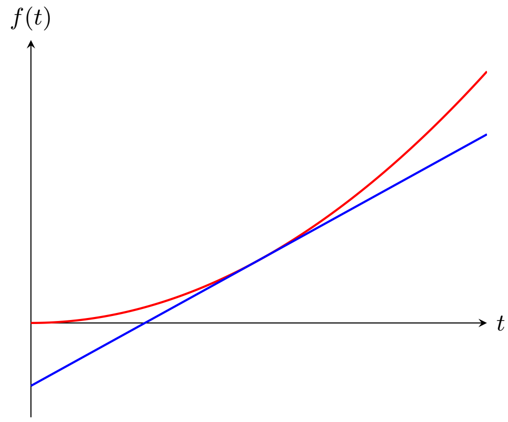
> *Civilization advances by extending the number of important operations which we can perform without thinking about them.* – Alfred Whitehead, mathematician & philosopher

- **Derivative of a Constant:** $\frac{d}{dx}(c) = 0$
- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Constant Multiple Rule:** If $c$ is a constant, then $\frac{d}{dx}[cf(x)] = cf'(x)$
- **Derivative of the Natural Exponential Function:** $\frac{d}{dx}(e^x) = e^x$
- **Sum Rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)g(x)}$

1. *Warm-up.* At noon one day, Jack plants a magical bean. To his delight, it grows quickly into a beanstalk. Let $f(t)$ be the beanstalk's height in inches $t$ hours after noon.

   (a) *If $f'(3) = 7$, roughly how much does the beanstalk grow between 3:00 pm and 3:03 pm?*

   **Solution.** The derivative tells us the instantaneous rate of change; $f'(3) = 7$ means that the instantaneous rate of change is 7 feet per hour. We do not have a good idea of what the graph of $f(x)$ looks like except that it has a slope of 7 at 3. Here is a possible graph for $f$:




There are two observations to make:

- the tangent line is a good approximation for $f(t)$ near $t = 3$.
- the slope of the tangent line is 7

These two facts show use that the beanstalk has roughly grown $\frac{6}{60} \cdot 7$ in those 6 minutes.

(b) *More generally, if $h$ is a fairly small positive number, write an approximation for the amount that the beanstalk grows between times $t$ and $t + h$.*

**Solution.** $f(t+h) - f(t) \approx hf'(t)$

2. **The Product Rule.** If we have two differentiable functions $f(x)$ and $g(x)$, we'd like to find the derivative of their product, $P(x) = f(x)g(x)$.

(a) *Gottfried Leibniz, one of the inventors of calculus, initially thought (incorrectly!) that $P'(x) = f'(x)g'(x)$. Verify that this is incorrect when $f(x) = x^2$ and $g(x) = x^3$.*

**Solution.** Here $P(x) = x^5$, so $P'(x) = 5x^4$. On the other hand, $f'(x)g'(x) = (2x)(3x^2) = 6x^3$, which is not equal to $P'(x)$.

**Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)$

(b) *Find the derivative of $f(x) = (x^2 + 1)(3\sqrt{x} - 1)$ in two ways: first by multiplying it out and then using the Product Rule. Make sure your answers agree.*

**Solution.** If we expand out the expression for $f$ and then differentiate, we get

$$\begin{aligned} f(x) &= (x^2+1)(3x^{1/2} - 1) \\ &= 3x^{5/2} - x^2 + 3x^{1/2} - 1 \\ f'(x) &= \boxed{\frac{15}{2}x^{3/2} - 2x + \frac{3}{2}x^{-1/2}} \end{aligned}$$

On the other hand, the Product Rule gives

$$f'(x) = 2x(3x^{1/2} - 1) + (x^2 + 1) \cdot \frac{3}{2}x^{-1/2}$$

To see that this is the same as our previous expression for $f'$, let's expand it out:

$$\begin{aligned} &= (6x^{3/2} - 2x) + \left(\frac{3}{2}x^{3/2} + \frac{3}{2}x^{-1/2}\right) \\ &= \frac{15}{2}x^{3/2} - 2x + \frac{3}{2}x^{-1/2}, \end{aligned}$$

which is the same as our previous expression for $f'$.

(c) Now, let's understand why the Product Rule is true; as usual when coming up with a new derivative rule, we'll need to go back to the definition of the derivative. Suppose we have two differentiable functions $f$ and $g$, and let $P(x) = f(x)g(x)$ be their product.

i. *According to the limit definition of the derivative, what is $P'(x)$?* **Solution.** To calculate the derivative use the limit definition:

$$P'(x) = (f(x)g(x))' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{x + h - x}$$

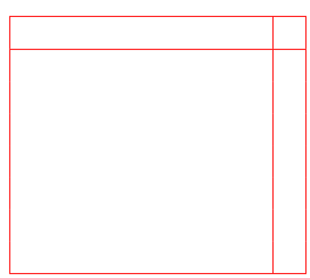
To help us visualize what's going on, let's imagine that we have a rectangle that is growing over time. Let $f(x)$ be its width in cm at time $x$ and $g(x)$ be its length in cm at time $x$, where $x$ is measured in seconds.

A. *Draw a picture to help you understand the limit you wrote in (c)i.*

**Solution.**

The set up to this problem is to about a rectangle that is growing. The width of the rectangle is given by $f(x)$ and the length is given by $g(x)$. So the area is given by $f(x)g(x)$.

Below is a picture of what the rectangle looks like after we let time increment from $x$ to $x + h$



We can use what we learned from the warm-up problem to think about how the area of the rectangle has changed from time $x$ to time $x+h$

$$f(x+h) \approx f(x) + hf'(x)$$

and

$$g(x+h) \approx g(x) + hg'(x)$$

.

$$f(x+h)g(x+h) - f(x)g(x) \approx f'(x)hg(x) + f(x)g'(x)h + f'(x)hg'(x)h$$

So now we can work on the limit

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{x+h-x} &= \lim_{h\to 0} \frac{f'(x)hg(x) + f(x)g'(x)h + f'(x)hg'(x)h}{h} \\ &= \lim_{h\to 0} f'(x)g(x) + f(x)g'(x) + f'(x)hg'(x) \\ &= f'(x)g(x) + g'(x)f(x)\end{aligned}$$

This is the all-important rule called the product rule.

(c) *Find* $\frac{d}{dx}\left(\sqrt[3]{x}e^x\right)$.

**Solution.** By the Product Rule, $\frac{d}{dx}\left(x^{1/3}e^x\right) = \boxed{\frac{1}{3}x^{-2/3}e^x + x^{1/3}e^x}$.

3. ***The Quotient Rule.*** *If $f(x)$ and $g(x)$ are differentiable functions, we might like to know how to differentiate the quotient $\frac{f(x)}{g(x)}$. In this problem, we'll see how. If $Q(x) = \frac{f(x)}{g(x)}$, then*

$$g(x)Q(x) = f(x).$$

*Differentiate both sides of this equation, and use that to find an expression for $Q'(x)$ in terms of $f(x), f'(x), g(x), g'(x)$.*

**Solution.** We have that $g(x)Q(x) = f(x)$. Let's differentiate both sides with respect to $x$. We'll need to use the product rule on the left hand side:

$$\begin{aligned}[g(x)Q(x)]' &= [f(x)]' \\ g(x)Q'(x) + g'(x)Q(x) &= f'(x)\end{aligned}$$

We need to solve for $Q'(x)$

$$\begin{aligned}g(x)Q'(x) &= f'(x) - g'(x)Q(x) \\ Q'(x) &= \frac{f'(x) - g'(x)Q(x)}{g(x)}\end{aligned}$$

Let's replace $Q(x)$ by $\frac{f(x)}{g(x)}$

$$\begin{aligned}Q'(x) &= \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)} \\ Q'(x) &= \frac{\frac{f'(x)g(x) - g'(x)f(x)}{g(x)}}{g(x)} \\ Q'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{g(x)g(x)}\end{aligned}$$

There we have it– that's the quotient rule! If you forget the quotient rule, you can always use the strategy we just used, of converting a quotient to a product to find the derivative!

**Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

4. *Let* $f(x) = \frac{x}{x^2+1}$. *Find* $f'(x)$

   **Solution.** We use the Quotient Rule to differentiate: $f'(x) = \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \boxed{\frac{1-x^2}{(x^2+1)^2}}$.

5. *Let* $f(x) = x^4e^{-x}$. *Find* $f'(x)$. *(*Hint: *You should know the derivative of* $e^x$ *very well, but the derivative of* $e^{-x}$ *is not one of the basic derivatives we've learned. Can you rewrite* $f(x)$ *in terms of* $e^x$ *rather than* $e^{-x}$*?)*

   **Solution.** We can rewrite $f(x) = \frac{x^4}{e^x}$ and use the Quotient Rule to differentiate: $f'(x) = \frac{e^x \cdot 4x^3 - x^4 \cdot e^x}{(e^x)^2} = \boxed{\frac{4x^3 - x^4}{e^x}}$.