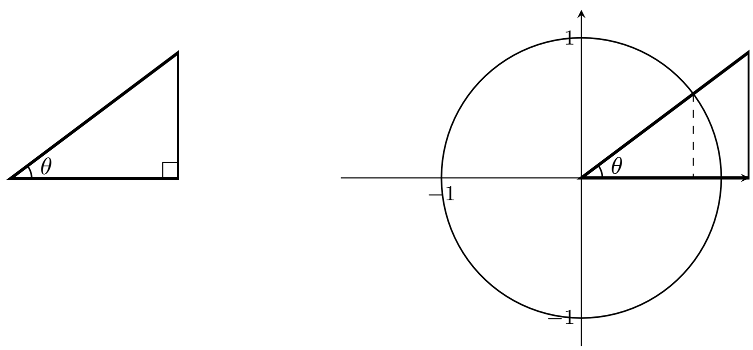
# Right Triangle Trigonometry

*Trigonometry means "measuring triangles," something people have been doing for centuries in astronomy, navigation, surveying, optics and other areas.*

To see the connection between a right triangle and the unit circle, we imagine drawing the triangle together with the unit circle like this:




So, in terms of **right** triangles:

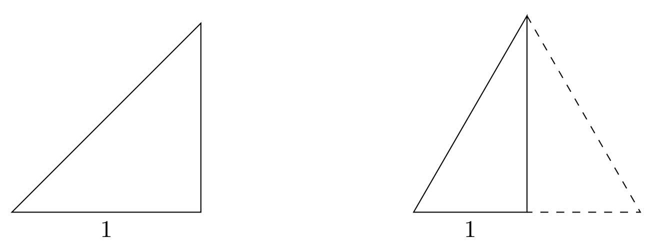
$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

We also define

$$\text{"cosecant of } \theta\text{"} : \csc\theta = \frac{1}{\sin\theta} \qquad \text{"secant of } \theta\text{"} : \sec\theta = \frac{1}{\cos\theta} \qquad \text{"cotangent of } \theta\text{"} : \cot\theta = \frac{\cos\theta}{\sin\theta}$$

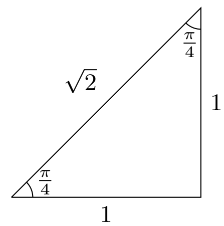
1. Here are two special right triangles, each shown with base length 1:

a right isosceles triangle　　　half of an equilateral triangle

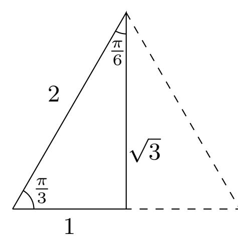



(a) *Find and label the lengths of all other sides of the triangles, as well as the size of the angles in each triangle.*

**Solution.** Let's start with the right isosceles triangle. "Isosceles" means the two legs have the same length, so both legs have length 1. By the Pythagorean Theorem, the hypotenuse therefore has length $\sqrt{2}$. The angles of this triangle are $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$. Thus, here is a completely labeled triangle:




For the equilateral triangle, since half of one side has length 1, the sides of the equilateral triangle are all 2. Therefore, the left half is a right triangle with base 1 and hypotenuse 2; by the Pythagorean Theorem, its height is $\sqrt{3}$. The angles of the equilateral triangle are all $\frac{\pi}{3}$, and half of this is $\frac{\pi}{6}$:

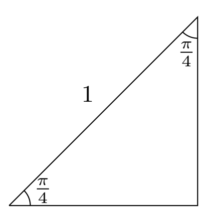


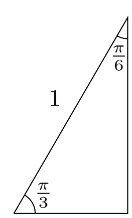

(b) *From your completed pictures, you should be able to determine sine, cosine, and tangent of the acute angles in each triangle. Do this.*

**Solution.** From the right isosceles triangle, we see that $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, and $\tan\frac{\pi}{4} = 1$.

From the other triangle, we see that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$, $\tan\frac{\pi}{3} = \sqrt{3}$, $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

2. *It's often useful to think about the two special right triangles as having hypotenuse 1 rather than base length 1. Find and label the lengths of the legs of these two triangles:*

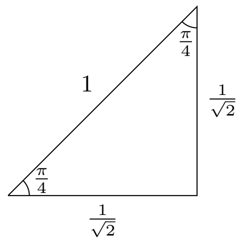


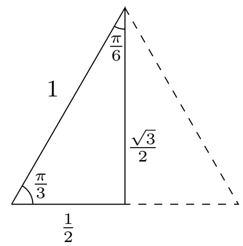




**Solution.** The first triangle is similar[(1)] to the right isosceles triangle in #1. Since this triangle has hypotenuse 1 and the one in #1 had hypotenuse $\sqrt{2}$, the sides of this one are all $\frac{1}{\sqrt{2}}$ as big as the ones in #1; therefore, the two legs both have length $\frac{1}{\sqrt{2}}$.

The second triangle is similar to the "half of an equilateral triangle" in #1; the sides of this one are $\frac{1}{2}$ as long as the ones in #1:

(1) Remember that two triangles are similar if they have the same angles; this means that one is simply a scaled up version of the other.



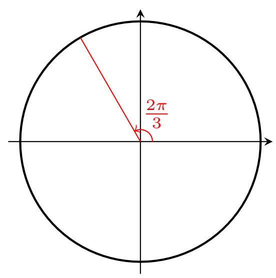




3. *Combining the right triangle and unit circle perspectives.*

   Suppose we'd like to find $\cos\left(\frac{2\pi}{3}\right)$. We can't draw a right triangle with an angle of $\frac{2\pi}{3}$ (why not?), so we'll combine our knowledge of the unit circle and right triangles.

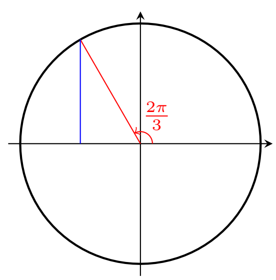
   (a) *Sketch the unit circle and the angle $\frac{2\pi}{3}$ in standard position. (The most important thing to get right in your sketch is the quadrant that the terminal side of the angle is in.)*

   **Solution.**



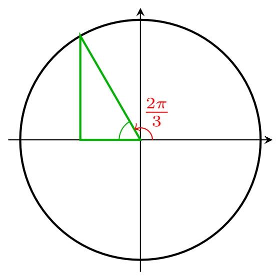

   (b) *On your sketch from (a), draw a perpendicular from the point $P\left(\frac{2\pi}{3}\right)$ to the $x$-axis. You should now see a triangle in your picture. We call this the reference triangle for the angle $\frac{2\pi}{3}$.*

   **Solution.** We draw in the blue line to the $x$-axis:



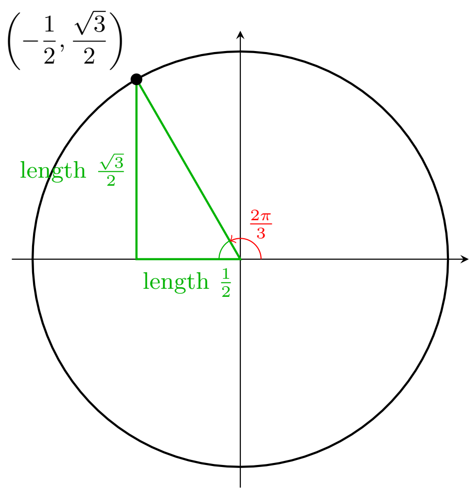

   (c) *What is the angle in the reference triangle between the hypotenuse and the $x$-axis? This is called the reference angle for $\frac{2\pi}{3}$.*

   **Solution.** The reference angle is the one marked in green, and it is $\frac{2\pi}{3}$ less than $\pi$, so $\boxed{\frac{\pi}{3}}$.




(d) *Using your reference triangle, you should now be able to see what* $\cos\left(\frac{2\pi}{3}\right)$ *is. What is it? How about* $\sin\left(\frac{2\pi}{3}\right)$*?*

**Solution.** The reference triangle is a 30°-60°-90° triangle (or $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ radians) with hypotenuse 1, so it's exactly the second triangle in #2. Therefore, the leg adjacent to the reference angle has length $\frac{1}{2}$, and the leg opposite the reference angle has length $\frac{\sqrt{3}}{2}$. Taking into account the quadrant the angle is in, the coordinates of $P\left(\frac{2\pi}{3}\right)$ are therefore $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

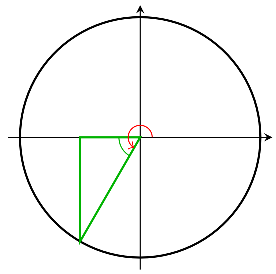



So, $\boxed{\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}}$ and $\boxed{\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}}$.

4. Use the same idea as in #3 to find **exact** values for the following:

(a) $\sin\left(\frac{4\pi}{3}\right)$

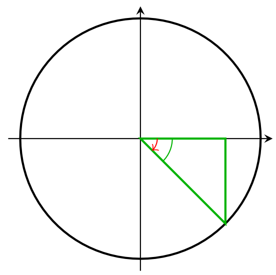
**Solution.** Here is the angle $\frac{4\pi}{3}$ (in red) and the reference triangle and reference angle (in green):



The reference angle is $\frac{\pi}{3}$ and the reference triangle has hypotenuse 1, so the side opposite the reference angle has length $\frac{\sqrt{3}}{2}$. Since the point is below the $x$-axis, $\sin\left(\frac{4\pi}{3}\right) = \boxed{-\dfrac{\sqrt{3}}{2}}$.

(b) $\sin\left(-\dfrac{\pi}{4}\right)$

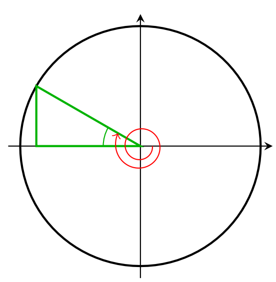
**Solution.** Here is the angle $-\frac{\pi}{4}$ (in red) and the reference triangle and reference angle in green:



It may look like the red angle and the green angle are the same, but we interpret them differently; since the red angle is in standard position, it's negative. We are thinking of the green angle as an acute angle in a right triangle, so it's positive: $\frac{\pi}{4}$. So, the reference triangle is like the first one in #2: its legs both have length $\frac{1}{\sqrt{2}}$. Therefore, $\sin\left(-\frac{\pi}{4}\right) = \boxed{-\dfrac{1}{\sqrt{2}}}$.

(c) $\tan\left(-\dfrac{19\pi}{6}\right)$

**Solution.** Here is $-\frac{19\pi}{6}$ (in red) and the reference triangle and reference angle (in green):



The reference angle is $\frac{\pi}{6}$, so the horizontal side of the reference triangle has length $\frac{\sqrt{3}}{2}$ and the vertical side has length $\frac{1}{2}$. Therefore, $\cos\left(-\frac{19\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{19\pi}{6}\right) = \frac{1}{2}$, so $\tan\left(-\frac{19\pi}{6}\right) =$